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NAVIER-STOKES SPACE TIME DECAY REVISITED

In memory of Tetsuro Miyakawa,
a dear friend and wonderful mathematician that left us too soon.

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Abstract. In this paper we analyze the decay in space and time of strong solutions to the Navier-Stokes equations is n space dimensions. We give first a brief review of known results, focusing on some of the Miyakawa techniques, which are then extended to cover the decay for higher derivatives.
*

1 Introduction

This paper is concerned with the space-time decay of solutions and derivatives of the Navier-Stokes equations.

$$\begin{cases} \partial_t u + \nabla \cdot (u \otimes u) + \nabla p = \nu \Delta u \\ \nabla \cdot u = 0 \qquad\qquad\qquad\qquad x \in \mathbb{R}^n, t \in \mathbb{R}^+ \\ u|_{t=0} = u_0. \end{cases} \tag{1.1}$$

Here $u \colon \mathbb{R}^n \times \mathbb{R}^+ \to \mathbb{R}^n$ is the velocity field. The scalar fields $p \colon \mathbb{R}^n \times \mathbb{R}^+ \to \mathbb{R}$ denotes the pressure. The viscosity ν needs to be strictly positive By rescaling the unknowns, we without loss of generality, that $\nu = 1$. As in Miyakawa's paper [18] we work with the integral form of the solutions

$$u(t) = e^{-tA}a - e^{(t-s)A}\nabla \cdot \mathcal{P}(u \otimes u)(s)ds, \tag{1.2}$$

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where $\mathcal{P}$ is the projection onto divergence free fields, $A = -\Delta$ is the Laplacian and $\nabla = (\partial_1, ...\partial_n)$, $\partial_j = \partial/\partial_j$.

After this short introduction we give a brief review of well known results on space-time decay. We then focus and describe one of Miyakawa's [18] results on the subject. The goal of the present paper is to show how the techniques used in [18] to establish the algebraic space-time decay of the solutions, can be extended to obtain algebraic space-time decay of derivatives of all order for strong solutions. The results can naturally be applied to weak solutions starting from a sufficiently large time where the solutions have become strong.

1.1 Notation

The following notation will be used $\alpha = (\alpha_1, \dots, \alpha_n), \alpha_i \geq 0,\ |\alpha| = \alpha_1 + \cdots + \alpha_n$,

$$D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}},\ ,\quad \text{and}\quad D_i = \frac{\partial}{\partial x_i} \tag{1.3}$$

For any integer $m \geq 0$, we set

$$D^m f(x) = \left(\sum_{|\alpha|=m} |D^\alpha f(x)|^2 \right)^{1/2} , x \in \mathbb{R}^n$$

The spaces for L^r, H^m are the standart normed Sobolev spaces with norms which we indicate as $\|...\|_r$ and $\|...\|_{H^m}$ respectively. Here $\mathcal{H}^p$, with $0 < p < \infty$ are the Hardy spaces. We recall the definition as can be found in [23].

$$\mathcal{H}^p = \{f:\ f \text{ vector valued distribution and } \exists\, \phi \in \mathcal{M} \text{ so that } \sup_{t>0} |\phi_t * f| \in L^p\}$$

where $\mathcal{M} = \{\phi : \phi \in \mathcal{S},\ \text{with} \int_{\mathbb{R}^n} \phi = 1\},\ \phi_t(x) = t^{-n}\phi(x/t)$ and $\mathcal{S}$ is the Schwartz space.

From [23] we know that if there exist $\phi \in \mathcal{M}$ so that $\sup_{t>0} |\phi_t * f| \in L^p$, then this condition holds for all $\phi \in \mathcal{M}$. We also recall that $\mathcal{H}^p = L^p,\ \forall\, p \in (1, \infty)$.
The quasi norm of $\mathcal{H}^p$ is as usual defined up to equivalences by

$$\|f\|_{\mathcal{H}^p} = \left\| \sup_{t>0} |\phi_t * f| \right\|_p$$

We recall the definition of $\mathcal{H}^p_w$

$$\mathcal{H}^p_w = \{f : \exists \phi \in \mathcal{M},\ \text{and} \sup_{t>0} |\phi_t * f| \in L^p_w\}$$

It is also known that if there is one $\phi \in \mathcal{M}$ so that the condition above insures that $f \in \mathcal{H}^p_w$, then it also holds for all $\phi \in \mathcal{M}$. The corresponding seminorm is defined as

$$\|f\|_{\mathcal{H}^p_w} = \left\| \sup_{t>0} |\phi_t * f| \right\|_{p<w}$$

As usual L^p_w is defined as the functions for which the weak L^p norm is bounded, that is

$$\|f\|_{p,w} = \sup_{t>0} tE(|g| > t)|^{1/p}$$

where$|E|$ is the Lebesgue measure of the measurable set E. For more details on properties of $\mathcal{H}^p$ and $\mathcal{H}^p_w$ spaces see [18] and [19].
All integrals in the sequel unless otherwise specified are over $\mathbb{R}^n$. Constants C will change from line to line.

2 Brief review of known results

There is a vast literature concerning decay in different norms for the solutions to the Navier-Stokes equations. For completeness we give a sample of papers, more information can be found in the references within: [7], [8], [9], [15], [17], [16] [10], [22] [21], [25], and [26].

The present paper focuses on questions related to the combined space and time decay of the solutions and derivatives to the (1.1) equations. For background we describe, in chronological order, a few of the many results that have been obtained for this type of decay.

We start by the work in [24] that studies the pointwise decay of solutions and derivatives with zero initial data and non zero external forces. Using a weighted-equation approach, pointwise decay rates are obtained both in time and space. The external forces are assumed to have an algebraic space-time decay rate and the solutions are assumed to be bounded in some weighted $L^{q,s}$ norms, with $n/q + 2/s = 1$ and $q, s \in [2, \infty]$, (the limiting Serrin class), where $L^{q,s}$ denotes the space of all $u : \mathbb{R}^n \times (0, \infty) \to \mathbb{R}^n$ such that

$$\{\int_0^\infty (\int_{\mathbb{R}^n} |u(x,t)|^q dx)^{s/q} dt\}^{1/s} < \infty.$$

The reader can also refer to [24] for an outline of previous work in the field.
Our results in [1] complement and extend the results in [24] . We consider nonzero initial data and no external force and establish decay for derivatives of all orders. This decay is optimal in the sense that it coincides with the decay of the underlying linear equation. Namely with the space time decay rate of the heat equation. Since the decay results in [1] are also for derivatives, it is done for strong solutions. The decay can be derived for weak solutions provided one starts at a sufficiently large time.
The results in [11] establish the space time decay of strong solutions and derivatives in $L^p(\mathbb{R}^3)$, $p \in [2, \infty]$ of the Navier-Stokes equations, under conditions for the solutions. Specifically it is shown that provided

$$\|u(x,.)\|_2 = O(t^{-\mu}) \text{ as } t \to \infty, \ \|u(x,.)|x|^r\|_2 = O(t^{-\mu+r/2}) \text{ as } t \to \infty, \ \ \mu \geq 0, r \geq 1$$

then

$$\|D^b u(t)|x|^a\|_p = O(t^{-(\mu+n/4(1-2/p))-a/2} \text{ as } t \to \infty.$$

Here $a \in (1, r),\ a < n/2 + 1, b \in N,$ and $p \in [2, \infty]$. The result in $2D$ requires only that $r > 1/2$

For decay of weak solutions with data in weighted L^2 spaces we refer the reader to [6]. In this case the spatial and time decays are obtained separately. The paper [12] improves the results in [11] by obtaining similar decay results with $r \geq 0$ and the decay rates are also obtained for the derivatives of the vorticity.

In [13] the decay in L^p-weighted spaces is obtained with the rates depending on the algebraic L^2 norm decay rate of the solution.

A very interesting result which looks at a different angle to the space-time decay can be found in [2]. "Here it is shown that for the solutions to the non-stationary NavierStokes equations in R^d for (d = 2, 3) which are left invariant under the action of discrete subgroups of the orthogonal group O(d) decay much faster as $|x| \to \infty$ or $t \to \infty$ than in the generic case." These better rates are obtained in detail.

In the paper [4] the authors work in unbounded domains and obtain algebraic space-time decay for weak solutons. The main problem to solve this case is that there is very little information on the pressure term near the boundary. In the calculations in [4] they manage to come up with ideas that avoid the pressure problem. Decay estimates are also obtained for strong solutions. For space decay results we also refer the reader to [5].

In [14] the author extends the results obtained in the former papers [11], [12], and [13]. The main question here deals with the influence of the decay of the solution in L^2 on the decay of the weighted spaces.

3 Miyakawa's results

In this section we recall some results that where obtained in [18] by Miyakawa. These results and techniques are going to be the basis in the next section to derive space time decay for higher derivatives. We first recall Theorem 1.1. from [18]:

Theorem 3.1. *Let $1 \leq \gamma \leq n+1$ and let a be a soleinoidal vector field on $\mathbb{R}^n$ satisfying*

$$|e^{-tA}a(x)| \leq C_0(1+|x|)^{-\alpha}(1+t)^{-\beta/2},\ \forall\ \alpha \text{ and } \beta \geq 0 \text{ with } \alpha+\beta=\gamma \tag{3.1}$$

If C_0 is small, there exists a solution u of (1.2) such that

$$|u(x,t)| \leq C(1+|x|)^{-\alpha}(1+t)^{-\beta/2},\ \forall\ \alpha \text{ and } \beta \geq 0 \text{ with } \alpha+\beta=\gamma \tag{3.2}$$

The solution u satisfies the initial condition $u\big|_{t=0} = a$ in the sense that

$$\lim_{t\to 0} u(x,t) = a(x) \ \text{ a.e. } \ x \in \mathbb{R}^n$$

Proof. See [18] □

To establish the above theorem the first step is to introduce the auxiliary integral field

$$v(t) = e^{-tA}a - \int_0^t \nabla e^{(t-s)A} \cdot \mathcal{P}(u \otimes u)(s)ds, \tag{3.3}$$

for the solenoidal fields $a = a(x)$ and $u = u(x,t)$. The following two Lemmas established in [18], were then used to analyze the auxiliary field v:

Lemma 3.1. *The kernel function $F(x,t)$ of $\nabla e^{-tA}\mathcal{P}$ satisfies*

$$|F(x,t)| \le C|x|^{-\alpha}t^{-\beta/2}, \quad (\alpha \ge 0, \beta \ge 0, \alpha+\beta = n+1) \tag{3.4}$$

and

$$\begin{aligned} \|F(\cdot,t)\|_p \le Ct^{-(n+1-\frac{n}{p})/2}, \qquad (1 \le p \le \infty) \\ \|F(\cdot,t)\|_{n-\frac{n}{p},1,\infty} \le C\|F(\cdot,t)\|_p \le Ct^{-(n+1-\frac{n}{p})/2}, \qquad (\frac{n}{n+1} < p < 1) \\ \|F(\cdot,t)\|_{-1,1,\infty} \le C\|F(\cdot,t)\|_{\frac{n}{n+1},w} \le C \end{aligned} \tag{3.5}$$

Proof. See [18]

Lemma 3.2. *Let $1 \le \gamma \le n+1$. If*

$$|e^{-tA}a(x)| \le C(1+|x|)^{-\alpha}(1+t)^{-\beta/2}, \ \forall \ \alpha \ \textit{and} \ \beta \ge 0 \ \textit{with} \ \alpha+\beta = \gamma \tag{3.6}$$

$$|u(x,t)| \le C(1+|x|)^{-\alpha}(1+t)^{-\beta/2}, \ \forall \ \alpha \ \textit{and} \ \beta \ge 0 \ \textit{with} \ \alpha+\beta = \gamma \tag{3.7}$$

then

$$|v(x,t)| \le C(1+|x|)^{-\alpha}(1+t)^{-\beta/2}, \ \forall \ \alpha \ \textit{and} \ \beta \ge 0 \ \textit{with} \ \alpha+\beta = \gamma \tag{3.8}$$

Once one has the estimates for v, define $\Phi(u)(x) = v(x)$ and show that $\Phi(u)$ has a fixed point. This fixed point will satisfy the conclusion of the theorem.

We also recall part of Theorem 1.2 in [18], which shows a class of solenoidal functions that give the necessary decay of the solution to the Heat equation.

Theorem 3.2. *If a is a bounded solenoidal field with compact support, then*

$$|e^{-tA}a(x)| \le C(1+|x|)^{-\alpha}(1+t)^{-\beta/2}, \ \forall \ \alpha \ \textit{and} \ \beta \ge 0 \ \textit{with} \ \alpha+\beta = n+1$$

Proof. See [18]

4 Extension to higher derivatives

In this section it is shown that the techniques used to establish Miyakawa's theorems and lemmas recorded in the last section, are the basis to obtain the decay for higher derivatives. These techniques are adapted and modified to fit our new hypothesis. Only some special cases are proved, we expect that other cases follow in similar fashion.

The next theorem is a "derivative version" of Theorem 3.1. For the decay of derivatives, we need to suppose that the solutions are strong. Recall that for $n = 2,3,4,5$ it is known that for t sufficiently large the solutions to (1.1) become smooth. Thus we could also start with large time t.

First an auxiliary lemma that extends part of Lemma 3.1 is recalled.

Let $K = ((4\pi t)^{n/2})^{-1}\exp -\frac{|x|^2}{4t}$ be the heat kernel, then

Lemma 4.1. *Let β, γ be multi-indices, $|\gamma| < |b| + 2\alpha\max\{j,1\}$, $j = 0,1,2,....$, $1 \le p \le \infty$. Then*

$$\|x^\gamma D_t^j D^\beta K(t)\|_p = Ct^{\frac{|\gamma|-|\beta|}{2}-j-\frac{n(p-1)}{2p}}$$

where C depends on $\alpha, \beta, \gamma, j, p$ and the space dimendion n.

Proof. See [22] □

Corollary 4.1. *The kernel function $F(x,t)$ of $\nabla e^{-tA}\mathcal{P}$ satisfies*

$$|\partial_i^k F(x,t)| \leq C|x|^{-\alpha} t^{-\beta/2}, \quad (\alpha \geq 0, \beta \geq 0, \alpha+\beta = n+1+k) \tag{4.1}$$

Proof. Follows by the last lemma, with $p=\infty$ and since $|\partial_i^k F(x,t)| \leq \|D^{k+1}K(t)\|_\infty$. □

Remark 4.1. *In the sequel we suppose we have smooth data. Otherwise we work with approximations and then pass to the limit.*

Theorem 4.1. *Let $\gamma = n+1+k$ and let $a \in C^k$ be a soleinoidal vector field on $\mathbb{R}^n$ satisfying*

$$|e^{-tA}\partial_i^k a(x)| \leq C_0(1+|x|)^{-\alpha}(1+t)^{-\beta/2},\ \forall\ \alpha \text{ and } \beta \geq 0 \text{ with } \alpha+\beta=\gamma. \tag{4.2}$$

If C_0 is small enough, there exists a solution u of (1.2) with data a such that

$$|\partial_i^k u(x,t)| \leq C(1+|x|)^{-\alpha}(1+t)^{-\beta/2},\ \forall\ \alpha \text{ and } \beta \geq 0 \text{ with } \alpha+\beta=\gamma \tag{4.3}$$

The solution u and derivatives satisfies the initial condition $u\big|_{t=0} = a$ in the sense that

$$\lim_{t\to 0} \partial_i^k u(x,t) = \partial_i^k a(x) \ \text{ a.e. } \ x \in \mathbb{R}^n. \tag{4.4}$$

Proof. The first step is to proof an analog to Lemma 3.2

Lemma 4.2. *If*

$$|e^{-tA}\partial_i^k a(x)| \leq C(1+|x|)^{-\alpha}(1+t)^{-\beta/2},\ \forall\ \alpha \ \ \beta \geq 0, \alpha+\beta = n+1+k \tag{4.5}$$

$$|\partial_i^m u(x,t)| \leq C(1+|x|)^{-\alpha_m}(1+t)^{-\beta_m/2},\ \forall\ \alpha_m \ \ \beta_m \geq 0 \tag{4.6}$$

$$\text{where } \ \alpha_m+\beta_m = n+1+m, \text{ for } m = 1,2,3,...k.$$

then

$$|\partial_i^k v(x,t)| \leq C(1+|x|)^{-\alpha}(1+t)^{-\beta/2},\ \forall\ \alpha \text{ and } \beta \geq 0 \text{ with } \alpha+\beta = n+1+k. \tag{4.7}$$

Proof. We suppose v as defined in (3.3). Note that since a and u are solenoidal so are $\partial_i^j a$ and ∂_i^j, with $i = 1,2,...n$ and $j = 1,...k$. $\mathcal{P}$ commutes with derivatives, hence

$$\partial_i^k v(t) = e^{-tA}\partial_i^k a - \int_0^t \nabla e^{(t-s)A} \cdot \mathcal{P}\partial_i^k (u\otimes u)(s)ds = \mathcal{A}_1(a) + \mathcal{A}_2(u) \tag{4.8}$$

Note that

$$|\partial_i^k (u\otimes u)(s)| \leq \sum_{j=0}^{k} |\partial^j u| \otimes |\partial^{k-j} u| \leq C(1+|y|)^{-(\alpha_j+\alpha_{k-j})}(1+|s|)^{-(\beta_j/2+\beta_{k-j}/2)}, \tag{4.9}$$

$$\text{with } \alpha_j + a_{k-j} + \beta_j + \beta_{k-j} = 2(n+1)+k$$

With this estimate in hand we proceed as in [18]. By the second estimate in (3.5) with $p = 1$ and Young's inequality

$$|\partial_i^k v(x,t)| \le C + \int_0^t \|F(t-s)\|_1 \|\partial_i^k (u \otimes u)(s)\|_\infty ds \le C + C_1 \int_0^t (t-s)^{-1/2} s^{-1/2} \le C$$

Thus $|\partial_i^k v(x,t)|$ is bounded for all x and t. To obtain the decay we only need to estimate the second term in (4.8) since the first decays by hypothesis. The same subdivision from the proof of Theorem 3.1 in [18] yields in our case

$$\mathcal{A}_2(|u|) = \int_0^t \left(\int_{|x-y|\le|x|/2} + \int_{|x-y|>|x|/2} \right) |F(x-y,t-s)||\partial_i^k (u\otimes u)(s)| ds \le \mathcal{B}_1(|u|) + \mathcal{B}_2(|u|).$$

Without loss of generality we assume that $|x| \ge 1$, hence by (4.1) and (4.9)

$$\mathcal{B}_1(|u|) \le C \int_0^t \int_{|x-y|\le|x|/2} (t-s)^{-3/4} |x-y|^{1/2-n} (1+|y|)^{-3/2-2n-k} (1+s)^{-1/4} dy ds$$

Since we have $|x| \le |x-y| + |y| \le |y| + |x|/2$ and thus $|x| \le 2|y|$, it follows that

$$\mathcal{B}_1(|u|) \le C|x|^{1/2} (1+|x|)^{-3/2-2n-k} \le C(1+|x|)^{-1-2n-k} \tag{4.10}$$

By (4.1) and (4.9) it follows that

$$\mathcal{B}_2(|u|) \le C \int_0^t \int_{|x-y|\ge|x|/2} |x-y|^{-(n+1)} (1+|y|)^{1-2n-k} (1+s)^{-3/2}$$

$$\le C|x|^{-(n+1)} \int_0^t \int_{|x-y|\ge|x|/2} |x-y|^{-(n+1)} (1+|y|)^{1-2n} (1+|y|)^{-k} (1+s)^{-3/2}$$

since as before we have $|x| \le 2|y|$ we have as $|x| \ge 1$

$$\mathcal{B}_2(|u|) \le C(1+|x|)^{-(n+1+k)} \tag{4.11}$$

Combining the estimates from (4.10),(4.11) with the hypothesis we obtain the conclusion of the Lemma in the case $\beta = 0$

The case $\alpha = 0$ is handled as follows. We show first that $\partial_i^k v(x,t) \in \dot{B}^{-1}_{1,\infty} \cap L^\infty$. We already have shown that $\partial_i^k v(x,t) \in L^\infty$. An easy estimate shows that

$$|\partial_i^k e^{-tA} a(x)| \le C(1+|x|)^{-(n+1+k)} \le C(1+|x|)^{-(n+1)} \;\Rightarrow\; a \in \mathcal{H}_w^{n/(n+1)}$$

Since $\mathcal{H}^{n/(n+1)} \subset \dot{B}^{-1}_{1,\infty}$. Thus $e^{-tA}a(x)$ is bounded in $\dot{B}^{-1}_{1,\infty}$. And by (3.5) we have that $F(x,t)$ also remains bounded in $\dot{B}^{-1}_{1,\infty}$. Combine this with estimate with (4.9) in the case

$$|\partial_i^k (u \otimes u)(s)| \le C(1+|y|)^{1/2-(2n+k)} (1+|s|)^{-5/4}$$

then we have

$$\|\partial_i^k v(t)\|_{-1;1,\infty} \le C_0 + C_1 \int_0^t \int |\partial_i^k (u\otimes u)(s)| dx ds$$

$$\le C_0 + C_2 \int_0^t (1+|s|)^{-5/4} ds \le C$$

Hence $\partial_I^k v \in \dot{B}^{-1}_{1,\infty}$. Using the same subdivision as in the proof of Theorem 1.1 in [18] (which we have recorded as Theorem 3.1).

$$\partial_i^k v(t) = e^{-tA/2}\left[e^{-tA/2}\partial_i^k a(x) - \int_0^t \nabla e^{-(t/2-s)A}\mathcal{P}\partial_i^k(u\otimes u)ds\right]$$

$$-\int_0^t \nabla e^{-(t-s)A}\mathcal{P}\partial_i^k(u\otimes u)ds = e^{-tA/2}\partial_i^k v(t/2) - \int_{t/2}^t \nabla e^{-(t-s)A}\mathcal{P}\partial_i^k(u\otimes u)ds.$$

Since $\|\partial_i^k v(t/2)\|_\infty \le C(1+t)^{-(n+1+k)/2}$, and since $v(t)$ is bounded in $L^\infty \cap \dot{B}^{-1}_{1,\infty}$, combined with the second estimate in (3.5) when $p=1$ and Young's inequality yields

$$\|\partial_i^k v(t)\|_\infty \le C(1+t)^{-(n+1+k)/2} + C\int_{t/2}^t (t-s)^{-1/2}(1+s)^{-(1+n+k)}ds \le C(1+t)^{-(n+1+k)/2}$$

From here follows the case for $\alpha = 0$. The intermediate cases follow by interpolation. This concludes the proof of the Lemma. $\square$.

To finish the theorem we need the following fixed point theorem for bilinear forms [3]:

Theorem 4.2. *Let $\mathcal{X}$ be a Banach space with norm denoted by $\|\cdot\|_{\mathcal{X}}$, and $B : \mathcal{X}\times\mathcal{X} \to \mathcal{X}$ a bilinear map, such that one has for all $(u,v) \in \mathcal{X}\times\mathcal{X}$*

$$\|B(u,v)\|_{\mathcal{X}} \le \eta \|u\|_{\mathcal{X}}\|v\|_{\mathcal{X}}$$

Then for all $u_0 \in \mathcal{X}$, satisfying $4\eta\|u_0\|_{\mathcal{X}} < 1$, the equation

$$u = u_0 + B(u,u)$$

admits a solution $u \in \mathcal{X}$ and this solution is the unique one that satisfies $\|u\|_{\mathcal{X}} \le 2\|u_0\|_{\mathcal{X}}$.

We will use the above theorem in the Banach space $\mathcal{X}$ of solenoidal fields u where the norm is given by

$$\|u\|_{\mathcal{X}} = \sup_{\substack{\alpha_m\ge 0,\beta_m\ge 0,\ m=0,1,2,3,\dots k,\\ a_m+\beta_m=n+1+m,}} \left\{(1+|x|)^{\alpha_m}(1+t)^{\beta_m/2}|\partial_i^m u(x,t)|\right\}$$

and the bilinear form Φ and initial field u_0 are given by

$$\Phi(u,v)(t) = \int_0^t \nabla e^{-(t-s)A}\mathcal{P}\partial_i^k(u\otimes v)ds,$$

By Corollary 4.1 and Lemma 4.2 it follows that there is a constant η so that

$$\|\Phi(u,v)\|_{\mathcal{X}} \le \eta\|u\|_{\mathcal{X}}\|v\|_{\mathcal{X}}$$

Now let $u_0(t) = e^{-tA}\partial_i^k a(t)$. By hypothesis (4.2) we have that

$$\|u_0(t)\|_{\mathcal{X}} \le C_0$$

If choose C_0 so small that $4\eta C_0 < 1$ then there is a unique solution $u(x,t)$ of the integral equation

$$u(x,t) = e^{-tA}a(t) + \int_0^t e^{-(t-s)A}\nabla\mathcal{P}(u\otimes)u)ds, \tag{4.12}$$

satisfying $\|u\|_{\mathcal{X}} \leq 2C_0$. This establishes conclusion (4.3) of the theorem.

We recall that for the solution to the heat equation we have that

$$e^{-tA}\partial_i^k a \to \partial_i^k a, \text{ as } t \to 0 \text{ almost every } x \in \mathbb{R}^n,$$

Hence we only need to show

$$\|\partial_i^k u(t) - e^{-tA}\partial_i^k a\|_\infty \to 0, \text{ as } t \to 0$$

From the integral form of the solution taking partial derivatives, and the arguments above, it follows that

$$\partial_i^k u(x,t) = e^{-tA}\partial_i^k a(t) + \int_0^t \partial_i^k e^{-(t-s)A}\nabla\mathcal{P}(u\otimes)u)ds$$

Since we are supposing that the derivatives of our solutions are bounded, after integration by parts it follows that

$$\|\partial_i^k u(x,t) - e^{-tA}\partial_i^k a(t)\|_\infty \leq C\sum_{j=1}^n \int_0^t (t-s)^{-1/2}\|\partial_i^{k-j}u(s)\partial_i^j u(s)\|_\infty \, ds \leq Ct^{1/2} \to 0.$$

This concludes the proof of the theorem. □

Remark 4.2. *We established Theorem 4.1 in the case when $\gamma = n+k+1$, we expect that a similar result can be obtained, combining the steps we have here with the proof of Theorem 3.1, in the case that $1 \leq \gamma \leq n+1+k$.*

The following is an immediate consequence of the last Theorem.

Corollary 4.2. *Under the hypothesis of Theorem 4.1*

$$D^k u(x,t)| \leq C(1+|x|)^{-\alpha}(1+t)^{\beta/2}, \ \alpha \ \text{ and } \ \beta \geq 0, \alpha+\beta = n+1+k$$

Proof. *Follows by Theorem 4.1* □

Corollary 4.3. *Under the hypothesis of Theorem 4.1 we have the following weighted L^p decay*

$$\||x|^a D^k u(t)\|_p \leq C(1+t)^{-b/2}, \ \alpha \text{ and } b \geq 0, \ \ a+b = (n+1)(1-\frac{1}{p}) + k + \frac{1-\epsilon}{p} \tag{4.13}$$

Where C depends on the data and p.

Proof. By Theorem 4.1 we have if $a+b+c=n+1+k$

$$\left||x|^a D^k u(t)\right|^p \le C(1+|x|)^{-cp}(1+t)^{-pb/2}$$

Let $cp=n+\epsilon$, then $\int(1+x)^{-cp}dx \le C < \infty$. Integrating (4.13) and taking the "p-th" root yields the conclusion of the Corollary. $\square$

Remark 4.3. *We note, that in this fashion, we can recuperate some of the decay rates obtained in [11], [14] [12] and [13]. In particular if the above results are extended to the case $1 \le \gamma \le n+1+k$. One difference is that we do not require conditions on the solutions, but directly on the data*

The next theorem which is the analog to Theorem 3.2, adapted to higher derivatives, describes data $a(x)$, so that "$e^{-tA}\partial_i^k a$", for$i=1,..,n$ satisfies condition (4.5) from the last theorem. We recall that the argument in [18] for the proof of Theorem 3.2 is based on a result in [23]. Our proof follows similar steps.

Theorem 4.3. *Let a be a bounded solenoidal fileld with compact support,. Suppose in addition that the derivatives ∂_i^k for any $i=1,2,...n$ and positive integer k are also bounded, then*

$$|e^{-tA}\partial_i^k a(x)| \le C(1+|x|)^{-\alpha}(1+t)^{-\beta/2},\ \forall\ \alpha \text{ and } \beta \ge 0 \text{ with} \alpha+\beta=n+1+k$$

Proof. Let B_r be a ball so that supp $a \subset B_r$. The boundedness $|e^{-tA}a(x)|$ allows to suppose without loss of generality that $|x| \ge 2r$. Since a is solenoidal, so is $\partial_i^k a$, since by hypothesis $\partial_i^k a \in L^1$ standart arguments (see [19], [20]) insure that $\int \partial_i^k a = 0$. Let $E(x)=(4\pi)^{-n/2}e^{-|x|^2/4}$. Therefor e

$$e^{-tA}\partial_i^k a(x) = t^{-n/2}\int [E\left(\frac{x-y}{\sqrt{t}}\right) - E\left(\frac{x}{\sqrt{t}}\right)]\partial_i^k a(y)dy.$$

Note that

$$E\left(\frac{x-y}{\sqrt{t}}\right) - E\left(\frac{x}{\sqrt{t}}\right) = -\int_0^1 \nabla E\left(\frac{x-\theta y}{\sqrt{t}}\right)\cdot\frac{y}{\sqrt{t}}d\theta.$$

Since $|x|>2r$, and $|y|\le r$ it follows that $|x-\theta y| \ge |x|/2$. Hence it is easy to show that

$$\partial_i^k\left[\nabla E\left(\frac{x-\theta y}{\sqrt{t}}\right)\right] \le C[|y|+1]e^{-c|x|^2/t}(t+1)^{-(n+1+k)}$$

By integration by parts we have that

$$|e^{-tA}\partial_i^k a(x)| \le Ce^{-c|x|^2/t}(t+1)^{-(n+1+k)}\int[|y|+1]|a(y)|dy$$

From where

$$|e^{-tA}\partial_i^k a(x)| \le C(1+|x|)^{-(n+1+k)} \tag{4.14}$$

Hence we have the result for $\beta=0$. Estimate (4.14) yields that $a \in \mathcal{H}_w^{n/(n+1+k)} \cap L^\infty$ thus

$$|e^{-tA}\partial_i^k a(x)| \le C(1+t)^{-\frac{n+1+k}{2}}\ \forall\, t>0 \tag{4.15}$$

Combining the results from (4.14) and (4.15) with the hypothesis yields the conclusion of the Theorem. $\square$

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