Math 207 Winter '07 Homework #4

3.8.1. Using the formula $\sin \pi z = (e^{i\pi z} - e^{-i\pi z})/2i$ show that the complex zeros of $\sin \pi z$ are exactly at the integers $\mathbb{Z}$, and that they are each of order 1. Calculate the residue of $1/\sin \pi z$ at $z = n \in \mathbb{Z}$.

3.8.2. Evaluate the integral

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^4}.$$

Where are the poles of $1/(1+z^4)$?

3.8.5 (simplified). Show that for $a > 0$

$$\int_{-\infty}^{\infty} \frac{\cos ax}{(1+x^2)^2} = \frac{\pi}{2}(1+a)\, e^{-a}.$$

[Hint: Integrate $e^{iaz}/(1+z^2)^2$ over the usual semi-circular contour. Use the general formula for evaluating a residue at a pole of higher order.]

3.8.8 (simplified). Prove that

$$\int_0^{2\pi} \frac{d\theta}{a + \cos\theta} = \frac{2\pi}{\sqrt{a^2-1}}$$

if $a > 1$, $a \in \mathbb{R}$. [Hint: Multiply numerator and denominator by $e^{i\theta}$ and rewrite the integral as an integral over the unit circle parametrized by $z = e^{i\theta}$.]

3.8.13. Suppose f is holomorphic in a punctured disc $D_r \backslash \{z_0\}$ and that

$$|f(z)| \le A\,|z - z_0|^{-1+\epsilon}$$

for some $\epsilon > 0$ and all z near z_0. Show that z_0 is a removable singularity for f.

3.9.3 (Not). Find the coefficients a_n in the Laurent expansion for $f(z) = (z + z^{-1})\, e^{1/z}$,

$$(z + z^{-1})\, e^{1/z} = \sum_{n=-\infty}^{\infty} a_n\, z^n,$$

valid for $z \neq 0$. What is the residue of $f(z)$ at $z = 0$?