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NORM INFLATION FOR INCOMPRESSIBLE MAGNETO-HYDRODYNAMIC SYSTEM IN $\dot{B}_{\infty}^{-1,\infty}$

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Abstract. Based on the construction of Bourgain and Pavlović [1] we show that the solutions to the Cauchy problem for the three-dimensional incompressible magneto-hydrodynamics (MHD) system can develop different types of norm inflations in $\dot{B}_{\infty}^{-1,\infty}$. In particular the magnetic field can develop norm inflation in a short time even when the velocity remains small and vice versa. Efforts are made to present a very expository development of the ingenious construction of Bourgain and Pavlović in [1].

1. Introduction

In this paper we consider the three-dimensional incompressible magneto-hydro-dynamics (MHD) system:

$$\begin{aligned} u_t - \triangle u + u \cdot \nabla u - b \cdot \nabla b + \nabla p = 0, \\ b_t - \triangle b + u \cdot \nabla b - b \cdot \nabla u = 0, \\ \nabla \cdot u = 0, \quad \nabla \cdot b = 0, \end{aligned} \tag{1.1}$$

with the initial conditions

$$u(x,0) = u_0(x), \quad b(x,0) = b_0(x), \tag{1.2}$$

where $x \in \mathbb{R}^3$, $t \geq 0$, u is the fluid velocity, b is the magnetic field. The initial data u_0 and b_0 are divergence free. When the magnetic field $b(x,t)$ vanishes, the incompressible MHD system is reduced to the incompressible

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Navier-Stokes equations. The solutions to the MHD system share the same scaling properties of solutions to the Navier-Stokes equations; that is,

$$u_\lambda(x,t) = \lambda u(\lambda x, \lambda^2 t),\ b_\lambda(x,t) = \lambda b(\lambda x, \lambda^2 t),\ p_\lambda(x,t) = \lambda^2 p(\lambda x, \lambda^2 t)$$

solves the MHD system (1.1) with initial data

$$u_{0\lambda} = \lambda u_0(\lambda x),\ b_{0\lambda} = \lambda b_0(\lambda x),$$

if $u(x,t)$ and $b(x,t)$ solve the MHD system (1.1) with initial data $u_0(x)$ and $b_0(x)$. The spaces that are invariant under the above scaling are called the critical spaces. Examples of critical spaces in three dimension are

$$\dot{H}^{\frac{1}{2}} \hookrightarrow L^3 \hookrightarrow BMO^{-1} \hookrightarrow \dot{B}^{-1,\infty}_{\infty}$$

(see [2], for example, for the discussions of the embeddings).

The study of solutions to the Navier-Stokes equations as well as of the MHD system in critical spaces has been one of the foci of research activities since the pioneering work of Kato [5]. In case of the Navier-Stokes equations, Koch and Tataru [6] in 2001 established global well posedness of the Navier-Stokes equations with small initial data in the space BMO^{-1}. Recently, Bourgain and Pavlović [1] showed ill posedness for Navier-Stokes equations in $\dot{B}^{-1,\infty}_{\infty}$. More precisely, Bourgain and Pavlović constructed some arbitrarily small initial data in $\dot{B}^{-1,\infty}_{\infty}$ and produced so-called norm inflation in the sense that the solution becomes arbitrarily large in $\dot{B}^{-1,\infty}_{\infty}$ after an arbitrarily short time.

In a recent work [8], Miao, Yuan and Zhang established the existence of a global mild solution in BMO^{-1} for small initial data and the uniqueness of such solutions in $C([0,\infty); BMO^{-1})$. It is then an interesting problem to study the solutions to the MHD system with initial data in the space $\dot{B}^{-1,\infty}_{\infty}$. In this paper, we discuss different cases of norm inflation phenomena for the MHD system in $\dot{B}^{-1,\infty}_{\infty}$. We construct arbitrarily small initial data (u_0, b_0) in $\dot{B}^{-1,\infty}_{\infty} \times \dot{B}^{-1,\infty}_{\infty}$. This data when evolved in time through the MHD system give rise to "norm inflation" in $\dot{B}^{-1,\infty}_{\infty}$ for the corresponding solutions (u, b). One particularly interesting scenario is that the magnetic field b shows norm inflation while the velocity u remains small. Namely, we show the following.

Theorem 1.1. *For any $\delta > 0$ there exists a solution (u, b, p) to the MHD system* (1.1) *with data $u_0 \in \mathcal{S}$ and $b_0 \in \mathcal{S}$ which satisfy*

$$\|u(0)\|_{\dot{B}^{-1,\infty}_{\infty}} \lesssim \delta, \quad \|b(0)\|_{\dot{B}^{-1,\infty}_{\infty}} \lesssim \delta,$$

such that for some $0 < T < \delta$

$$\|b(T)\|_{\dot{B}_{\infty}^{-1,\infty}} \gtrsim \tfrac{1}{\delta}$$

but for any $0 < t < T < \delta$

$$\|u(t)\|_{B_{\infty}^{-1,\infty}} \lesssim \delta.$$

Remark 1.2. We refer the reader to the beginning of the section of preliminaries for the definition of the symbol $\lesssim$.

Our proof follows the methods introduced by Bourgain and Pavlović in [1]. Efforts are made to give a very expository development of the ingenious ideas of Bourgain and Pavlović in [1].

We now recall some auxiliary concepts related to plane waves, which are necessary in the sequel:

- The "diffusion" of a plane wave $v \sin(k \cdot x)$ in R^3 is given by
$$e^{\Delta t} v \sin(k \cdot x) = e^{-|k|^2 t} v \sin(k \cdot x).$$
Thus the magnitude of the diffusion of a plane wave dies down in time in the scale that is measured by the square of the magnitude of the wave vector k.
- It is easy to see that $u = b = e^{-|k|^2 t} v \sin(k \cdot x)$ solves the MHD system when the wave vector k is orthogonal to the amplitude vector v.
- The nonlinear interaction of two such diffusions in the MHD system can be captured, and it only produces a slower diffusion if the two wave vectors are close.

We note that these observations are the basis of the original argument of Bourgain and Pavlović in [1]. We will use them to construct a combination of such "diffusions" with minimum nonlinear interactions yet producing enough slower "diffusions" to cause the norm inflation in short time.

Remark 1.3. It is interesting to observe that even though the initial velocity is zero the velocity can be triggered to develop norm inflation while the magnetic field stays under control. More precisely we can show that, for any $\delta > 0$, there exists a solution (u, b, p) to the MHD system (1.1) with vanishing initial velocity and some $b_0 \in \mathcal{S}$ which satisfies

$$\|b(0)\|_{\dot{B}_{\infty}^{-1,\infty}} \lesssim \delta$$

and for some $0 < T < \delta$

$$\|u(T)\|_{\dot{B}_{\infty}^{-1,\infty}} \gtrsim 1/\delta,$$

while for all $0 < t < T < \delta$

$$\|b(t)\|_{B_\infty^{-1,\infty}} \lesssim \delta.$$

Remark 1.4. We also note that due to the interaction between the velocity and the magnetic field, if initially they are the same, they may restrain each other from norm inflations.

In our paper, we present our results in $\mathbb{T}^3$. But, as pointed out in [1], the proof can be modified to $\mathbb{R}^3$.

2. PRELIMINARIES

2.1. **Notation.** We denote by $A \lesssim B$ an estimate of the form $A \le CB$ with some constant C, and by $A \sim B$ an estimate of the form $C_1 B \le A \le C_2 B$ with some constants C_1, C_2. For completeness we recall the defining norms for the Besov space $\dot{B}_\infty^{-1,\infty}$ and the BMO^{-1} space

$$\|f\|_{\dot{B}_\infty^{-1,\infty}} = \sup_{t>0} t^{1/2} \|e^{t\triangle} f\|_{L^\infty}; \tag{2.1}$$

$$\|f\|_{BMO^{-1}} = \sup_{x_0 \in R^3, R>0} \left(\frac{1}{|B(x_0, \sqrt{R})|} \int_0^R \int_{B(x_0,\sqrt{R})} |e^{t\triangle} f(y)|^2 \, dy \, dt \right)^{\frac{1}{2}}. \tag{2.2}$$

We will also work with the so-called inhomogeneous Besov space $B_\infty^{-1,\infty}$ with the norm

$$\|f\|_{B_\infty^{-1,\infty}} = \sup_{0<t<1} \sqrt{t} \|e^{t\Delta} f\|_{L^\infty}. \tag{2.3}$$

Clearly,

$$\|f\|_{B_\infty^{-1,\infty}} \le \|f\|_{\dot{B}_\infty^{-1,\infty}} \quad \text{and} \quad \|f\|_{B_\infty^{-1,\infty}} \le \|f\|_{L^\infty},$$

since $\|e^{t\Delta} f\|_{L^\infty} \le \|f\|_{L^\infty}$.

2.2. **The well-posedness result of the incompressible MHD system in BMO^{-1}.** We recall the well-posedness result of C. Miao, B. Yuan and B. Zhang in BMO^{-1} in [8]. For this we introduce the spaces X_T and the corresponding norm.

Definition 2.1. Let $u(x,t)$ be a measurable function on $\mathbb{R}^3 \times [0,T)$ for $T > 0$ and let

$$\begin{aligned} \|u(\cdot,\cdot)\|_{X_T} = & \sup_{0<t<T} t^{1/2} \|u(\cdot,t)\|_{L^\infty} \\ & + \sup_{x_0 \in R^3, 0<R<T} \left(\frac{1}{|B(x_0,\sqrt{R})|} \int_0^R \int_{B(x_0,\sqrt{R})} |u(y,t)|^2 \, dy \, dt \right)^{\frac{1}{2}}. \end{aligned} \tag{2.4}$$

Then the space-time space X_T is defined by

$$X_T = \{f(x,t) \in L^2(0,T;L^2(\mathbb{R}^3)) : \|f\|_{X_T} < \infty\}$$

It is worth mentioning that, for each $t \in (0,T]$,

$$\|f(\cdot,t)\|_{L^\infty} \leq \frac{1}{\sqrt{t}}\|f\|_{X_T}.$$

In [8], Miao, Yuan and Zhang proved the following existence theorem.

Theorem 2.2. (Miao, Yuan, and Zhang) *Let $(u_0(x), b_0(x)) \in BMO^{-1} \times BMO^{-1}$ with $\nabla \cdot u_0 = 0$ and $\nabla \cdot b_0 = 0$. Then, there exists a positive constant ε such that if $\|(u_0,b_0)\|_{BMO^{-1}} < \varepsilon$, then the MHD system has a unique global mild solution $(u(x,t), b(x,t)) \in X_T \times X_T$ satisfying*

$$\|(u(x,t), b(x,t))\|_{X_T\times X_T} \leq 2\varepsilon \quad \text{for all } T > 0.$$

2.3. **Bilinear operators.** Let $\mathbb{P}$ denote the projection on divergence-free vector fields, which acts on a function ϕ as

$$\mathbb{P}(\phi) = \phi + \nabla \cdot (-\triangle)^{-1} div\phi.$$

As shown in [6, 8] the bilinear operator

$$\mathcal{B}(u,v) = \int_0^t e^{(t-\tau)\triangle}\mathbb{P}(u\cdot\nabla v)\, d\tau$$

maps $X_T \times X_T$ into X_T continuously. More precisely we have

$$\|\mathcal{B}(u,v)\|_{X_T} \lesssim \|u\|_{X_T}\|v\|_{X_T}. \tag{2.5}$$

2.4. **Rewriting the MHD system.** Following ideas from [1] we rewrite the MHD system (1.1) introducing the expression

$$u = e^{t\triangle}u_0 - u_1 + y \tag{2.6}$$

$$b = e^{t\triangle}b_0 - b_1 + z, \tag{2.7}$$

where

$$u_1(x,t) = \mathcal{B}(e^{t\triangle}u_0(x), e^{t\triangle}u_0(x)) - \mathcal{B}(e^{t\triangle}b_0(x), e^{t\triangle}b_0(x)), \tag{2.8}$$

$$b_1(x,t) = \mathcal{B}(e^{t\triangle}u_0(x), e^{t\triangle}b_0(x)) - \mathcal{B}(e^{t\triangle}b_0(x), e^{t\triangle}u_0(x)). \tag{2.9}$$

An easy calculation shows that

$$\begin{aligned} y_t - \triangle y + G_0 + G_1 + G_2 = 0, \\ z_t - \triangle z + K_0 + K_1 + K_2 = 0, \end{aligned} \tag{2.10}$$

where

$$G_0 = \mathbb{P}[(e^{t\triangle}u_0\cdot\nabla)u_1 + (u_1\cdot\nabla)e^{t\triangle}u_0 + (u_1\cdot\nabla)u_1]$$

$$
\begin{aligned}
&- \mathbb{P}[(e^{t\triangle}b_0 \cdot \nabla)b_1 + (b_1 \cdot \nabla)e^{t\triangle}b_0 + (b_1 \cdot \nabla)b_1] \\
G_1 &= \mathbb{P}[(e^{t\triangle}u_0 \cdot \nabla)y + (u_1 \cdot \nabla)y + (y \cdot \nabla)e^{t\triangle}u_0 + (y \cdot \nabla)u_1] \\
&- \mathbb{P}[(e^{t\triangle}b_0 \cdot \nabla)z + (b_1 \cdot \nabla)z + (z \cdot \nabla)e^{t\triangle}b_0 + (z \cdot \nabla)b_1] \\
G_2 &= \mathbb{P}[(y \cdot \nabla)y] - \mathbb{P}[(z \cdot \nabla)z]
\end{aligned}
$$

and

$$
\begin{aligned}
K_0 &= \mathbb{P}[(e^{t\triangle}u_0 \cdot \nabla)b_1 + (u_1 \cdot \nabla)e^{t\triangle}b_0 + (u_1 \cdot \nabla)b_1] \\
&- \mathbb{P}[(e^{t\triangle}b_0 \cdot \nabla)u_1 + (b_1 \cdot \nabla)e^{t\triangle}u_0 + (b_1 \cdot \nabla)u_1] \\
K_1 &= \mathbb{P}[(e^{t\triangle}u_0 \cdot \nabla)z + (u_1 \cdot \nabla)z + (y \cdot \nabla)e^{t\triangle}b_0 + (y \cdot \nabla)b_1] \\
&- \mathbb{P}[(e^{t\triangle}b_0 \cdot \nabla)y + (b_1 \cdot \nabla)y + (z \cdot \nabla)e^{t\triangle}u_0 + (z \cdot \nabla)u_1] \\
K_2 &= \mathbb{P}[(y \cdot \nabla)z] - \mathbb{P}[(z \cdot \nabla)y].
\end{aligned}
$$

Here G_0 and K_0 are constants and G_1 and K_1 are linear, while G_2 and K_2 are quadratic in terms of y and z.

Remark 2.3. Note that although the second equation in the MHD system (1.1) has no pressure, since u and b are both divergence free, the term $u \cdot \nabla b - b \cdot \nabla u$ is automatically divergence free. Hence the projector $\mathbb{P}$ acting on this term does not change the second equation and hence we can write b_1 and the K_i's as described above.

3. Interactions of plane waves

In this section we show how the diffusions of plane waves interact in MHD system. These interactions are the basis for the constructions of initial data which will evolve into the different cases of velocity and magnetic field norm inflations .

3.1. **Diffusion of a plane wave.** As a first step, we consider the same initial data for velocity and the magnetic field using one single plane wave. Suppose $k \in \mathbb{R}^3$, $v \in \mathbb{S}^2$ and $k \cdot v = 0$. Let

$$u_0 = b_0 = v \cos(k \cdot x).$$

Then $\nabla \cdot u_0 = 0, \nabla \cdot b_0 = 0$, and

$$e^{t\triangle} v \cos(k \cdot x) = e^{-|k|^2 t} v \cos(k \cdot x). \tag{3.1}$$

In fact the "diffusions" $(e^{t\Delta} v \cos(k \cdot x), e^{t\Delta} v \cos(k \cdot x))$ of a plane wave solve the MHD system with vanishing pressure. It is important to notice that

- $\|v\cos(k\cdot x)\|_{\dot{B}^{-1,\infty}_{\infty}} \sim \frac{1}{|k|}$,
- $\|e^{t\Delta}v\cos(k\cdot x)\|_{X_T} \lesssim \frac{1}{|k|}$,

which says that the size of a plane wave in the space $\dot{B}^{-1,\infty}_{\infty}$ is reciprocal to the magnitude of its wave vector, and in X_T it is bounded by this same reciprocal.

3.2. **Interaction of plane waves.** Now we consider different plane wave initial data for the velocity and magnetic field. Suppose $k_i \in \mathbb{R}^3$, $v_i \in \mathbb{S}^2$ and $k_i \cdot v_i = 0$, for $i = 1, 2$. Let

$$u_0 = \cos(k_1 \cdot x)v_1, \qquad b_0 = \cos(k_2 \cdot x)v_2.$$

Using the decomposition given in Section 2.4,

$$u = e^{t\Delta}u_0, \quad b = e^{t\Delta}b_0 - b_1 + z,$$

solve the MHD system with vanishing pressure. To simplify our calculations we assume that

$$k_2 \cdot v_1 = 0, \quad \text{and} \quad k_1 \cdot v_2 = \tfrac{1}{2},$$

which eliminates the term $e^{t\Delta}u_0 \cdot \nabla(e^{t\Delta}b_0)$ and gives

$$\begin{aligned} e^{t\triangle}b_0 \cdot \nabla(e^{t\triangle}u_0) &= -e^{-(|k_1|^2+|k_2|^2)t}v_1 \sin(k_1 \cdot x)\cos(k_2 \cdot x)(k_1 \cdot v_2) \\ &= -\tfrac{1}{4}e^{-(|k_1|^2+|k_2|^2)t}v_1(\sin((k_1-k_2)\cdot x) + \sin((k_1+k_2)\cdot x)). \end{aligned}$$

Hence

$$\begin{aligned} b_1 &= \frac{1}{4}v_1 \sin((k_1-k_2)\cdot x)\int_0^t e^{-(|k_1|^2+|k_2|^2)\tau}e^{-|k_1-k_2|^2(t-\tau)}d\tau \\ &+ \frac{1}{4}v_1 \sin((k_1+k_2)\cdot x)\int_0^t e^{-(|k_1|^2+|k_2|^2)\tau}e^{-|k_1+k_2|^2(t-\tau)}d\tau \\ &= b_{1,0} + b_{1,1}, \end{aligned}$$

where

$$b_{1,0} = \frac{1}{4}v_1 \sin((k_1-k_2)\cdot x)\frac{-e^{-(|k_1|^2+|k_2|^2)t} + e^{-|k_1-k_2|^2 t}}{2k_1 \cdot k_2}$$

and

$$b_{1,1} = \frac{1}{4}v_1 \sin((k_1+k_2)\cdot x)\frac{e^{-(|k_1|^2+|k_2|^2)t} - e^{-|k_1+k_2|^2 t}}{2k_1 \cdot k_2}.$$

Therefore, if we can manage to control z in light of the continuity of the bilinear operator $\mathcal{B}$ in X_T, then the interaction of two plane waves is small in $\dot{B}^{-1,\infty}_{\infty}$ if neither the sum nor the difference of their wave vectors is small in magnitude. In the meantime, the interaction is sizable in $\dot{B}^{-1,\infty}_{\infty}$ if either the sum or the difference of their wave vectors is small in magnitude.

4. Proof of theorem 1.1

In this section we will follow the idea from [1] to construct initial data to produce norm inflation for solutions to MHD systems. From the discussions in the previous sections we know that the interaction of two plane waves is not enough to show the norm inflation. We need to build interactions of more plane waves. The construction in [1] depends on the rather sophisticated choices of plane waves. We will use a similar scheme.

4.1. **Construction of initial data for the MHD system.** For a fixed small number $\delta > 0$ we will specify later the following initial data:

$$u_0 = \frac{Q}{\sqrt{r}} \sum_{s=1}^{r} |k_s| v_s \cos(k_s \cdot x) \tag{4.1}$$

and

$$b_0 = \frac{Q}{\sqrt{r}} \sum_{s=1}^{r} |k'_s| v'_s \cos(k'_s \cdot x). \tag{4.2}$$

We expect, for each s, that the interaction of the two plane waves $v_s \cos(k_s \cdot x)$ and $v'_s \cos(k'_s \cdot x)$ is sizable in $\dot{B}^{-1,\infty}_{\infty}$, while the interactions of plane waves of different s is small, as demonstrated in Section 3.2. Hence we have the following.

- Wave vectors: The wave vectors $k_s \in \mathbb{R}^3$ are parallel to a given vector $k_0 \in \mathbb{R}^3$. The modulo $|k_0|$ will be taken to be large, depending on Q. The magnitude of k_s is defined by,

  $$|k_s| = 2^s |k_0||k_{s-1}|, \qquad s = 1, 2, 3, ..., r. \tag{4.3}$$

  The wave vectors $k'_s \in \mathbb{R}^3$ are defined by

  $$k_s - k'_s = \eta \tag{4.4}$$

  for a given vector $\eta \in \mathbb{S}^2$.
- Amplitude vectors: The amplitude vectors $v_s, v'_s \in \mathbb{S}^2$ satisfy

  $$k_s \cdot v_s = k'_s \cdot v'_s = 0 \tag{4.5}$$

  to ensure the initial data are divergence free.

- Auxiliary assumptions: We also require that
$$\eta \cdot v_s = 0, \quad \eta \cdot v_s' = \tfrac{1}{2} \tag{4.6}$$
to simplify our calculations. In fact we will choose $v_s = v$ a fixed vector.

We first point out the following simple facts to further motivate the choices of the magnitudes of k_s.

Lemma 4.1.
$$\sum_{l<s} |k_l| \sim |k_{s-1}| \quad \textit{and} \; \sum_{l<s} |k_l'| \sim |k_{s-1}'|; \tag{4.7}$$
$$\sum_{s=1}^{r} |k_s| e^{-|k_s|^2 t} \lesssim \frac{1}{\sqrt{t}} \quad \textit{and} \; \sum_{s=1}^{r} |k_s'| e^{-|k_s'|^2 t} \lesssim \frac{1}{\sqrt{t}}; \tag{4.8}$$
$$v_i \cdot k_j = v_i \cdot k_j' = v_i \cdot \eta = 0, \quad \forall \quad i, j = 1, 2, \ldots, r; \tag{4.9}$$
$$\sum_{i=1}^{r} |k_i| e^{-\frac{|k_i|^2}{|k_0|^2}} \lesssim 1, \quad \textit{and} \; \sum_{i=1}^{r} |k_i'| e^{-\frac{|k_i'|^2}{|k_0|^2}} \lesssim 1. \tag{4.10}$$

Proof of Lemma. By the definition (4.3), it is clear that $|k_{l-1}| < \frac{1}{2}|k_l|$, which easily implies the first statement. For the second statement, again due to the definition (4.3), we know that $|k_s| \sim |k_s| - |k_{s-1}|$. Thus,
$$\sum_{s=1}^{r} |k_s| e^{-|k_s|^2 t} \sim \sum_{s=1}^{r} (|k_s| - |k_{s-1}|) e^{-|k_s|^2 t},$$
while the later one can be considered as a finite Riemman sum of the function $e^{-x^2 t}$. Therefore,
$$\sum_{s=1}^{r} |k_s| e^{-|k_s|^2 t} \lesssim \int_0^\infty e^{-x^2 t} dx = \frac{1}{\sqrt{t}} \int_0^\infty e^{-x^2 t} d(x\sqrt{t}) = \frac{\sqrt{\pi}}{2\sqrt{t}}.$$
For the third statement, we note that the k_s are parallel to the given vector k_0 for all s and $v_s = v$ is a fixed vector by the above choice. Hence, by (4.5)
$$v_i \cdot k_j = 0, \quad \text{for all } i, j = 1, 2, ..., r.$$
On the other hand, from (4.4) and (4.6), we have
$$v_i \cdot k_j' = v_i \cdot (k_j - \eta) = v_i \cdot k_j - v_i \cdot \eta = 0.$$
The forth statement in the lemma follows from the second one with an appropriate choice of k_0. This completes the proof of the lemma. $\square$

Next we calculate the norm of our initial data.

Lemma 4.2. *For u_0 and b_0 given in* (4.1) *and* (4.2)*, we have*

$$\|u_0\|_{\dot{B}_\infty^{-1,\infty}} \lesssim \frac{Q}{\sqrt{r}}, \quad \|b_0\|_{\dot{B}_\infty^{-1,\infty}} \lesssim \frac{Q}{\sqrt{r}}. \tag{4.11}$$

Proof of Lemma. For the given initial data u_0, we have, due to(3.1),

$$e^{\tau\triangle} u_0 = \frac{Q}{\sqrt{r}} \sum_{s=1}^{r} |k_s| v_s \cos(k_s \cdot x) e^{-|k_s|^2\tau}. \tag{4.12}$$

Hence by Lemma 4.1

$$\|u_0\|_{\dot{B}_\infty^{-1,\infty}} \sim \frac{Q}{\sqrt{r}} \sup_{t>0} \sqrt{t} \sum_{s=1}^{r} |k_s| e^{-|k_s|^2 t} \lesssim \frac{Q}{\sqrt{r}}. \tag{4.13}$$

The bound for $\|b_0\|_{\dot{B}_\infty^{-1,\infty}}$ follows in a similar way. □

Lemma 4.3. *For u_0 and b_0 given in* (4.1) *and* (4.2)*, we have*

$$\|e^{t\Delta} u_0\|_{X_T} \lesssim Q, \quad \|e^{t\Delta} b_0\|_{X_T} \lesssim Q. \tag{4.14}$$

Proof of Lemma. We only need to verify one of the two.

$$\|e^{t\Delta} u_0\|_{X_T} \lesssim \frac{Q}{\sqrt{r}} \Big(1 + \sup_{t\in[0,T]} \Big(\int_0^t \Big(\sum_{i=1}^{r} |k_i| e^{-|k_i|^2\tau} \Big)^2 d\tau \Big)^{\frac{1}{2}} \Big),$$

where

$$\begin{aligned} \Big(\sum_{i=1}^{r} |k_i| e^{-|k_i|^2\tau} \Big)^2 &\lesssim \sum_{i=1}^{r} |k_i|^2 e^{-2|k_i|^2\tau} + 2 \sum_{i=1}^{r} |k_i| e^{-|k_i|^2\tau} \sum_{j<i} |k_j| \\ &\lesssim \sum_{i=1}^{r} |k_i|^2 e^{-|k_i|^2\tau}. \end{aligned}$$

Hence,

$$\int_0^t \Big(\sum_{i=1}^{r} |k_i|^2 e^{-|k_i|^2\tau} \Big)^2 d\tau \lesssim \sum_{i=1}^{r} (1 - e^{-|k_i|^2 t}) \lesssim r,$$

which implies

$$\|e^{t\Delta} u_0\|_{X_T} \lesssim \frac{Q}{\sqrt{r}} + Q.$$

This completes the proof of the lemma. □

Finally we note the following.

Lemma 4.4. *For $t \in [0, +\infty)$,*

$$\|e^{t\Delta}u_0\|_{\dot{B}_{\infty}^{-1,\infty}} \lesssim \frac{Q}{\sqrt{r}} e^{-|k_0|^2 t}, \quad \|e^{t\Delta}b_0\|_{\dot{B}_{\infty}^{-1,\infty}} \lesssim \frac{Q}{\sqrt{r}} e^{-|k_0|^2 t}. \tag{4.15}$$

4.2. **Analysis of u_1.** As demonstrated in Section 3.2 we consider the decomposition

$$u = e^{t\Delta}u_0 - u_1 + y, \qquad b = e^{t\Delta}b_0 - b_1 + z.$$

We want to handle u_1 first. Recall the definition (2.8)

$$u_1 = \mathcal{B}(e^{t\Delta}u_0, e^{t\Delta}u_0) - \mathcal{B}(e^{t\Delta}b_0, e^{t\Delta}b_0).$$

By our discussions in Section 3.2 the interactions should be small. By the fact that $v_i \cdot k_j = 0$ it is immediately seen that

$$e^{t\triangle}u_0 \cdot \nabla e^{t\triangle}u_0 = 0.$$

Then a straightforward calculation shows

$$\begin{aligned} e^{t\triangle}b_0 \cdot \nabla e^{t\triangle}b_0 &= -\frac{Q^2}{r}\sum_{i,j=1}^{r} |k_i'||k_j'|e^{-(|k_i'|^2+|k_j'|^2)t}(v_i' \cdot k_j')v_j' \cos(k_i' \cdot x)\sin(k_j' \cdot x) \\ &= -\frac{Q^2}{2r}\sum_{i,j=1}^{r} |k_i'||k_j'|e^{-(|k_i'|^2+|k_j'|^2)t}(v_i' \cdot k_j')v_j' \sin(k_i' + k_j') \cdot x \\ &\quad - \frac{Q^2}{2r}\sum_{i,j=1}^{r} |k_i'||k_j'|e^{-(|k_i'|^2+|k_j'|^2)t}(v_i' \cdot k_j')v_j' \sin(k_j' - k_i') \cdot x \end{aligned}$$

and

$$\begin{aligned} \mathbb{P}e^{t\triangle}b_0 \cdot \nabla e^{t\triangle}b_0 &= -\frac{Q^2}{2r}\sum_{i,j=1}^{r} |k_i'||k_j'|e^{-(|k_i'|^2+|k_j'|^2)t}(v_i' \cdot k_j')u_j \sin(k_j' + k_i') \cdot x \\ &\quad - \frac{Q^2}{2r}\sum_{i,j=1}^{r} |k_i'||k_j'|e^{-(|k_i'|^2+|k_j'|^2)t}(v_i' \cdot k_j')w_j \sin(k_j' - k_i') \cdot x \\ &= E_1 + E_2, \end{aligned}$$

where u_j is the projection of v_j' onto the orthogonal to $k_j' + k_i'$ and w_j is the projection of v_j' onto the orthogonal to $k_j' - k_i'$. Hence,

$$\mathcal{B}((e^{t\Delta}b_0, e^{t\Delta}b_0) = \int_0^t e^{(t-\tau)\Delta}E_1 d\tau + \int_0^t e^{(t-\tau)\Delta}E_2 d\tau = F_1 + F_2.$$

We give the estimate for F_1 in detail. The bounds for F_2 follow similarly and as such are omitted.

$$F_1 = \frac{Q^2}{2r}\sum_{i,j=1}^{r} |k_i'||k_j'|(v_i'\cdot k_j')u_j \sin(k_j'+k_i')\cdot x \frac{e^{-(|k_i'|^2+|k_j'|^2)t}-e^{-|k_i'+k_j'|^2 t}}{|k_i'+k_j'|^2-(|k_i'|^2+|k_j'|^2)}.$$

By the fact that $k_i'\cdot v_i'=0$ and since the function $\frac{1-e^{-x}}{x}$ is bounded for $x>0$, we have

$$|e^{\tau\Delta}F_1| \lesssim \frac{Q^2}{r}\sum_{j=1}^{r}\sum_{i<j}|k_i'||k_j'|^2 t e^{-(|k_i'|^2+|k_j'|^2)t} e^{-|k_i'+k_j'|^2\tau}$$

and

$$|F_1| \lesssim \frac{Q^2}{r}\sum_{j=1}^{r}\sum_{i<j}|k_i'||k_j'|^2 t e^{-(|k_i'|^2+|k_j'|^2)t}.$$

Hence,

$$|F_1| \lesssim \frac{Q^2}{r}\sum_{j=1}^{r}\sum_{i<j}|k_j'|^2 t e^{-\frac14|k_j'|^2 t}|k_i'|e^{-\frac14|k_j'|^2 t} \lesssim \frac{Q^2}{r}\sum_{j=1}^{r}|k_{j-1}'|e^{-\frac14|k_j'|^2 t}$$

and

$$|e^{\tau\Delta}F_1| \lesssim \frac{Q^2}{r}\sum_{j=1}^{r}|k_{j-1}'|e^{-\frac14|k_j'|^2 t}e^{-\frac12|k_j'|^2\tau},$$

where we use the fact that xe^{-x} is bounded for $x>0$ and Lemma 4.1.

$$\begin{aligned}\|F_1\|_{X_T} \lesssim{}& \frac{Q^2}{r}\sum_{j=1}^{r}|k_{j-1}'|\sup_{t\in[0,T]}\sqrt{t}e^{-\frac14|k_j'|^2 t}\\ &+\sup_{t\in[0,T]}\Big(\int_0^t\Big(\sum_{j=1}^{r}|k_{j-1}'|e^{-\frac14|k_j'|^2\tau}\Big)^2 d\tau\Big)^{\frac12}\end{aligned}$$

where

$$\sum_{j=1}^{r}|k_{j-1}'|\sup_{t\in[0,T]}\sqrt{t}e^{-\frac14|k_j'|^2 t} \lesssim \sum_{j=1}^{r}\frac{|k_{j-1}'|}{|k_j'|}\lesssim 1$$

and

$$\sup_{t\in[0,T]}\Big(\int_0^t\Big(\sum_{j=1}^{r}|k_{j-1}'|e^{-\frac14|k_j'|^2\tau}\Big)^2 d\tau\Big)^{\frac12} \lesssim \Big(\sum_{j=1}^{r}\frac{|k_{j-1}'|}{|k_j'|}\Big)^{\frac12}\lesssim 1$$

by the same argument as used in the proof of Lemma 4.3. Similarly

$$\|F_1\|_{\dot{B}_\infty^{-1,\infty}} = \sup_{\tau>0} \sqrt{\tau} \|e^{\tau\Delta} F_1\|_{L^\infty} \lesssim \frac{Q^2}{r} \sum_{j=1}^{r} \frac{|k'_{j-1}|}{|k'_j|} \lesssim \frac{Q^2}{r}.$$

Therefore, we conclude the following.

Lemma 4.5.

$$\|u_1\|_{\dot{B}_\infty^{-1,\infty}} \lesssim \frac{Q^2}{r}, \quad \|u_1\|_{X_T} \lesssim \frac{Q^2}{r}. \tag{4.16}$$

Proof. F_2 can be handled just as we did F_1. □

4.3. **Analysis of b_1.** First we recall that from (2.9)

$$b_1(x,t) = \mathcal{B}(e^{t\triangle} u_0(x), e^{t\triangle} b_0(x)) - \mathcal{B}(e^{t\triangle} b_0(x), e^{t\triangle} u_0(x)).$$

Similar to the calculations in the previous section, first, due to the fact that $v_i \cdot k'_j = 0$, we have

$$e^{\tau\triangle} u_0 \cdot \nabla e^{\tau\triangle} b_0 = 0.$$

Also

$$\begin{aligned} e^{\tau\triangle} b_0 \cdot \nabla e^{\tau\triangle} u_0 = &-\frac{Q^2}{2r} \sum_{i,j=1}^{r} |k'_i||k_j| e^{-(|k'_i|^2+|k_j|^2)t} (v'_i \cdot k_j) v_j \sin(k_j + k'_i) \cdot x \\ &- \frac{Q^2}{2r} \sum_{i\neq j}^{r} |k'_i||k_j| e^{-(|k'_i|^2+|k_j|^2)t} (v'_i \cdot k_j) v_j \sin(k_j - k'_i) \cdot x \\ &- \frac{Q^2}{2r} \sum_{i=1}^{r} |k_i||k'_i| e^{-(|k_i|^2+|k'_i|^2)t} (v'_i \cdot k_i) v_i \sin(k_i - k'_i) \cdot x. \end{aligned}$$

We will write

$$e^{\tau\triangle} u_0 \cdot \nabla e^{\tau\triangle} b_0 - e^{\tau\triangle} b_0 \cdot \nabla e^{\tau\triangle} u_0 = \tilde{b}_{1,0} + \tilde{b}_{1,1},$$

where

$$\begin{aligned} \tilde{b}_{1,0} &= \frac{Q^2}{2r} \sum_{i=1}^{r} |k_i||k'_i| e^{-(|k_i|^2+|k'_i|^2)t} \sin(k_i - k'_i) \cdot x (v'_i \cdot k_i) v_i \\ &= \frac{Q^2}{4r} \sin(\eta \cdot x) \sum_{i=1}^{r} |k_i||k'_i| e^{-(|k_i|^2+|k'_i|^2)t} v_i, \end{aligned}$$

due to our choices of wave vectors and amplitude vectors in Section 4.1. We then set $\tilde{b}_{1,1} = \tilde{d}_{1,1} + \tilde{e}_{1,1}$, where

$$\begin{aligned}\tilde{d}_{1,1} &= \frac{Q^2}{2r}\sum_{i=1}^{r}|k_i||k_i'|e^{-(|k_i|^2+|k_i'|^2)t}\sin(k_i+k_i')\cdot x(v_i'\cdot k_i)v_i\\ &= \frac{Q^2}{4r}\sum_{i=1}^{r}|k_i||k_i'|e^{-(|k_i|^2+|k_i'|^2)t}v_i\sin(k_i+k_i')\cdot x.\end{aligned}$$

By the choices of k_i', which behave more or less like k_i for each i when $|k_0|$ is very large, we conclude that $\tilde{e}_{1,1}$ can be handled just as we did E_1 in the previous section. We then have

$$\begin{aligned}d_{1,1} &= \int_0^t e^{\Delta(t-\tau)}\mathbb{P}\tilde{d}_{1,1}(\tau)d\tau\\ &= \frac{Q^2}{4r}\sum_{i=1}^{r}|k_i||k_i'|v_i\sin(k_i+k_i')\cdot x\frac{e^{-(|k_i|^2+|k_i'|^2)t}-e^{-|k_i+k_i'|^2t}}{|k_i+k_i'|^2-(|k_i|^2+|k_i'|^2)}\end{aligned}$$

which gives

$$|d_{1,1}| \lesssim \frac{Q^2}{r}\sum_{i=1}^{r}|k_i|^2te^{-\frac{1}{2}|k_i|^2t} \lesssim \frac{Q^2}{r}\sum_{i=1}^{r}e^{-\frac{1}{4}|k_i|^2t}$$

and

$$|e^{\tau\Delta}d_{1,1}| \lesssim \frac{Q^2}{r}\sum_{i=1}^{r}e^{-\frac{1}{4}|k_i|^2t}e^{-\frac{1}{2}|k_i|^2\tau}.$$

Hence it is even easier to handle $d_{1,1}$ than it is to handle F_1 in the previous section. Now, if we denote

$$b_{1,1} = \int_0^t e^{\Delta(t-\tau)}\mathbb{P}\tilde{b}_{1,1}(\cdot,\tau)d\tau,$$

we may conclude the following.

Lemma 4.6.

$$\|b_{1,1}\|_{\dot{B}^{-1,\infty}_\infty} \lesssim \frac{Q^2}{r}, \quad \|b_{1,1}\|_{X_T} \lesssim \frac{Q^2}{r}. \tag{4.17}$$

The focus is now on

$$\begin{aligned}b_{1,0} &= \int_0^t e^{\Delta(t-\tau)}\mathbb{P}\tilde{b}_{1,0}(\cdot,\tau)d\tau \qquad (4.18)\\ &= \frac{Q^2}{4r}e^{-t}\sin(\eta\cdot x)v\sum_{i=1}^{r}|k_i||k_i'|\int_0^t e^{(1-(|k_i|^2+|k_i'|^2)\tau}d\tau\end{aligned}$$

$$= \frac{Q^2}{4r} \sin(\eta \cdot x) v \sum_{i=1}^{r} |k_i||k_i'| \frac{e^{-t} - e^{-(|k_i|^2+|k_i'|^2)t}}{|k_i|^2 + |k_i'|^2 - 1}$$

since $v_i = v$ is fixed. Therefore we have the following.

Lemma 4.7. *Suppose* $\frac{1}{|k_1|^2} \ll T \ll 1$. *Then*

$$\|b_{1,0}(\cdot, T)\|_{B_\infty^{-1,\infty}} = \sup_{\tau \in (0,1)} \sqrt{\tau} \|e^{\tau \Delta} b_{1,0}\|_{L^\infty} \sim Q^2, \quad \|b_{1,0}\|_{X_T} \lesssim \sqrt{T} Q^2. \tag{4.19}$$

Proof. By (4.18), it follows that

$$b_{1,0} \sim Q^2 \sin(\eta \cdot x) v, \tag{4.20}$$

for $\frac{1}{|k_1|^2} \ll T \ll 1$. Indeed, $T \ll 1$ insures $e^{-t} \sim 1$, for $t \le T$; $\frac{1}{|k_1|^2} \ll T$ insures $e^{-(|k_s|^2+|k_s'|^2)t} \sim 0$. And, since $|k_1|$ is very large, $|k_s| \sim |k_s'|$ for every s by (4.4). Thus,

$$\begin{aligned} \|b_{1,0}\|_{B_\infty^{-1,\infty}} &\sim Q^2 \sup_{0<t<1} \sqrt{t} \|e^{t\Delta} \sin(\eta \cdot x)\|_{L^\infty} \\ &\sim Q^2 \sup_{0<t<1} \sqrt{t} e^{-|\eta|^2 t} \sim Q^2. \end{aligned} \tag{4.21}$$

Also

$$\begin{aligned} \|b_{1,0}\|_{X_T} &\sim Q^2 \sup_{0<t<T} \sqrt{t} \|\sin(\eta \cdot x)\|_{L^\infty} \\ &+ Q^2 \sup_{x_0, 0<R<T} \Big(\frac{1}{|B(x_0, \sqrt{R})|} \int_0^R \int_{B(x_0,\sqrt{R})} |\sin(\eta \cdot x)|^2 dx dt \Big)^{\frac{1}{2}} \\ &\lesssim \sqrt{T} Q^2. \end{aligned} \tag{4.22}$$

□

4.4. **Analysis of y and z.** In this section we analyze the parts y and z of the solution. The idea is to control y and z using the boundedness of the bilinear operator $\mathcal{B}$ in the space X_T. Naively one would hope that nonlinear terms turn out to be even smaller. But the trouble is at the linear term G_1 and K_1 since

$$\|e^{t\Delta} u_0\|_{X_T} \lesssim Q, \quad \text{and} \quad \|e^{t\Delta} b_0\|_{X_T} \lesssim Q,$$

by Lemma 4.3, which is the best we can have. The problem is the plane waves should not be lumped together in one single time scale. Plane waves that have much bigger wave vectors diffuse much quicker. Therefore it makes sense to analyze how y and z evolve in different time scales and see how different plane waves contribute. In [1], Bourgain and Pavlović very skillfully designed time steps to group appropriately the plane waves. We will use the same idea. We now introduce the time step division as used in [1]. Let $0 < T_1 < T_2 < \cdots < T_\beta$, where $\beta = Q^3$ and $T_\alpha = |k_{r_\alpha}|^{-2}$, $r_\alpha = r - \alpha Q^{-3} r$, $\alpha = 1, 2, \dots$. In particular, $r_\beta = 0$ and $T_\beta = |k_0|^{-2}$. The following are the key estimates for the time step design in [1].

Lemma 4.8. *Suppose that r is sufficiently large for a fixed large number Q. Then*

$$\|(e^{t\triangle}u_0)\chi_{[T_\alpha,T_{\alpha+1}]}(t)\|_{X_{T_{\alpha+1}}} \lesssim Q^{-1/2}, \quad \|(e^{t\triangle}b_0)\chi_{[T_\alpha,T_{\alpha+1}]}(t)\|_{X_{T_{\alpha+1}}} \lesssim Q^{-1/2}.$$

Proof. The proof is the same as in [1]. For the convenience of the reader we outline the proof showing the design of the time steps. First use the decomposition

$$(e^{t\triangle}u_0)\chi_{[T_\alpha,T_{\alpha+1}]}(t) \approx L_1 + L_2 + L_3,$$

where

$$L_1 = \frac{Q}{\sqrt{r}} \sum_{s<r_{\alpha+1}} |k_s| v_s \cos(k_s \cdot x) e^{-|k_s|^2 t} \chi_{[T_\alpha,T_{\alpha+1}]}(t)$$

$$L_2 = \frac{Q}{\sqrt{r}} \sum_{s=r_{\alpha+1}}^{r_\alpha} |k_s| v_s \cos(k_s \cdot x) e^{-|k_s|^2 t} \chi_{[T_\alpha,T_{\alpha+1}]}(t)$$

$$L_3 = \frac{Q}{\sqrt{r}} \sum_{r_\alpha<s\le r} |k_s| v_s \cos(k_s \cdot x) e^{-|k_s|^2 t} \chi_{[T_\alpha,T_{\alpha+1}]}(t).$$

The first group are those plane waves whose sizes are small. Provided $\frac{Q}{\sqrt{r}} \lesssim Q^{-\frac{1}{2}}$ we have

$$\|L_1\|_{X_{T_{\alpha+1}}} \lesssim \frac{Q}{\sqrt{r}} \sqrt{T_{\alpha+1}} |k_{r_{\alpha+1}-1}| + \frac{Q}{\sqrt{r}} (T_{\alpha+1}|k_{r_{\alpha+1}-1}|^2)^{\frac{1}{2}} \lesssim \frac{Q}{\sqrt{r}} \lesssim Q^{-\frac{1}{2}}.$$

The second group is the group of active plane waves in the time scale $[T_\alpha, T_{\alpha+1}]$. But, by the design, the number of plane waves in this group is small:

$$\|L_2\|_{X_{T_{\alpha+1}}} \lesssim \frac{Q}{\sqrt{r}} + \frac{Q}{\sqrt{r}} (r_\alpha - r_{\alpha+1})^{\frac{1}{2}} = \frac{Q}{\sqrt{r}} + \frac{Q}{\sqrt{r}} (Q^{-3} r)^{\frac{1}{2}} \lesssim Q^{-1/2},$$

if $\frac{Q}{\sqrt{r}} \lesssim Q^{-\frac{1}{2}}$. The last group of plane waves are those that have diffused too much and become small in size:

$$\|L_3\|_{X_{T_{\alpha+1}}} \lesssim \frac{Q}{\sqrt{r}} \sup_{t<T_{\alpha+1}} \sum_{s=r_\alpha+1}^{r} |k_s| \sqrt{t} e^{-|k_s|^2 t}$$

$$+\frac{Q}{\sqrt{r}}\sup_{t<T_{\alpha+1}}\Big(\int_0^t\Big|\sum_{s=r_\alpha+1}^{r}|k_s|^2e^{-|k_s|^2t}\chi_{[T_\alpha,T_{\alpha+1}]}(\tau)\Big|d\tau\Big)^{\frac{1}{2}}.$$

The first supremum of the last line is controlled by the integral

$$\begin{aligned}\int_{|k_{r_\alpha+1}|}^{|k_r|}\sqrt{t}e^{-x^2t}dx=\int_{|k_{r_\alpha+1}|\sqrt{t}}^{|k_r|\sqrt{t}}e^{-y^2}dy\le\int_{|k_{r_\alpha+1}|\sqrt{t}}^{|k_r|\sqrt{t}}e^{-y}dy\\ \le e^{-|k_{r_\alpha+1}|\sqrt{t}}\le e^{-|k_{r_\alpha+1}|/|k_{r_\alpha}|}\ll 1,\end{aligned}$$

where we used the fact that $|k_{r_\alpha+1}|\sqrt{t}>1$ for $t\in[T_\alpha,T_{\alpha+1}]$ with the second step and the last step follows from (4.3).

The second supremum is controlled by

$$\begin{aligned}\Big(\sum_{s=r_\alpha+1}^{r}e^{-|k_s|^2T_{\alpha+1}}\Big)^{1/2}&\le(r-r_\alpha)e^{-|k_{r_\alpha+1}|^2/|k_{r_{\alpha+1}}|^2}\\ &\lesssim(\alpha Q^{-3}r)e^{-4^{r-\alpha Q^{-3}r}}\ll 1.\end{aligned}$$

In the same way one can show the same estimate for b_0. The proof of the lemma is complete. □

Lemma 4.9. *For $T>T_\beta$,*

$$\|(e^{t\triangle}u_0)\chi_{[T_\beta,T]}(t)\|_{X_T}\lesssim\frac{Q}{\sqrt{r}},\tag{4.23}$$

$$\|(e^{t\triangle}b_0)\chi_{[T_\beta,T]}(t)\|_{X_T}\lesssim\frac{Q}{\sqrt{r}}.\tag{4.24}$$

Proof. From Lemma 4.1 and (4.12), we see that

$$\|(e^{t\triangle}u_0)\chi_{[T_\beta,T]}(t)\|_{X_T}\lesssim\frac{Q}{\sqrt{r}}+\frac{Q}{\sqrt{r}}\sum_{s=1}^{r}|k_s|e^{-\frac{|k_s|^2}{|k_0|^2}}|(T-T_\beta)^{\frac{1}{2}}\lesssim\frac{Q}{\sqrt{r}}.$$

The second one follows in the same way. □

Recall the equations for y and z from Section 2.4:

$$\begin{aligned}y_t-\triangle y+G_0+G_1+G_2=0,\\ z_t-\triangle z+K_0+K_1+K_2=0.\end{aligned}$$

Note that $y(0)=z(0)=0$. Hence, $t\in[T_\alpha,T_{\alpha+1}]$ and

$$y(t)=-\int_0^t e^{(t-\tau)\triangle}G(\tau)d\tau\tag{4.25}$$

$$= -\int_0^t e^{(t-\tau)\triangle} G(\tau)\chi_{[0,T_\alpha]}(\tau)d\tau - \int_0^t e^{(t-\tau)\triangle} G(\tau)\chi_{[T_\alpha,T_{\alpha+1}]}(\tau)d\tau,$$

where $G = G_0 + G_1 + G_2$. So we can write

$$\|y\|_{X_{T_{\alpha+1}}} \leq I_1 + I_2 \tag{4.26}$$

to see how y develops in the time step $[T_\alpha, T_{\alpha+1}]$.

Similarly for z, we have

$$\begin{aligned} \|z\|_{X_{T_{\alpha+1}}} \leq & \|\int_0^t e^{(t-\tau)\triangle} K(\tau)\chi_{[0,T_\alpha]}d\tau\|_{X_{T_{\alpha+1}}} \\ & + \|\int_0^t e^{(t-\tau)\triangle} K(\tau)\chi_{[T_\alpha,T_{\alpha+1}]}d\tau\|_{X_{T_{\alpha+1}}} = J_1 + J_2, \end{aligned} \tag{4.27}$$

where $K = K_0 + K_1 + K_2$. Now we are ready to estimate the increments of y and z during the time scale $[T_\alpha, T_{\alpha+1}]$.

Lemma 4.10. *With appropriate choices of r and T, we have*

$$\|y\|_{X_{T_{\alpha+1}}} + \|z\|_{X_{T_{\alpha+1}}} \lesssim Q^3(\frac{1}{r} + \sqrt{T_\beta}) + Q(\|y\|_{X_{T_\alpha}} + \|z\|_{X_{T_\alpha}}). \tag{4.28}$$

Proof. Applying the bilinear estimate (2.5), estimates in the space X_{T_α} from Lemma 4.3, (4.19), (4.17), and (4.16), we have

$$\begin{aligned} I_1 \lesssim & (Q + \frac{Q^2}{r} + \|y\|_{X_{T_\alpha}})\|y\|_{X_{T_\alpha}} \\ & + (Q + Q^2\sqrt{T_\alpha} + \frac{Q^2}{r} + \|z\|_{X_{T_\alpha}})\|z\|_{X_{T_\alpha}} + (Q + \frac{Q^2}{r})\frac{Q^2}{r} \\ & + (Q + Q^2\sqrt{T_\alpha} + \frac{Q^2}{r})(Q^2\sqrt{T_\alpha} + \frac{Q^2}{r}). \end{aligned} \tag{4.29}$$

We next apply Lemma 4.8 and estimate

$$\begin{aligned} I_2 \lesssim & (Q^{-1/2} + \frac{Q^2}{r} + \|y\|_{X_{T_{\alpha+1}}})\|y\|_{X_{T_{\alpha+1}}} \\ & + (Q^{-1/2} + Q^2\sqrt{T_{\alpha+1}} + \frac{Q^2}{r} + \|z\|_{X_{T_{\alpha+1}}})\|z\|_{X_{T_{\alpha+1}}} + (Q^{-1/2} + \frac{Q^2}{r})\frac{Q^2}{r} \\ & + (Q^{-1/2} + Q^2\sqrt{T_{\alpha+1}} + \frac{Q^2}{r})(Q^2\sqrt{T_{\alpha+1}} + \frac{Q^2}{r}). \end{aligned} \tag{4.30}$$

We choose r sufficiently large and T appropriately small such that

$$\frac{Q^2}{r} < Q^{-1/2}, \quad Q^2\sqrt{T} < Q^{-1/2}. \tag{4.31}$$

Hence we combine (4.26), (4.29) and (4.30) to arrive at

$$\|y\|_{X_{T_{\alpha+1}}} \lesssim Q^3(\frac{1}{r} + \sqrt{T_\beta}) + Q(\|y\|_{X_{T_\alpha}} + \|z\|_{X_{T_\alpha}}) \tag{4.32}$$
$$+Q^{-1/2}(\|y\|_{X_{T_{\alpha+1}}} + \|z\|_{X_{T_{\alpha+1}}}) + \|y\|^2_{X_{T_{\alpha+1}}} + \|z\|^2_{X_{T_{\alpha+1}}}.$$

Similarly we can obtain

$$\|z\|_{X_{T_{\alpha+1}}} \lesssim Q^3(\frac{1}{r} + \sqrt{T_\beta}) + Q(\|y\|_{X_{T_\alpha}} + \|z\|_{X_{T_\alpha}}) \tag{4.33}$$
$$+ Q^{-1/2}(\|y\|_{X_{T_{\alpha+1}}} + \|z\|_{X_{T_{\alpha+1}}}) + \|y\|_{X_{T_{\alpha+1}}}\|z\|_{X_{T_{\alpha+1}}}.$$

Therefore, adding (4.32) and (4.33), we have

$$\|y\|_{X_{T_{\alpha+1}}} + \|z\|_{X_{T_{\alpha+1}}} \lesssim Q^3(\frac{1}{r} + \sqrt{T_\beta}) + Q(\|y\|_{X_{T_\alpha}} + \|z\|_{X_{T_\alpha}})$$
$$+ (\|y\|_{X_{T_{\alpha+1}}} + \|z\|_{X_{T_{\alpha+1}}})^2.$$

So, for much larger r and $|k_0|$, we have $\|y\|_{X_{T_{\alpha+1}}} + \|z\|_{X_{T_{\alpha+1}}}$ small and

$$\|y\|_{X_{T_{\alpha+1}}} + \|z\|_{X_{T_{\alpha+1}}} \lesssim Q^3(\frac{1}{r} + \sqrt{T_\beta}) + Q(\|y\|_{X_{T_\alpha}} + \|z\|_{X_{T_\alpha}}). \qquad \square$$

By iterating (4.28) the following then follows easily.

Lemma 4.11.

$$\|y\|_{X_{T_\beta}} + \|z\|_{X_{T_\beta}} \lesssim Q^{\beta+2}(\frac{1}{r} + \sqrt{T_\beta}). \tag{4.34}$$

Next, for $T > T_\beta$, in light of Lemma 4.9, one may repeat the argument in the proof of (4.28) and obtain the following.

Lemma 4.12. *For appropriate choice of r and T,*

$$\|y\|_{X_T} + \|z\|_{X_T} \lesssim 4Q^4 T. \tag{4.35}$$

This implies that

$$\|y(\cdot, T)\|_{B^{-1,\infty}_\infty} \lesssim \|y(\cdot, T)\|_{L^\infty} \lesssim T^{-1/2}\|y\|_{X_T} \lesssim 4Q^4\sqrt{T} \tag{4.36}$$

and

$$\|z(\cdot, T)\|_{B^{-1,\infty}_\infty} \lesssim \|z(\cdot, T)\|_{L^\infty} \lesssim T^{-1/2}\|z\|_{X_T} \lesssim 4Q^4\sqrt{T}. \tag{4.37}$$

4.5. **Finishing the Proof.** Now we are ready to complete the proof of Theorem 1.1. Since (4.17) implies

$$\|b_{1,1}(\cdot,T)\|_{B^{-1,\infty}_\infty} \lesssim \|b_{1,1}(\cdot,T)\|_{L^\infty} \lesssim T^{-\frac{1}{2}}\|b_{1,1}\|_{X_T} \lesssim \frac{Q^2}{r\sqrt{T}}, \tag{4.38}$$

from (2.7) we combine (4.19), (4.38) and (4.37) to obtain

$$\begin{aligned}\|b(\cdot,T)-e^{T\triangle}b_0\|_{\dot{B}^{-1,\infty}_\infty} &\geq \|b(\cdot,T)-e^{T\triangle}b_0\|_{B^{-1,\infty}_\infty}\\ &\geq \|b_{1,0}(\cdot,T)\|_{B^{-1,\infty}_\infty} - \|b_{1,1}(\cdot,T)\|_{B^{-1,\infty}_\infty} - \|z(\cdot,T)\|_{B^{-1,\infty}_\infty}\\ &\gtrsim Q^2 - \|b_{1,1}\|_{L^\infty} - \|z\|_{L^\infty}\\ &\gtrsim Q^2\left(1-\frac{1}{r\sqrt{T}} - 4Q^2\sqrt{T}\right).\end{aligned}$$

Therefore, in light of (4.15),

$$\|b(T)\|_{\dot{B}^{-1,\infty}_\infty} \gtrsim Q^2.$$

On the other hand, from (2.6), we combine (4.15), (4.16), and (4.36) and have, for any $t\in[0,T]$,

$$\|u(\cdot,t)\|_{B^{-1,\infty}_\infty} \lesssim \frac{Q}{\sqrt{r}} + \frac{Q^2}{r} + Q^4\sqrt{T} \tag{4.39}$$

and remains small in $B^{-1,\infty}_\infty$. Thus we proved Theorem 1.1.

Remark 4.13. We would like to note the following simple chart to indicate how the choices of the several parameters are made.

$$\delta \longrightarrow Q \longrightarrow T \longrightarrow |k_0| \longrightarrow |k_s|, \quad Q \longrightarrow r.$$

5. Other scenarios of norm inflations

In this section, we consider other interesting norm inflation phenomena for MHD systems. The essence of the construction introduced by Bourgain and Pavlović in [1] and used in this paper is that the collisions of plane waves with similar but large wave vectors can cause norm inflations in NSE as well as in MHD systems. More precisely we see that the collisions in the quadratic terms trigger the norm inflations. Hence we can arrange the initial data to have collisions of plane waves with similar wave vectors either in u_1 or in b_1 to produce various norm inflation modes for MHD systems. For example, even the initial velocity is zero, if the initial magnetic field contains enough plane waves to collide, we can produce the scenario where the velocity develops norm inflation in $\dot{B}_{\infty}^{-1,\infty}$ while the magnetic field remains small in the space $B_{\infty}^{-1,\infty}$. Namely, we have the following.

Theorem 5.1. *Let* $u_0 \equiv 0$ *and*

$$b_0 = \frac{Q}{\sqrt{r}} \sum_{i=1}^{r} (|k_i| v_i \cos(k_i \cdot x) + |k_i'| v_i' \cos(k_i' \cdot x)).$$

Then, for any $\delta > 0$*, there exists a solution* (u, b, p) *to the MHD system* (1.1) *with initial data* u_0 *and* b_0 *as in the above satisfying*

$$\|b(0)\|_{\dot{B}_{\infty}^{-1,\infty}} \lesssim \delta,$$

such that for some $0 < T < \delta$

$$\|u(T)\|_{\dot{B}_{\infty}^{-1,\infty}} \gtrsim 1/\delta,$$

while for any $0 < t < T < \delta$

$$\|b(t)\|_{B_{\infty}^{-1,\infty}} \lesssim \delta.$$

The proof will be more or less the same as the proof in [1] in light of our discussions in the previous section. Another interesting case mentioned in Remark 1.4 is that when the initial velocity and the initial magnetic field are the same, although they both include many plane waves that are to collide, the collisions cancel each other in the evolution of the MHD system and produce no norm inflations.

Remark 5.2. Finally we would like to mention that, in [3], Cheskidov and Shvydkoy introduced a different construction of initial data to prove the ill posedness of NSE in certain Besov spaces. There are two more works about the ill-posedness results for Navier-Stokes equations by Germain [4] and Yoneda [9].

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