Stability and Instability. The concepts of stability, asymptotic stability, and instability have already been mentioned several times in this book. It is now time to give a precise mathematical definition of these concepts, at least for autonomous systems of the form

$$\mathbf{x}' = \mathbf{f}(\mathbf{x}). \tag{5}$$

In the following definitions, and elsewhere, we use the notation $\|\mathbf{x}\|$ to designate the length, or magnitude, of the vector $\mathbf{x}$.

The points, if any, where $\mathbf{f}(\mathbf{x}) = \mathbf{0}$ are called **critical points** of the autonomous system (5). At such points $\mathbf{x}' = \mathbf{0}$ also, so critical points correspond to constant, or equilibrium, solutions of the system of differential equations. A critical point $\mathbf{x}^0$ of the system (5) is said to be **stable** if, given any $\epsilon > 0$, there is a $\delta > 0$ such that every solution $\mathbf{x} = \boldsymbol{\phi}(t)$ of the system (1), which at $t = 0$ satisfies

$$\|\boldsymbol{\phi}(0) - \mathbf{x}^0\| < \delta, \tag{6}$$

exists for all positive t and satisfies

$$\|\boldsymbol{\phi}(t) - \mathbf{x}^0\| < \epsilon \tag{7}$$

for all $t \geq 0$. This is illustrated geometrically in Figures 9.2.1*a* and 9.2.1*b*. These mathematical statements say that all solutions that start "sufficiently close" (that is, within the distance δ) to $\mathbf{x}^0$ stay "close" (within the distance ϵ) to $\mathbf{x}^0$. Note that in Figure 9.2.1*a* the trajectory is within the circle $\|\mathbf{x} - \mathbf{x}^0\| = \delta$ at $t = 0$ and, although it soon passes outside of this circle, it remains within the circle $\|\mathbf{x} - \mathbf{x}^0\| = \epsilon$ for all $t \geq 0$. However, the trajectory of the solution does not have to approach the critical point $\mathbf{x}^0$ as $t \to \infty$, as illustrated in Figure 9.2.1*b*. A critical point that is not stable is said to be **unstable**.

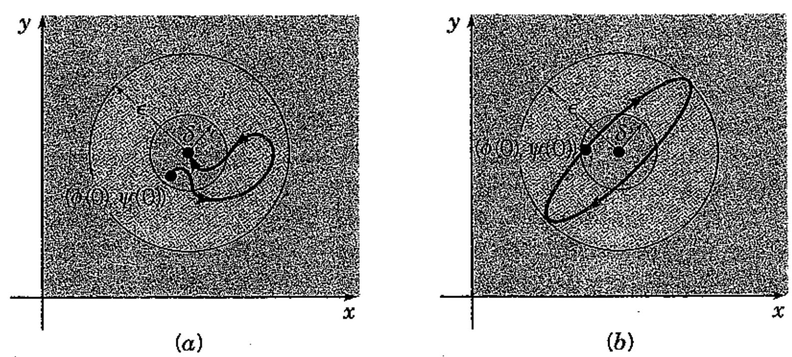

FIGURE 9.2.1 (*a*) Asymptotic stability. (*b*) Stability.

A critical point $\mathbf{x}^0$ is said to be **asymptotically stable** if it is stable and if there exists a δ_0 ($\delta_0 > 0$) such that if a solution $\mathbf{x} = \boldsymbol{\phi}(t)$ satisfies

$$\|\boldsymbol{\phi}(0) - \mathbf{x}^0\| < \delta_0, \tag{8}$$

then

$$\lim_{t\to\infty} \boldsymbol{\phi}(t) = \mathbf{x}^0. \tag{9}$$

Let V be defined on some domain D containing the origin. Then V is said to be **positive definite** on D if $V(0,0) = 0$ and $V(x,y) > 0$ for all other points in D. Similarly, V is said to be **negative definite** on D if $V(0,0) = 0$ and $V(x,y) < 0$ for all other points in D. If the inequalities $>$ and $<$ are replaced by $\geq$ and $\leq$, then V is said to be **positive semidefinite** and **negative semidefinite**, respectively. We emphasize that when we speak of a positive definite (negative definite, ...) function on a domain D containing the origin, the function must be zero at the origin in addition to satisfying the proper inequality at all other points in D.

Theorem 9.6.1 Suppose that the autonomous system (6) has an isolated critical point at the origin. If there exists a function V that is continuous and has continuous first partial derivatives, is positive definite, and for which the function $\dot{V}$ given by Eq. (7) is negative definite on some domain D in the xy-plane containing $(0,0)$, then the origin is an asymptotically stable critical point. If $\dot{V}$ is negative semidefinite, then the origin is a stable critical point.

Theorem 9.6.2 Let the origin be an isolated critical point of the autonomous system (6). Let V be a function that is continuous and has continuous first partial derivatives. Suppose that $V(0,0) = 0$ and that in every neighborhood of the origin there is at least one point at which V is positive (negative). If there exists a domain D containing the origin such that the function $\dot{V}$ given by Eq. (7) is positive definite (negative definite) on D, then the origin is an unstable critical point.