Final 207 Fall 2009

December 3, 2009

Here is formula that you might need

$$\frac{\sin \pi z}{\pi} = z\Pi\left(1 - \frac{z^2}{n^2}\right)$$

1. Calculate

   a. $\int_0^\infty \frac{x^{-a}}{x+1}\, dx,\ 0 < a < 10$

   b. $\int_0^{2\pi} \frac{d\theta}{(1+a\sin\theta)}\ ,\ -1 < a < 1.$

2.a.Let g be analytic on a domain containing the simple closed curve γ. Then

$$\frac{1}{2\pi i}\int_\gamma \frac{h'(z)}{h(z)} g(z)dz = \sum_{i=1}^{N} g(z_i) - \sum_{j=1}^{M} g(w_j),$$

where $z-1, ...z_N$ are the zeros h and $w_1, ...w_M$ are the poles of h inside γ listed with multiplicities.
b. Let f be analytic and one-one in a domain D. let γ be a piecewice smooth simple closed curve inD, inside γ lies in D. Let $\Omega = f(D)$. Let w be a point of $|Omega$ and lies inside the curve $\Gamma = f \circ \gamma$. Show that

$$f^{-1}(w) = \frac{1}{2\pi i}\int_\gamma \frac{zf'(z)}{f(w) - w} dz$$

is the RHS analytic?

3. Suppose f is analytic in the unit disc $\mathbb{D}$ ($f : \mathbb{D} \to \mathbb{D}$), and $|f(z)| < |z|$. If there exists two points a, and $b \in \mathbb{D}$ such that $f(a) = a$ and $f(b) = b$, then $f(z) = z$.

4. Construct conformal maps that map

a. First quadrant ($\{z : z = x + iy,\ x > 0, y > 0\}$ into the unit disc.

b. The upper half disc into the first quadrant.

5. Let F be a fractional linear transformation which fixes 3 points. Show that F is the identity.

6.

7. a. Let $0 < |a| < 1$ and $|z| \leq r < 1$; show that

$$\left| \frac{a + |a|z}{(1 - \bar{a}z)a} \right| \leq \frac{1 + r}{1 - r}.$$

b. Let $\{a_n\}$ be a sequence of complex numbers with
$0 < |a_n| < 1$ and $\sum(1 - |a_n|) < \infty$. Show that the infinite product

$$B(z) = \Pi_{n=1}^{\infty} \frac{|a_n|}{a_n} \left(\frac{a_n - z}{1 - \bar{a}_n z} \right)$$

converges for $|z| \leq r < 1$ and that $B(z) \leq 1$. What are the zeros of B

8. Show that

$$\Pi_{n=2}^{\infty} \left(1 - \frac{z^2}{n^2} \right) = \frac{1}{2}$$