On the Oberbeck-Boussinesq approximation on unbounded domains

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Abstract We study the Oberbeck-Boussinesq approximation describing the motion of an incompressible, heat-conducting fluid occupying a general unbounded domain in R^3. We provide a rigorous justification of the model by means of scale analysis of the full Navier-Stokes-Fourier system in the low Mach and Froude number regime on large domains, the diameter of which is proportional to the speed of sound. Finally, we show that the total energy of any solution of the resulting Oberbeck-Boussinesq system tends to zero with growing time.

Key words: Oberbeck-Boussinesq system, singular limit, unbounded domain

1 Introduction

Stratified flows occur frequently in the atmosphere or oceans. The Oberbeck-Boussinesq approximation is a mathematical model of a stratified fluid flow, where the fluid is assumed to be incompressible and yet convecting a diffusive quantity creating positive or negative buoyancy force. The diffusive quantity is identified with the deviation of temperature from its equilibrium value. The resulting system of equations reads:

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$$\mathrm{div}_x \mathbf{U} = 0, \tag{1}$$

$$\overline{\rho}\Big(\partial_t \mathbf{U} + \mathrm{div}_x \mathbf{U} \otimes \mathbf{U}\Big) + \nabla_x \Pi = \mu \Delta \mathbf{U} + r \nabla_x G, \tag{2}$$

$$\overline{\rho} c_p \Big(\partial_t \Theta + \mathrm{div}_x (\Theta \mathbf{U})\Big) - \kappa \Delta \Theta - \overline{\rho}\,\overline{\vartheta} \alpha \mathrm{div}_x (G \mathbf{U}) = 0, \tag{3}$$

$$r + \overline{\rho} \alpha \Theta = 0, \tag{4}$$

where the unknowns are the fluid velocity $\mathbf{U} = \mathbf{U}(t,x)$ and the temperature deviation $\Theta = \Theta(t,x)$. The symbol Π denotes the pressure, $\mu > 0$ is the viscosity coefficient, $\kappa > 0$ the heat conductivity coefficient, $\overline{\rho} > 0$ stands for the fluid density, and $\overline{\vartheta} > 0$ is the reference temperature. Here, $c_p > 0$ is the specific heat at constant pressure and $\alpha > 0$ denotes the coefficient of thermal expansion of the fluid, both evaluated at the reference density $\overline{\rho}$ and temperature $\overline{\vartheta}$. The function $G = G(x)$ is a given gravitational potential acting on the fluid. Thus the fluid density is constant in the Oberbeck-Boussinesq approximation except in the buoyancy force, where it is inter-related to the temperature deviation through *Boussinesq relation* (4), cf. Zeytounian [30], [31].

In real world applications, it is customary to take the x_3-coordinate to be vertical parallel to the gravitational force $\nabla_x G = g[0,0,-1]$. This is indeed a reasonable approximation provided the fluid occupies a bounded domain $\Omega \subset R^3$, where the gravitational field can be taken constant. Recently, several authors studied system (1 - 4) on the whole space $\Omega = R^3$, with $\nabla_x G = g[0,0,-1]$, see [4], Danchin and Paicu [7]. Such an “extrapolation” of the model is quite natural from the mathematical viewpoint, however, a bit awkward physically. Indeed, if the self-gravitation of the fluid is neglected, the origin of the gravitational force must be an object placed *outside* the fluid domain Ω therefore

$$G(x) = \int_{R^3} \frac{1}{|x-y|} m(y)\ \mathrm{d}y, \text{ with } m \geq 0, \ \mathrm{supp}[m] \subset R^3 \setminus \Omega, \tag{5}$$

where m denotes the mass density of the object acting on the fluid by means of gravitation. In other words, G is a harmonic function in Ω, $G(x) \approx 1/|x|$ as $|x| \to \infty$.

Motivated by the previous observations, we consider the Oberbeck-Boussinesq system on a domain $\Omega = R^3 \setminus K$ exterior to a compact set K. Accordingly, we take G such that

$$-\Delta G = m \text{ in } R^3, \ \nabla_x G \in L^2(R^3; R^3), \ \mathrm{supp}[m] \subset K. \tag{6}$$

In particular, introducing a new variable $\theta = \Theta - \overline{\vartheta} \alpha G / c_p$ we can rewrite the system (1 - 4) in the more frequently used form

$$\mathrm{div}_x \mathbf{U} = 0, \tag{7}$$

$$\overline{\rho}\Big(\partial_t \mathbf{U} + {\rm div}_x \mathbf{U}\otimes\mathbf{U}\Big) + \nabla_x P = \mu\Delta\mathbf{U} - \overline{\rho}\alpha\theta\nabla_x G, \tag{8}$$

$$\overline{\rho}c_p\Big(\partial_t\theta + {\rm div}_x(\theta\mathbf{U})\Big) - \kappa\Delta\theta = 0, \tag{9}$$

where we have set $P = \Pi - G^2\overline{\rho}\overline{\vartheta}\alpha^2/2c_p$.

We will show in Section 2 that the Oberbeck-Boussinesq approximation (1 - 4), supplemented with suitable boundary conditions, may be viewed as a singular limit of the full Navier-Stokes-Fourier system considered on a family of "large domains", where the Mach and Froude numbers tend simultaneously to zero. This part of the paper can be viewed as an application of the abstract method developed in [13] in order to control the propagation and the final filtering of acoustic waves in the limit system. Furthermore, we discuss the basic properties of the limit system (1 - 4), in particular, validity of the energy inequality, see Section 3. Finally, in Section 4, we show that the total energy of any weak solution to the Oberbeck-Boussinesq approximation (7 - 9), supplemented with the homogeneous Dirichlet boundary conditions, tends to zero with growing time. To this end, we first establish the result for the temperature deviations represented by θ, and then use the standard estimates for the incompressible Navier-Stokes in the spirit of Miyakawa and Sohr [24].

1.1 Notation and preliminaries

We use the symbol $< \cdot,\cdot >$ to denote duality product, in particular,

$$< f, g > = \int_O fg,$$

provided f, g are square integrable on a set O.

The symbol $L^p(O)$ denotes the space of measurable functions v, with $|v|^p$ integrable in O. $W^{k,p}$ denotes the Sobolev space of functions having derivatives up to order k in L^p. Finally, we introduce the homogeneous Sobolev spaces:

$$\widehat{W}^{m,p} = \Big\{ v \in L^1_{loc}(\Omega),\ D^\alpha u \in L^p(\Omega), |\alpha| = m \Big\},\ m \geq 0,\ p \geq 1.$$

By the symbol c we denote a generic constant that may change line by line.

Most of the results of the paper concern problems on an exterior domain $\Omega \subset R^3$. In order to avoid technicalities, we assume that the boundary $\partial\Omega$ is smooth, say of class $C^{2+\nu}$, in particular, Ω satisfies the *cone property*:

The domain Ω is said to satisfy the *cone property* if there exists a finite cone $\mathscr{C}$ such that each point $x \in \Omega$ is the vertex of a finite cone $\mathscr{C}_x$ contained in Ω and congruent to $\mathscr{C}$.

To conclude the preliminary part, we record a variant of the Gagliardo-Nirenberg inequalities for exterior domains proved by Crispo and Maremonti [6].

Proposition 1.1 *Let $\Omega \subset R^N$ be an exterior domain with cone property. Let $w \in \widehat{W}^{m,p}(\Omega) \cap L^q(\Omega)$, with $1 \leq p \leq \infty$, $1 \leq q < \infty$.*

Then

$$\|D^k w\|_{L^r(\Omega)} \leq c \|D^m w\|^a_{L^p(\Omega)} \|w\|^{1-a}_{L^q(\Omega)} \tag{10}$$

for any integer $k \in [0, m-1]$, where

$$\frac{1}{r} = \frac{k}{N} + a\left(\frac{1}{p} - \frac{m}{N}\right) + (1-a)\frac{1}{q},$$

with $a \in \left[\frac{k}{m}, 1\right]$, either if $p = 1$ or $p > 1$ and $m - k - \frac{N}{p} \notin \mathcal{N} \cup \{0\}$, while $a \in \left[\frac{k}{m}, 1\right)$ if $p > 1$ and $m - k - \frac{N}{p} \in \mathcal{N} \cup \{0\}$.

2 The Oberbeck-Boussinesq approximation as a singular limit of the full Navier-Stokes-Fourier system

Motivated by the mathematical theory developed in [14], we introduce a scaled *Navier-Stokes-Fourier system* in the form:

MASS CONSERVATION

$$\partial_t \rho + \mathrm{div}_x(\rho \mathbf{u}) = 0, \tag{11}$$

MOMENTUM BALANCE

$$\partial_t(\rho \mathbf{u}) + \mathrm{div}_x(\rho \mathbf{u} \otimes \mathbf{u}) + \frac{1}{\varepsilon^2}\nabla_x p(\rho, \vartheta) = \mathrm{div}_x \mathbb{S}(\vartheta, \nabla_x \mathbf{u}) + \frac{1}{\varepsilon}\rho \nabla_x G, \tag{12}$$

ENTROPY BALANCE

$$\partial_t(\rho s(\rho, \vartheta)) + \mathrm{div}_x(\rho s(\rho, \vartheta)) + \mathrm{div}_x\left(\frac{\mathbf{q}(\vartheta, \nabla_x \vartheta)}{\vartheta}\right) = \sigma, \tag{13}$$

TOTAL ENERGY CONSERVATION

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \left(\frac{1}{2}\rho|\mathbf{u}|^2 + \frac{1}{\varepsilon^2}\rho e(\rho, \vartheta) - \frac{1}{\varepsilon}\rho G\right)\, \mathrm{d}x = 0, \tag{14}$$

where S is the viscous stress given by *Newton's rheological law*

$$\mathbb{S}(\vartheta, \nabla_x \mathbf{u}) = \mu(\vartheta)\left(\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{3}\mathrm{div}_x \mathbf{u}\mathbb{I}\right) + \eta(\vartheta)\mathrm{div}_x \mathbf{u}\mathbb{I}, \tag{15}$$

q is the heat flux determined by *Fourier's law*

$$\mathbf{q}(\vartheta, \nabla_x \vartheta) = -\kappa(\vartheta)\nabla_x \vartheta, \tag{16}$$

whereas the *entropy production rate* σ satisfies

$$\sigma \geq \frac{1}{\vartheta}\left(\varepsilon^2 \mathbb{S} : \nabla_x \mathbf{u} - \frac{\mathbf{q}\cdot\nabla_x\vartheta}{\vartheta}\right). \tag{17}$$

The unknowns in (11 - 14) are the fluid mass density $\rho = \rho(t,x)$, the velocity field $\mathbf{u} = \mathbf{u}(t,x)$, and the absolute temperature $\vartheta = \vartheta(t,x)$. The pressure p, the specific internal energy e, and the specific entropy s are given numerical functions of ρ and ϑ interrelated through *Gibbs' equation*

$$\vartheta Ds = De + pD\left(\frac{1}{\rho}\right). \tag{18}$$

The system (11 - 14) is supplemented with the *conservative* boundary conditions, specifically,

$$\mathbf{u}\cdot\mathbf{n}|_{\partial\Omega} = 0,\ \beta[\mathbf{u}]_{\tan} + [\mathbb{S}\mathbf{n}]_{\tan}|_{\partial\Omega} = 0,\ \mathbf{q}\cdot\mathbf{n}|_{\partial\Omega} = -\beta|\mathbf{u}|^2|_{\partial\Omega},\ \beta > 0, \tag{19}$$

where $\mathbf{n}$ denotes the outer normal vector to $\partial\Omega$. The first two conditions in (19) are usually termed *Navier's slip boundary condition* with a friction coefficient $\beta > 0$, see Málek and Rajagopal [22]. In accordance with (19), the total energy of the fluid is a conserved quantity as stated in (14).

The small parameter ε appearing in (12), (14), and (17) results from the scaling analysis of the Navier-Stokes-Fourier system, where the *Mach number* and the *Froude number* are proportional to ε, see [14, Chapters 4,5], Klein et al. [16], Zeytounian [32]. Physically this means that the characteristic speed of the fluid is largely dominated by the speed of sound and the fluid is stratified. Note that a similar system of equations may be obtained by *constitutive* scaling, where the rheological properties of the fluid are changing rather than the characteristic geometrical parameters of the flow, see Novotný, Růžička, Thaeter [25], Rajagopal, Růžička, and Srinivasa [26].

2.1 Weak solutions

In the framework of *weak solutions*, the equation of continuity (11) is replaced by a family of integral identities

$$\int_\Omega \Big[\rho(\tau,\cdot)\varphi(\tau,\cdot) - \rho_0\varphi(0,\cdot)\Big]\,\mathrm{d}x \tag{20}$$

$$= \int_0^\tau \int_\Omega \Big(\rho \partial_t \varphi + \rho \mathbf{u} \cdot \nabla_x \varphi \Big) \ \mathrm{d}x \ \mathrm{d}t \text{ for any } \tau \in [0,T],$$

for any test function $\varphi \in C^1([0,T] \times \overline{\Omega})$. In particular, the mapping $\tau \mapsto \rho(\tau, \cdot)$ is weakly continuous, and ρ satisfies the initial condition

$$\rho(0, \cdot) = \rho_0.$$

Similarly, the momentum equation (12), together with Navier's slip boundary conditions (19), read

$$\int_\Omega \Big[\rho \mathbf{u}(\tau, \cdot) \cdot \varphi(\tau, \cdot) - \rho_0 \mathbf{u}_0 \cdot \varphi(0, \cdot) \Big] \ \mathrm{d}x \tag{21}$$

$$= \int_0^\tau \int_\Omega \Big(\rho \mathbf{u} \cdot \partial_t \varphi + \rho \mathbf{u} \otimes \mathbf{u} : \nabla_x \varphi + \frac{p}{\varepsilon^2} \mathrm{div}_x \varphi - \mathbb{S} : \nabla_x \varphi + \frac{\rho}{\varepsilon} \nabla_x G \cdot \varphi \Big) \ \mathrm{d}x \ \mathrm{d}t$$

$$+ \int_0^\tau \int_{\partial\Omega} \beta \mathbf{u} \cdot \varphi \ \mathrm{dS}_x \ \mathrm{d}t,$$

for any $\tau \in [0,T]$, and any $\varphi \in C^1([0,T] \times \overline{\Omega}; R^3)$, $\varphi \cdot \mathbf{n}|_{\partial\Omega} = 0$. Thus the momentum $\tau \mapsto (\rho \mathbf{u})(\tau, \cdot)$ is weakly continuous and

$$(\rho \mathbf{u})(0, \cdot) = \rho_0 \mathbf{u}_0$$

Finally, we may write the entropy balance (13) in the form

$$\int_\Omega \Big[\rho s(\rho, \vartheta)(\tau, \cdot) \varphi(\tau, \cdot) - \rho_0 s(\rho_0, \vartheta_0) \varphi(0, \cdot) \Big] \ \mathrm{d}x = < \sigma, 1_{[0,\tau]} \varphi > + \tag{22}$$

$$\int_0^\tau \int_\Omega \Big(\rho s \partial_t \varphi + \rho s \mathbf{u} \cdot \nabla_x \varphi + \frac{\mathbf{q}}{\vartheta} \cdot \nabla_x \varphi \Big) \ \mathrm{d}x \ \mathrm{d}t + \varepsilon^2 \int_0^\tau \int_{\partial\Omega} \frac{\beta}{\vartheta} |\mathbf{u}|^2 \varphi \ \mathrm{dS}_x \ \mathrm{d}t,$$

for any test function $\varphi \in C^1([0,T] \times \overline{\Omega})$, where the entropy production rate σ is interpreted as a non-negative measure on $[0,T] \times \overline{\Omega}$ satisfying

$$\sigma \geq \frac{1}{\vartheta} \left(\varepsilon^2 \mathbb{S} : \nabla_x \mathbf{u} - \frac{\mathbf{q} \cdot \nabla_x \vartheta}{\vartheta} \right). \tag{23}$$

The total energy balance (14) reads

$$\int_\Omega \left(\frac{1}{2} \rho |\mathbf{u}|^2 + \frac{1}{\varepsilon^2} \rho e - \frac{1}{\varepsilon} \rho G \right) (\tau, \cdot) \ \mathrm{d}x \tag{24}$$

$$= \int_\Omega \left(\frac{1}{2} \rho_0 |\mathbf{u}_0|^2 + \frac{1}{\varepsilon^2} \rho_0 e(\rho_0, \vartheta_0) - \frac{1}{\varepsilon} \rho_0 G \right) \ \mathrm{d}x.$$

The interested reader may consult [14, Chapter 2] for a formal interpretation of the weak solutions to the Navier-Stokes-Fourier system. We only note that the entropy production rate σ associated to a weak solution that is sufficiently smooth necessarily satisfies

$$\sigma = \frac{1}{\vartheta}\left(\varepsilon^2 \mathbb{S} : \nabla_x \mathbf{u} - \frac{\mathbf{q}\cdot\nabla_x\vartheta}{\vartheta}\right),$$

in agreement with the classical theory.

Unlike (20), (21), relations (22), (24) are satisfied only for a.a. $\tau \in [0,T]$. In particular, the total entropy $\rho s(\rho,\vartheta)$ may not be a weakly continuous function of time due to hypothetical jumps in σ. Introducing a *time lifting* Σ of the measure σ in the form

$$< \Sigma, \varphi > \equiv < \sigma, I[\varphi] >,$$

where

$$I[\varphi](t,x) = \int_0^t \varphi(z,x) \ \mathrm{d}z \text{ for any } \varphi \in L^1(0,T;C(\overline{\Omega})),$$

we check easily that Σ can be identified with a mapping $\Sigma \in L^\infty_{\rm weak}(0,T;\mathscr{M}^+(\overline{\Omega}))$, where

$$< \Sigma(\tau), \varphi > = \lim_{\delta\to 0+} < \sigma, \psi_\delta \varphi >,$$

with

$$\psi_\delta(t) = \begin{cases} 0 \text{ for } t \in [0,\tau), \\ \frac{1}{\delta}(t-\tau), \text{ for } t \in (\tau, \tau+\delta), \\ 1 \text{ for } t \geq \tau + \delta. \end{cases}$$

In particular, the measure Σ is well-defined for *any* $\tau \in [0,T]$, and the mapping $\tau \mapsto \Sigma_\varepsilon$ is non-increasing in the sense of measures. Here the subscript in $L^\infty_{\rm weak}$ means "weakly measurable".

The entropy balance (22) can be therefore rewritten as

$$\int_\Omega \Big[\rho s(\rho,\vartheta)(\tau,\cdot)\varphi(\tau,\cdot) - \rho_0 s(\rho_0,\vartheta_0)\varphi(0,\cdot)\Big] \ \mathrm{d}x \tag{25}$$

$$+ < \Sigma(\tau), \varphi(\tau,\cdot) > - < \Sigma(0), \varphi(0,\cdot) >$$

$$= \int_0^\tau < \Sigma, \partial_t \varphi > \ \mathrm{d}t + \int_0^\tau \int_\Omega \left(\rho s(\rho,\vartheta)\partial_t\varphi + \rho s(\rho,\vartheta)\mathbf{u}\cdot\nabla_x\varphi + \frac{\mathbf{q}}{\vartheta}\cdot\nabla_x\varphi\right) \mathrm{d}x \ \mathrm{d}t$$

$$+\varepsilon^2 \int_0^\tau \int_{\partial\Omega} \frac{\beta}{\vartheta}|\mathbf{u}|^2 \varphi \ \mathrm{dS}_x \ \mathrm{d}t,$$

for any $\varphi \in C^1([0,T]\times\overline{\Omega})$, where the mapping

$$\tau \mapsto \rho s(\rho,\vartheta)(\tau,\cdot) + \Sigma(\tau) \text{ is continuous with values in } \mathscr{M}(\overline{\Omega})$$

provided the space of measures $\mathscr{M}$ is endowed with the *weak* $-(*)$ topology.

2.2 Existence theory for the Navier-Stokes-Fourier system

The framework of weak solutions introduced in Section 2.1 is broad enough to develop an existence theory without any essential restrictions imposed on the initial data as well as the length of the time interval $(0,T)$. We start with a list of technical hypotheses imposed on the constitutive equations and the transport coefficients. The reader may consult [14, Chapter 3] for the physical background and further discussion.

The pressure p will be given by a general formula

$$p(\rho,\vartheta) = \vartheta^{5/2} P\left(\frac{\rho}{\vartheta^{3/2}}\right) + \frac{a}{3}\vartheta^4,\ a > 0, \tag{26}$$

where

$$P \in C^1[0,\infty),\ P(0) = 0,\ P'(Z) > 0 \text{ for all } Z \geq 0, \tag{27}$$

in particular, the compressibility $\partial_\rho p(\rho,\vartheta)$ is always positive. The former component in (26) represents the standard molecular pressure of a general monoatomic gas while the latter is a contribution due to thermal radiation.

In accordance with Gibbs' relation (18), the specific internal energy can be taken in the form

$$e(\rho,\vartheta) = \frac{3}{2}\vartheta\left(\frac{\vartheta^{3/2}}{\rho}\right) P\left(\frac{\rho}{\vartheta^{3/2}}\right) + a\frac{\vartheta^4}{\rho}, \tag{28}$$

where, in addition to (27), we assume that

$$0 < \frac{\frac{5}{3}P(Z) - P'(Z)Z}{Z} < c \text{ for all } Z > 0. \tag{29}$$

The awkwardly looking condition (29) has a clear physical meaning, namely the specific heat at constant volume - $\partial_\vartheta e(\rho,\vartheta)$ - is positive and bounded. In particular, (29) implies that the function $Z \mapsto P(Z)/Z^{5/3}$ is decreasing, and we assume

$$\lim_{Z\to\infty} \frac{P(Z)}{Z^{5/3}} = P_\infty > 0. \tag{30}$$

We remark that the molecular pressure $\vartheta^{5/2}P(\rho/\vartheta^{3/2})$ coincides with the standard perfect gas law $\mathrm{R}\vartheta\rho$ as long as $P(Z) \approx \mathrm{R}Z$, see Eliezer, Ghatak, and Hora [11] and [14, Chapter 1].

In addition to the previous hypotheses, we suppose that the transport coefficients $\mu = \mu(\vartheta)$, $\eta = \eta(\vartheta)$, and $\kappa = \kappa(\vartheta)$ are continuously differentiable functions of $\vartheta \in [0,\infty)$ such that

$$0 < \underline{\mu}(1+\vartheta) \leq \mu(\vartheta),\ |\mu'(\vartheta)| \leq \mu_1 \text{ for all } \vartheta \geq 0, \tag{31}$$

$$0 \leq \eta(\vartheta) \leq \overline{\eta}(1+\vartheta) \text{ for all } \vartheta \geq 0, \tag{32}$$

and

$$0 < \underline{\kappa}(1+\vartheta^3) \leq \kappa(\vartheta) \leq \overline{\kappa}(1+\vartheta^3) \text{ for all } \vartheta \geq 0. \tag{33}$$

We report the following result (see [14, Chapter 3, Theorem 3.1]):

Theorem 2.1 *Assume that $\Omega \subset R^3$ is a bounded domain of class $C^{2+\nu}$. Let $\varepsilon > 0$ $\beta > 0$ be given, let the initial data satisfy*

$$\rho_0 \in L^\infty(\Omega), \rho_0 > 0,\ \vartheta_0 \in L^\infty(\Omega), \vartheta_0 > 0,\ \mathbf{u}_0 \in L^\infty(\Omega; R^3),$$

and let $G \in W^{1,\infty}(\Omega)$. Suppose that the thermodynamic functions p, e, and s satisfy Gibbs' equation (18), together with the structural hypotheses (26 - 30), and the transport coefficients comply with (31 - 33).

Then the Navier-Stokes-Fourier system possesses a weak solution ρ, ϑ, $\mathbf{u}$ on the set $(0,T)\times\Omega$ in the sense specified in Section 2.1.

Remark 2.1 *As a matter of fact, the existence theorem [14, Chapter 3, Theorem 3.1] is proved for $\beta = 0$, however, the case $\beta > 0$ requires only straightforward modifications.*

Remark 2.2 *The weak solution, the existence of which is claimed in Theorem 2.1, satisfies $\rho \geq 0$, $\vartheta > 0$ a.a. in $(0,T)\times\Omega$. In addition, the weak solutions can be constructed to satisfy the equation of continuity (11) in the sense of renormalized solutions introduced by DiPerna and Lions [9]. Other regularity properties of the weak solutions are discussed in [14, Chapter 3, Section 3.8].*

Remark 2.3 *The hypotheses imposed on the initial data in Theorem 2.1 are not optimal. As a matter of fact, it is enough to assume that the initial energy and entropy of the system is finite. see [14, Chapter 3]. Similarly, the hypotheses imposed on the structural properties of thermodynamic functions as well as the transport coefficients may be considerably relaxed, see [14, Chapter 3].*

2.3 Uniform bounds and stability with respect to the singular parameter

Our goal is to identify the Oberbeck-Boussinesq approximation (1 - 4) with the asymptotic limit for $\varepsilon \to 0$ of the scaled Navier-Stokes-Fourier system (11 - 14). Moreover, we want the limit system to be defined on an exterior (unbounded) domain $\Omega \subset R^3$. To this end, we consider the scaled Navier-Stokes-Fourier system on a family of (bounded) domains

$$\Omega_\varepsilon = \Omega \cap \{x \in R \mid |x| < \frac{1}{\varepsilon^r}\},\ r > 1, \tag{34}$$

supplemented, for simplicity, with the complete slip boundary condition (Navier's slip with $\beta = 0$),

$$\mathbf{u} \cdot \mathbf{n}|_{\partial\Omega_\varepsilon} = 0, \ [\mathbb{S}\mathbf{n}] \times \mathbf{n}|_{\partial\Omega_\varepsilon} = 0, \ \mathbf{q} \cdot \mathbf{n}|_{\partial\Omega_\varepsilon} = 0, \tag{35}$$

cf. (19).

Thus, at least formally, $\Omega_\varepsilon \to \Omega$ as $\varepsilon \to 0$. As we shall see, the major problem in the limit passage is filtering the acoustic waves represented by the gradient component of the velocity field. Since the speed of sound in the fluid is proportional to $1/\varepsilon$, hypothesis (34) ensures that the outer boundary of Ω_ε becomes irrelevant, at least for what concerns the behavior of acoustic waves on compact subsets of the physical space, and, accordingly, we may use the dispersive phenomena to eliminate the presence of acoustic waves in the asymptotic limit.

2.3.1 Uniform bounds based on energy dissipation

Let $\{\rho_\varepsilon, \vartheta_\varepsilon, \mathbf{u}_\varepsilon\}$ be a weak solution of the scaled Navier-Stokes-Fourier system on the set $(0,T) \times \Omega_\varepsilon$ in the sense of Section 2.1. We start by deriving *uniform* bounds independent of $\varepsilon \to 0$. The key quantity is the *ballistic free energy* introduced by Ericksen [12, Chapter 1.3]:

$$H(\rho, \vartheta) = \rho e(\rho, \vartheta) - \overline{\vartheta} \rho s(\rho, \vartheta),$$

where $\overline{\vartheta}$ is a positive constant. It is easy to check that

$$\frac{\partial^2 H(\rho, \overline{\vartheta})}{\partial \rho^2} = \frac{1}{\rho} \frac{\partial p(\rho, \overline{\vartheta})}{\partial \rho}, \ \frac{\partial H(\rho, \vartheta)}{\partial \vartheta} = \frac{\rho}{\vartheta} (\vartheta - \overline{\vartheta}) \frac{\partial e(\rho, \vartheta)}{\partial \vartheta},$$

in particular, hypotheses (27), (29) imply that

$$\left[\begin{array}{c} \rho \mapsto H(\rho, \overline{\vartheta}) \text{ is strictly convex,} \\ \\ \vartheta \mapsto H(\rho, \vartheta) \text{ is strictly decreasing for } \vartheta < \overline{\vartheta} \\ \\ \text{and strictly increasing for } \vartheta > \overline{\vartheta}. \end{array} \right]$$

Conditions (27), (29) guarantee *thermodynamic stability* of the system, see Bechtel, Rooney, and Forest [3]. As we will see, they are crucial to control the norm of solutions to the scaled system.

In the so-called *static* density and temperature distribution for the scaled Navier-Stokes-Fourier system, the temperature equals a positive constant $\overline{\vartheta}$ while the density $\tilde{\rho}_\varepsilon$ satisfies

$$\nabla_x p(\tilde{\rho}_\varepsilon, \overline{\vartheta}) = \varepsilon \tilde{\rho}_\varepsilon \nabla_x G.$$

It is easy to check that

$$\frac{\partial H(\tilde{\rho}_\varepsilon, \overline{\vartheta})}{\partial \rho} = \varepsilon G + \text{const in } \Omega_\varepsilon \tag{36}$$

provided $\tilde{\rho}_\varepsilon$ is strictly positive in Ω_ε.

Taking advantage of (36), we may combine total energy balance (24) with the entropy equation (22) to obtain

$$\begin{aligned}\int_{\Omega_\varepsilon} \left(\frac{1}{2}\rho_\varepsilon |\mathbf{u}_\varepsilon|^2 + \frac{1}{\varepsilon^2} \left(H(\rho_\varepsilon, \vartheta_\varepsilon) - \frac{\partial H(\tilde{\rho}_\varepsilon, \overline{\vartheta})}{\partial \rho} (\rho_\varepsilon - \tilde{\rho}_\varepsilon) - H(\tilde{\rho}_\varepsilon, \overline{\vartheta}) \right) \right) (\tau, \cdot) \ \mathrm{d}x \\ + \frac{\overline{\vartheta}}{\varepsilon^2} \sigma_\varepsilon [[0,\tau] \times \overline{\Omega}] = \int_{\Omega_\varepsilon} \left(\frac{1}{2} \rho_{0,\varepsilon} |\mathbf{u}_{0,\varepsilon}|^2 + \frac{1}{\varepsilon^2} \left(H(\rho_{0,\varepsilon}, \vartheta_{0,\varepsilon}) \right.\right. \\ \left.\left. - \frac{\partial H(\tilde{\rho}_\varepsilon, \overline{\vartheta})}{\partial \rho} (\rho_{0,\varepsilon} - \tilde{\rho}_\varepsilon) - H(\tilde{\rho}_\varepsilon, \overline{\vartheta}) \right) \right) \ \mathrm{d}x\end{aligned} \tag{37}$$

for a.a. $\tau \in (0,T)$ provided we fix the static density so that

$$\int_{\Omega_\varepsilon} \rho_\varepsilon(\tau, \cdot) \ \mathrm{d}x = \int_{\Omega_\varepsilon} \rho_{0,\varepsilon} \ \mathrm{d}x = \int_{\Omega_\varepsilon} \tilde{\rho}_\varepsilon \ \mathrm{d}x,$$

meaning the total mass of the fluid contained in Ω_ε coincides with the total mass of the static distribution $\tilde{\rho}_\varepsilon$.

As a matter of fact, it is more convenient to consider a static solution $\tilde{\rho}_\varepsilon$ defined on the whole space R^3, specifically,

$$\nabla_x p(\tilde{\rho}_\varepsilon, \overline{\vartheta}) = \varepsilon \tilde{\rho}_\varepsilon \nabla_x G \text{ in } R^3,$$

satisfying

$$\lim_{|x| \to \infty} \tilde{\rho}_\varepsilon(x) = \overline{\rho}.$$

Consequently, we have

$$\tilde{\rho}_\varepsilon - \overline{\rho} = \frac{\varepsilon}{\mathscr{P}'(\overline{\rho})} G + \varepsilon^2 h_\varepsilon G, \ \mathscr{P}'(\rho) = \frac{1}{\rho} \partial_\rho p(\rho, \overline{\vartheta}), \tag{38}$$

with

$$\|h_\varepsilon\|_{L^\infty(R^3)} \leq c, \ |\nabla_x \tilde{\rho}_\varepsilon(x)| \leq \varepsilon c |\nabla_x G(x)| \text{ for } x \in R^3. \tag{39}$$

In order to exploit (37), the initial data must be chosen in such a way that the right-hand side of (37) remains bounded uniformly for $\varepsilon \to 0$. To this end, we take

$$\rho_{0,\varepsilon} = \tilde{\rho}_\varepsilon + \varepsilon \rho^{(1)}_{0,\varepsilon}, \ \vartheta_{0,\varepsilon} = \overline{\vartheta} + \varepsilon \vartheta^{(1)}_{0,\varepsilon}, \tag{40}$$

where

$$\|\rho^{(1)}_{0,\varepsilon}\|_{L^2 \cap L^\infty(\Omega_\varepsilon)} \leq c, \ \|\vartheta^{(1)}_{0,\varepsilon}\|_{L^2 \cap L^\infty(\Omega_\varepsilon)} \leq c, \tag{41}$$

$$\int_{\Omega_\varepsilon} \rho^{(1)}_{0,\varepsilon} \ \mathrm{d}x = \int_{\Omega_\varepsilon} \vartheta^{(1)}_{0,\varepsilon} \ \mathrm{d}x = 0; \tag{42}$$

and

$$\|\mathbf{u}_{0,\varepsilon}\|_{L^2\cap L^\infty(\Omega_\varepsilon;R^3)} \leq c, \tag{43}$$

where all constants are independent of ε.

By virtue of (27), (29), the ballistic free energy possesses remarkable coercivity properties, specifically,

$$H(\rho,\vartheta) - \frac{\partial H(\tilde{\rho},\overline{\vartheta})}{\partial \rho}(\rho-\tilde{\rho}) - H(\tilde{\rho},\overline{\vartheta}) \tag{44}$$

$$\geq c(K)\Big(|\rho-\tilde{\rho}|^2 + |\vartheta-\overline{\vartheta}|^2\Big) \text{ for all } (\rho,\vartheta)\in K,$$

and

$$H(\rho,\vartheta) - \frac{\partial H(\tilde{\rho},\overline{\vartheta})}{\partial \rho}(\rho-\tilde{\rho}) - H(\tilde{\rho},\overline{\vartheta}) \tag{45}$$

$$\geq c(K)\Big(1+\rho|e(\rho,\vartheta)| + \rho|s(\rho,\vartheta)|\Big) \text{ for all } (\rho,\vartheta)\in (0,\infty)^2\setminus K,$$

for any compact $K\subset(0,\infty)^2$ containing $(\tilde{\rho},\overline{\vartheta})$, see [14, Chapter 5, Lemma 5.1]. Consequently, introducing the decomposition

$$h = h_{\rm ess} + h_{\rm res},\ h_{\rm ess} = \chi(\rho_\varepsilon,\vartheta_\varepsilon)h,\ h_{\rm res} = (1-\chi(\rho_\varepsilon,\vartheta_\varepsilon))h, \tag{46}$$

for any measurable function h, where $\chi\in C^\infty_c((0,\infty)^2)$ such that

$$0\leq\chi\leq 1,\ \chi\equiv 1 \text{ on the rectangle } [\overline{\vartheta}/2, 2\overline{\vartheta}]\times[\overline{\rho}/2, 2\overline{\rho}],$$

we deduce from (37) the following list of uniform bounds:

$$\operatorname{ess}\sup_{t\in(0,T)} \int_{\Omega_\varepsilon} \rho_\varepsilon|\mathbf{u}_\varepsilon|^2(t,\cdot)\ {\rm d}x \leq c, \tag{47}$$

and, by virtue of (44),

$$\operatorname{ess}\sup_{t\in(0,T)} \left\| \left[\frac{\rho_\varepsilon-\tilde{\rho}_\varepsilon}{\varepsilon}\right]_{\rm ess}(t,\cdot)\right\|_{L^2(\Omega_\varepsilon)} \leq c, \tag{48}$$

$$\operatorname{ess}\sup_{t\in(0,T)} \left\| \left[\frac{\vartheta_\varepsilon-\overline{\vartheta}}{\varepsilon}\right]_{\rm ess}(t,\cdot)\right\|_{L^2(\Omega_\varepsilon)} \leq c, \tag{49}$$

where we have used (38), (39) and the fact that the static density $\tilde{\rho}_\varepsilon$ remains uniformly close to the constant $\overline{\rho}$ as soon as ε is small enough.

Furthermore, by virtue of (45) and the hypotheses (26 - 30), it follows that

$$\operatorname{ess}\sup_{t\in[0,T]} \int_{\Omega_\varepsilon} [\rho_\varepsilon]^{5/3}_{\rm res}(t,\cdot)\ {\rm d}x \leq \varepsilon^2 c, \tag{50}$$

$$\operatorname*{ess\,sup}_{t\in[0,T]} \int_{\Omega_\varepsilon} [\vartheta_\varepsilon]^4_{\rm res}(t,\cdot)\ {\rm d}x \leq \varepsilon^2 c, \tag{51}$$

and

$$\operatorname*{ess\,sup}_{t\in[0,T]} \int_{\Omega_\varepsilon} \Big(|[\rho_\varepsilon e(\rho_\varepsilon,\vartheta_\varepsilon)]_{\rm res}| + |[p(\rho_\varepsilon,\vartheta_\varepsilon)]_{\rm res}| + |[\rho_\varepsilon s(\rho_\varepsilon,\vartheta_\varepsilon)]_{\rm res}| \Big)\ {\rm d}x \leq c. \tag{52}$$

Finally, by the same token, the measure of the "residual" set is also small, specifically,

$$\operatorname*{ess\,sup}_{t\in[0,T]} \int_{\Omega_\varepsilon} 1_{\rm res}(t,\cdot)\ {\rm d}x \leq \varepsilon^2 c, \tag{53}$$

where all the constants "c" are independent of ε. It is remarkable that the measure of the "residual" set remains small although the measure of Ω_ε tends to infinity as $\varepsilon \to 0$.

Going back to (37) we get

$$\|\sigma_\varepsilon\|_{\mathscr{M}^+([0,T]\times\overline{\Omega}_\varepsilon)} \leq \varepsilon^2 c; \tag{54}$$

whence, in view of (23) and hypotheses (31 - 33),

$$\int_0^T \left\| \nabla_x \mathbf{u}_\varepsilon + \nabla_x^t \mathbf{u}_\varepsilon - \frac{2}{3} {\rm div}_x \mathbf{u}_\varepsilon \mathbb{I} \right\|^2_{L^2(\Omega_\varepsilon;R^{3\times 3})} {\rm d}t \leq c, \tag{55}$$

and

$$\int_0^T \left\| \nabla_x \frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon} \right\|^2_{L^2(\Omega_\varepsilon;R^3)} {\rm d}t + \int_0^T \left\| \nabla_x \frac{\log(\vartheta_\varepsilon) - \log(\overline{\vartheta})}{\varepsilon} \right\|^2_{L^2(\Omega_\varepsilon;R^3)} {\rm d}t \leq c. \tag{56}$$

Moreover, since the measure of the residual set is small (see (53)), we can apply Poincaré's inequality to conclude that

$$\int_0^T \left\| \frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon} \right\|^2_{W^{1,2}(\Omega_\varepsilon)} {\rm d}t + \int_0^T \left\| \frac{\log(\vartheta_\varepsilon) - \log(\overline{\vartheta})}{\varepsilon} \right\|^2_{W^{1,2}(\Omega_\varepsilon)} {\rm d}t \leq c. \tag{57}$$

A similar argument, based on a generalized version of Korn's inequality due to Reshetnyak [28] (see also [14, Chapter 10, Theorem 10.16]), can be applied to (47), (48) to conclude that

$$\int_0^T \|\mathbf{u}_\varepsilon\|^2_{W^{1,2}(\Omega_\varepsilon;R^3)}\ {\rm d}t \leq c. \tag{58}$$

Here we have also used the fact that $[\rho]_{\rm ess}$ is bounded below away from zero on a set, the complement of which is of small measure (see (53)).

2.4 Convergence to the limit system - part I

Our goal now is to exploit the uniform bounds obtained in the previous part to pass to the limit in the sequence $\{\rho_\varepsilon, \mathbf{u}_\varepsilon, \vartheta_\varepsilon\}_{\varepsilon>0}$ for $\varepsilon \to 0$. To begin, we observe that (48), (50) yield

$$\operatorname*{ess\ sup}_{t\in(0,T)} \|\rho_\varepsilon(t,\cdot) - \tilde{\rho}_\varepsilon\|_{(L^2\oplus L^{5/3})(\Omega_\varepsilon)} \to 0 \text{ as } \varepsilon \to 0. \tag{59}$$

In particular, by virtue of (38),

$$\operatorname*{ess\ sup}_{t\in(0,T)} \|\rho_\varepsilon(t,\cdot) - \overline{\rho}\|_{L^{5/3}(K)} \to 0 \text{ as } \varepsilon \to 0 \text{ for any compact } K \subset \Omega. \tag{60}$$

Thus the fluid density becomes constant provided the Mach number tends to zero.

Similarly, relations (49), (51), and (53) yield

$$\operatorname*{ess\ sup}_{t\in(0,T)} \|\vartheta_\varepsilon(t,\cdot) - \overline{\vartheta}\|_{L^2(\Omega_\varepsilon)} \to 0 \text{ as } \varepsilon \to 0. \tag{61}$$

Next, in order to control the temperature deviations from the equilibrium state $\overline{\vartheta}$, we use (57), (58) to deduce that

$$\Theta_\varepsilon \equiv \frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon} \to \Theta \text{ weakly in } L^2(0,T;W^{1,2}(\Omega)). \tag{62}$$

Moreover, by the same token,

$$\mathbf{u}_\varepsilon \to \mathbf{U} \text{ weakly in } L^2(0,T;W^{1,2}(\Omega;R^3)), \tag{63}$$

passing to subsequences if necessary. Here, we have assumed that ϑ_ε, $\mathbf{u}_\varepsilon$ were extended to the whole domain Ω.

A short inspection of the scaled Navier-Stokes-Fourier system (11 - 13) reveals the most difficult step, namely we need to show strong (pointwise) convergence of the velocity in order to control the convective term. More specifically, we need to show that

$$\mathbf{u}_\varepsilon \to \mathbf{U} \text{ (strongly) in } L^2((0,T)\times K;R^3) \text{ for any compact } K \subset \Omega. \tag{64}$$

As a matter of fact, it is enough to prove that

$$\rho_\varepsilon \mathbf{u}_\varepsilon \to \overline{\rho}\mathbf{U} \text{ in } L^2(0,T;W^{-1,2}(K)). \tag{65}$$

Indeed, for any $\varphi \in C_c^\infty(\Omega)$, we have

$$\overline{\rho}\int_0^T \int_\Omega \varphi|\mathbf{u}_\varepsilon|^2 \, \mathrm{d}x\, \mathrm{d}t = \int_0^T \int_\Omega \varphi(\overline{\rho} - \rho_\varepsilon)|\mathbf{u}_\varepsilon|^2 \, \mathrm{d}x\mathrm{d}t + \int_0^T \int_\Omega \varphi\rho_\varepsilon \mathbf{u}_\varepsilon \cdot \mathbf{u}_\varepsilon \, \mathrm{d}x\, \mathrm{d}t,$$

where, by virtue of the previous estimates and the embedding relation $W^{1,2}(\Omega) \hookrightarrow L^6(\Omega)$,

$$\int_0^T \int_\Omega \varphi(\overline{\rho} - \rho_\varepsilon)|\mathbf{u}_\varepsilon|^2 \ \mathrm{d}x \ \mathrm{d}t \to 0,$$

while, as a consequence of (63), (65),

$$\int_0^T \int_\Omega \varphi \rho_\varepsilon \mathbf{u}_\varepsilon \cdot \mathbf{u}_\varepsilon \ \mathrm{d}x \ \mathrm{d}t \to \overline{\rho} \int_0^T \int_\Omega \varphi |\mathbf{U}|^2 \ \mathrm{d}x \ \mathrm{d}t.$$

The final observation is that for (65) to hold it is enough to show that

$$\left\{ t \mapsto \int_\Omega (\rho_\varepsilon \mathbf{u}_\varepsilon)(t,\cdot) \cdot \varphi \ \mathrm{d}x \right\} \text{ is precompact in } L^2(0,T) \tag{66}$$

for any fixed $\varphi \in C_c^\infty(\Omega)$ since, as a consequence of (47), (48), and (50),

$$\operatorname*{ess\,sup}_{t\in(0,T)} \|\rho_\varepsilon \mathbf{u}_\varepsilon\|_{L^{5/4}(K;R^3)} \le c(K) \text{ for any compact } K \subset \Omega$$

and the embedding $L^{5/4}(K) \hookrightarrow W^{-1,2}(K)$ is compact. Accordingly, we fix $\varphi \in C_c^\infty(\Omega)$ for the remaining part of this section and focus on proving (66).

2.5 Acoustic equation

As already pointed out, our main goal is to show (66) for any fixed $\varphi \in C_c^\infty(\Omega)$. To this end, we rewrite the Navier-Stokes-Fourier system in the form

$$\varepsilon \partial_t R_\varepsilon + \omega \mathrm{div}_x \mathbf{V}_\varepsilon = \varepsilon f_\varepsilon^1, \tag{67}$$

$$\varepsilon \partial_t \mathbf{V}_\varepsilon + \nabla_x R_\varepsilon = \varepsilon \mathbf{f}_\varepsilon^2, \tag{68}$$

where we have set

$$R_\varepsilon = A\left(\frac{\rho_\varepsilon - \overline{\rho}}{\varepsilon}\right) + B\left(\frac{\rho_\varepsilon s(\rho_\varepsilon, \vartheta_\varepsilon) - \overline{\rho} s(\overline{\rho}, \overline{\vartheta})}{\varepsilon}\right) - \overline{\rho} G, \ \mathbf{V}_\varepsilon = \rho_\varepsilon \mathbf{u}_\varepsilon,$$

$$f_\varepsilon^1 = B\left[\mathrm{div}_x \left(\rho_\varepsilon \frac{s(\overline{\rho}, \overline{\vartheta}) - s(\rho_\varepsilon, \vartheta_\varepsilon)}{\varepsilon} \mathbf{u}_\varepsilon\right) + \mathrm{div}_x \left(\frac{\kappa(\vartheta_\varepsilon)}{\vartheta_\varepsilon} \frac{\nabla_x \vartheta_\varepsilon}{\varepsilon}\right) + \frac{1}{\varepsilon}\sigma_\varepsilon\right],$$

and

$$\mathbf{f}_\varepsilon^2 = \frac{1}{\varepsilon} \nabla_x \left[A\left(\frac{\rho_\varepsilon - \overline{\rho}}{\varepsilon}\right) + B\left(\frac{\rho_\varepsilon s(\rho_\varepsilon, \vartheta_\varepsilon) - \overline{\rho} s(\overline{\rho}, \overline{\vartheta})}{\varepsilon}\right) - \left(\frac{p(\rho_\varepsilon, \vartheta_\varepsilon) - p(\overline{\rho}, \overline{\vartheta})}{\varepsilon}\right)\right]$$

$$-\mathrm{div}_x(\rho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon) + \mathrm{div}_x \mathbb{S}_\varepsilon + \frac{\rho_\varepsilon - \overline{\rho}}{\varepsilon} \nabla_x G,$$

and where the constants A, B, ω are chosen so that

$$B\overline{\rho}\partial_\vartheta s(\overline{\rho},\overline{\vartheta}) = \partial_\vartheta p(\overline{\rho},\overline{\vartheta}),\ A + B\partial_\rho(\rho s)(\overline{\rho},\overline{\vartheta}) = p_\rho(\overline{\rho},\overline{\vartheta}),$$

and

$$\omega = p_\rho(\overline{\rho},\overline{\vartheta}) + \frac{|p_\vartheta(\overline{\rho},\overline{\vartheta})|^2}{\overline{\rho}^2 s_\vartheta(\overline{\rho},\overline{\vartheta})} > 0.$$

System (67), (68) is usually termed *acoustic equation*, or, *Lighthill's acoustic analogy*, see Lighthill [19], [20].

The inevitable presence of the measure σ_ε in the forcing term f^1_ε may cause discontinuities (in time) in solutions of the system (67), (68); therefore it seems more convenient to use the time-lifting Σ_ε of the measure σ_ε introduced in Section 2.1. With the new variables

$$S_\varepsilon = A\left(\frac{\rho_\varepsilon - \overline{\rho}}{\varepsilon}\right) + B\left(\frac{\rho_\varepsilon s(\rho_\varepsilon,\vartheta_\varepsilon) - \overline{\rho} s(\overline{\rho},\overline{\vartheta})}{\varepsilon}\right) - \overline{\rho} G + \frac{B}{\varepsilon}\Sigma_\varepsilon,\ \mathbf{V}_\varepsilon = \rho_\varepsilon \mathbf{u}_\varepsilon, \quad (69)$$

we may write the acoustic equation in the form

$$\varepsilon\partial_t S_\varepsilon + \omega \mathrm{div}_x \mathbf{V}_\varepsilon = \varepsilon F^1_\varepsilon, \quad (70)$$

$$\varepsilon\partial_t \mathbf{V}_\varepsilon + \nabla_x S_\varepsilon = \varepsilon \mathbf{F}^2_\varepsilon, \quad (71)$$

with

$$F^1_\varepsilon = B\left[\mathrm{div}_x\left(\rho_\varepsilon \frac{s(\overline{\rho},\overline{\vartheta}) - s(\rho_\varepsilon,\vartheta_\varepsilon)}{\varepsilon}\mathbf{u}_\varepsilon\right) + \mathrm{div}_x\left(\frac{\kappa(\vartheta_\varepsilon)}{\vartheta_\varepsilon}\frac{\nabla_x\vartheta_\varepsilon}{\varepsilon}\right)\right], \quad (72)$$

and

$$\mathbf{F}^2_\varepsilon = \frac{1}{\varepsilon}\nabla_x\left[A\left(\frac{\rho_\varepsilon - \overline{\rho}}{\varepsilon}\right) + B\left(\frac{\rho_\varepsilon s(\rho_\varepsilon,\vartheta_\varepsilon) - \overline{\rho} s(\overline{\rho},\overline{\vartheta})}{\varepsilon}\right)\right. \quad (73)$$

$$\left. - \left(\frac{p(\rho_\varepsilon,\vartheta_\varepsilon) - p(\overline{\rho},\overline{\vartheta})}{\varepsilon}\right)\right] - \mathrm{div}_x(\rho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon) + \mathrm{div}_x \mathbb{S}_\varepsilon + \frac{\rho_\varepsilon - \overline{\rho}}{\varepsilon}\nabla_x G + \frac{B}{\varepsilon^2}\nabla_x \Sigma_\varepsilon,$$

supplemented with the homogeneous Neumann boundary condition

$$\mathbf{V}_\varepsilon \cdot \mathbf{n}|_{\partial\Omega_\varepsilon} = 0. \quad (74)$$

Of course, system (69 - 74) should be understood in the weak sense as specified in the following section.

2.5.1 Boundedness of the data in the acoustic equation

As suggested by the previous discussion, the system (70), (71) will describe the behavior of the velocity field or rather the momentum $\mathbf{V}_\varepsilon$, while the remaining quantities appearing F^1_ε, $\mathbf{F}^2_\varepsilon$ are given. Using the uniform bounds established in Section 2.3, we estimate the forcing terms as well as the initial data in the acoustic equation. To begin write, using the decomposition introduced by (46)

$$\frac{\rho_\varepsilon - \overline{\rho}}{\varepsilon} = \frac{\rho_\varepsilon - \tilde{\rho}_\varepsilon}{\varepsilon} + \frac{\tilde{\rho}_\varepsilon - \overline{\rho}}{\varepsilon} = \left[\frac{\rho_\varepsilon - \tilde{\rho}_\varepsilon}{\varepsilon}\right]_{\rm ess} + \left[\frac{\rho_\varepsilon - \tilde{\rho}_\varepsilon}{\varepsilon}\right]_{\rm res} + \frac{\tilde{\rho}_\varepsilon - \overline{\rho}}{\varepsilon},$$

where, in accordance with (48), (50), and (51), we have

$$\operatorname*{ess\ sup}_{t\in(0,T)} \left\| \left[\frac{\rho_\varepsilon - \tilde{\rho}_\varepsilon}{\varepsilon}\right]_{\rm ess} \right\|_{L^2(\Omega_\varepsilon)} \leq c,\ \operatorname*{ess\ sup}_{t\in(0,T)} \left\| \left[\frac{\rho_\varepsilon - \tilde{\rho}_\varepsilon}{\varepsilon}\right]_{\rm res} \right\|_{L^1(\Omega_\varepsilon)} \leq \varepsilon c, \tag{75}$$

and, moreover, using (38), (39) it follows that

$$\left\| \frac{\tilde{\rho}_\varepsilon - \overline{\rho}}{\varepsilon} \right\|_{(L^\infty \cap L^q)(R^3)} \leq c \text{ for any } q > 3,\ \left\| \nabla_x \left(\frac{\tilde{\rho}_\varepsilon - \overline{\rho}}{\varepsilon} \right) \right\|_{L^2(R^3;R^3)} \leq c, \tag{76}$$

The next step is to write

$$\frac{\rho_\varepsilon s(\rho_\varepsilon, \vartheta_\varepsilon) - \overline{\rho} s(\overline{\rho}, \overline{\vartheta})}{\varepsilon} = \frac{\rho_\varepsilon s(\rho_\varepsilon, \vartheta_\varepsilon) - \tilde{\rho}_\varepsilon s(\tilde{\rho}_\varepsilon, \overline{\vartheta})}{\varepsilon} + \frac{\tilde{\rho}_\varepsilon s(\tilde{\rho}_\varepsilon, \overline{\vartheta}) - \overline{\rho} s(\overline{\rho}, \overline{\vartheta})}{\varepsilon}$$
$$= \left[\frac{\rho_\varepsilon s(\rho_\varepsilon, \vartheta_\varepsilon) - \tilde{\rho}_\varepsilon s(\tilde{\rho}_\varepsilon, \overline{\vartheta})}{\varepsilon}\right]_{\rm ess} + \left[\frac{\rho_\varepsilon s(\rho_\varepsilon, \vartheta_\varepsilon) - \tilde{\rho}_\varepsilon s(\tilde{\rho}_\varepsilon, \overline{\vartheta})}{\varepsilon}\right]_{\rm res}$$
$$+ \frac{\tilde{\rho}_\varepsilon s(\tilde{\rho}_\varepsilon, \overline{\vartheta}) - \overline{\rho} s(\overline{\rho}, \overline{\vartheta})}{\varepsilon},$$

where, in accordance with the uniform bounds established in Section 2.3,

$$\operatorname*{ess\ sup}_{t\in(0,T)} \left\| \left[\frac{\rho_\varepsilon s(\rho_\varepsilon, \vartheta_\varepsilon) - \tilde{\rho}_\varepsilon s(\tilde{\rho}_\varepsilon, \overline{\vartheta})}{\varepsilon}\right]_{\rm ess} \right\|_{L^2(\Omega_\varepsilon)} \leq c,$$

$$\operatorname*{ess\ sup}_{t\in(0,T)} \left\| \left[\frac{\rho_\varepsilon s(\rho_\varepsilon, \vartheta_\varepsilon) - \tilde{\rho}_\varepsilon s(\tilde{\rho}_\varepsilon, \overline{\vartheta})}{\varepsilon}\right]_{\rm res} \right\|_{L^1(\Omega_\varepsilon)} \leq \varepsilon c,$$

and

$$\left\| \frac{\tilde{\rho}_\varepsilon s(\tilde{\rho}_\varepsilon, \overline{\vartheta}) - \overline{\rho} s(\overline{\rho}, \overline{\vartheta})}{\varepsilon} \right\|_{(L^\infty \cap L^q)(R^3)} \leq c \text{ for all } q > 3,$$

$$\left\| \nabla_x \left(\frac{\tilde{\rho}_\varepsilon s(\tilde{\rho}_\varepsilon, \overline{\vartheta}) - \overline{\rho} s(\overline{\rho}, \overline{\vartheta})}{\varepsilon} \right) \right\|_{L^2(R^3;R^3)} \leq c.$$

Furthermore, by virtue of (54),

$$\operatorname*{ess\ sup}_{t\in(0,T)} \left\| \frac{\Sigma_\varepsilon(t,\cdot)}{\varepsilon} \right\|_{\mathscr{M}^+(\overline{\Omega}_\varepsilon)} \leq \varepsilon c,$$

therefore we may write

$$S_\varepsilon(t) = S^1_\varepsilon(t) + S^2_\varepsilon(t) + S^3_\varepsilon,$$

with

$$\operatorname*{ess\ sup}_{t\in(0,T)} \|S^1_\varepsilon\|_{\mathscr{M}^+(\overline{\Omega}_\varepsilon)} \leq \varepsilon c,\ \operatorname*{ess\ sup}_{t\in(0,T)} \|S^2_\varepsilon\|_{L^2(\Omega_\varepsilon)} \leq c,\ \|S^3_\varepsilon\|_{D^{1,2}(R^3)} \leq c,$$

where the symbol $D^{1,2}$ denotes the homogeneous Sobolev space - a completion of compactly supported smooth functions with respect to the L^2-norm of their gradients.

Next, writing

$$\mathbf{V}_\varepsilon = [\rho_\varepsilon \mathbf{u}_\varepsilon]_{\rm ess} + [\rho_\varepsilon \mathbf{u}_\varepsilon]_{\rm res},$$

we have, in agreement with (47), (50), (53),

$$\operatorname*{ess\ sup}_{t\in(0,T)} \|[\rho_\varepsilon \mathbf{u}_\varepsilon]_{\rm ess}\|_{L^2(\Omega_\varepsilon;R^3)} \leq c, \operatorname*{ess\ sup}_{t\in(0,T)} \|[\rho_\varepsilon \mathbf{u}_\varepsilon]_{\rm res}\|_{L^1(\Omega_\varepsilon;R^3)} \leq \varepsilon c. \tag{77}$$

Other terms appearing in F^1_ε, $\mathbf{F}^2_\varepsilon$ can be treated in a similar manner. We focus only on the most complicated expression:

$$A\left(\frac{\rho_\varepsilon - \overline{\rho}}{\varepsilon^2}\right) + B\left(\frac{\rho_\varepsilon s(\rho_\varepsilon,\vartheta_\varepsilon) - \overline{\rho} s(\overline{\rho},\overline{\vartheta})}{\varepsilon^2}\right) - \left(\frac{p(\rho_\varepsilon,\vartheta_\varepsilon) - p(\overline{\rho},\overline{\vartheta})}{\varepsilon^2}\right)$$

$$= A\left(\frac{\rho_\varepsilon - \tilde{\rho}_\varepsilon}{\varepsilon^2}\right) + B\left(\frac{\rho_\varepsilon s(\rho_\varepsilon,\vartheta_\varepsilon) - \tilde{\rho}_\varepsilon s(\tilde{\rho}_\varepsilon,\overline{\vartheta})}{\varepsilon^2}\right) - \left(\frac{p(\rho_\varepsilon,\vartheta_\varepsilon) - p(\tilde{\rho}_\varepsilon,\overline{\vartheta})}{\varepsilon^2}\right)$$

$$+ A\left(\frac{\tilde{\rho}_\varepsilon - \overline{\rho}}{\varepsilon^2}\right) + B\left(\frac{\tilde{\rho}_\varepsilon s(\tilde{\rho}_\varepsilon,\overline{\vartheta}) - \overline{\rho} s(\overline{\rho},\overline{\vartheta})}{\varepsilon^2}\right) - \left(\frac{p(\tilde{\rho}_\varepsilon,\overline{\vartheta}) - p(\overline{\rho},\overline{\vartheta})}{\varepsilon^2}\right)$$

Seeing that

$$A + B\partial_\rho(\rho s)(\overline{\rho},\overline{\vartheta}) - \partial_\rho p(\overline{\rho},\overline{\vartheta}) = 0,$$

the quantity

$$A\left(\frac{\rho_\varepsilon - \tilde{\rho}_\varepsilon}{\varepsilon^2}\right) + B\left(\frac{\rho_\varepsilon s(\rho_\varepsilon,\vartheta_\varepsilon) - \tilde{\rho}_\varepsilon s(\tilde{\rho}_\varepsilon,\overline{\vartheta})}{\varepsilon^2}\right) - \left(\frac{p(\rho_\varepsilon,\vartheta_\varepsilon) - p(\tilde{\rho}_\varepsilon,\overline{\vartheta})}{\varepsilon^2}\right)$$

contains only quadratic terms proportional to $\rho_\varepsilon - \tilde{\rho}_\varepsilon$, $\vartheta - \overline{\vartheta}$ and as such may be estimated in terms of (48 - 53). Similarly,

$$\left\| A\left(\frac{\rho_\varepsilon - \tilde{\rho}_\varepsilon}{\varepsilon^2}\right) + B\left(\frac{\rho_\varepsilon s(\rho_\varepsilon,\vartheta_\varepsilon) - \tilde{\rho}_\varepsilon s(\tilde{\rho}_\varepsilon,\overline{\vartheta})}{\varepsilon^2}\right)\right.$$

$$-\left(\frac{p(\rho_\varepsilon,\vartheta_\varepsilon)-p(\tilde\rho_\varepsilon,\overline{\vartheta})}{\varepsilon^2}\right)\Big\|_{(L^\infty\cap L^q)(R^3)}\leq c \text{ for all } q>3/2.$$

Summing up the previous estimates we may write down a weak formulation of the acoustic equation in the form:

$$\varepsilon\int_0^T <S_\varepsilon(t,\cdot),\partial_t\varphi> \ \mathrm{d}t+\omega\int_0^T\int_{\Omega_\varepsilon}\mathbf{V}_\varepsilon\cdot\nabla_x\varphi\ \mathrm{d}x\ \mathrm{d}t \tag{78}$$

$$=-\varepsilon<S_{0,\varepsilon},\varphi(0,\cdot)>+\varepsilon\int_0^T\int_{\Omega_\varepsilon}\Big(\mathbf{H}^1_\varepsilon\cdot\nabla_x\varphi+\mathbf{H}^2_\varepsilon\cdot\nabla_x\varphi\Big)\ \mathrm{d}x\ \mathrm{d}t,$$

for any $\varphi\in C^1_c([0,T)\times\overline{\Omega}_\varepsilon)$,

$$\varepsilon\int_0^T\int_{\Omega_\varepsilon}\mathbf{V}_\varepsilon\cdot\partial_t\varphi\ \mathrm{d}x\ \mathrm{d}t+\int_0^T <S_\varepsilon(t,\cdot),\mathrm{div}_x\varphi> \ \mathrm{d}t \tag{79}$$

$$=-\varepsilon\int_{\Omega_\varepsilon}\mathbf{V}_{0,\varepsilon}\cdot\varphi(0,\cdot)\ \mathrm{d}x+\varepsilon\int_0^T <\mathbb{G}^1_\varepsilon(t,\cdot),\nabla_x\varphi> \ \mathrm{d}t$$

$$+\varepsilon\int_0^T\int_\Omega\mathbb{G}^2_\varepsilon:\nabla_x\varphi\ \mathrm{d}x\ \mathrm{d}t+\varepsilon\int_0^T\int_\Omega\mathbf{G}^3_\varepsilon\cdot\varphi\ \mathrm{d}x\ \mathrm{d}t$$

for any $\varphi\in C^1_c([0,T)\times\overline{\Omega}_\varepsilon;R^3)$, $\varphi\cdot\mathbf{n}|_{\partial\Omega}=0$, where

$$S_\varepsilon=S^1_\varepsilon+S^2_\varepsilon+S^{1,2}_\varepsilon,$$

$$\operatorname*{ess\ sup}_{t\in(0,T)}\|S^1_\varepsilon(t,\cdot)\|_{\mathscr{M}^1(\overline{\Omega}_\varepsilon)}\leq\varepsilon c,\ \operatorname*{ess\ sup}_{t\in(0,T)}\|S^2_\varepsilon(t,\cdot)\|_{L^2(\overline{\Omega}_\varepsilon)}+\|S^{1,2}_\varepsilon\|_{D^{1,2}(R^3)}\leq c, \tag{80}$$

$$S_{0,\varepsilon}=S^1_{0,\varepsilon}+S^2_{0,\varepsilon}+S^{1,2}_\varepsilon,$$

$$\|S^1_{0,\varepsilon}\|_{\mathscr{M}^1(\overline{\Omega}_\varepsilon)}\leq\varepsilon c,\ \|S^2_{0,\varepsilon}\|_{L^2(\Omega_\varepsilon)}\leq c, \tag{81}$$

and, moreover,

$$S_\varepsilon\in C_{\mathrm{weak}-(*)}([0,T];\mathscr{M}^+(\overline{\Omega}_\varepsilon)).$$

Furthermore,

$$\mathbf{V}_\varepsilon=\mathbf{V}^1_\varepsilon+\mathbf{V}^2_\varepsilon,$$

$$\operatorname*{ess\ sup}_{t\in(0,T)}\|\mathbf{V}^1_\varepsilon\|_{L^1(\Omega_\varepsilon;R^3)}\leq\varepsilon c,\ \operatorname*{ess\ sup}_{t\in(0,T)}\|\mathbf{V}^2_\varepsilon\|_{L^2(\Omega_\varepsilon;R^3)}\leq c, \tag{82}$$

$$\|\mathbf{V}_{0,\varepsilon}\|_{(L^\infty\cap L^2)(\Omega_\varepsilon;R^3)}\leq c, \tag{83}$$

and

$$\mathbf{V}_\varepsilon\in C_{\mathrm{weak}}([0,T];L^1(\Omega_\varepsilon)).$$

Finally,

$$\int_0^T\Big(\|\mathbf{H}^1_\varepsilon\|^2_{L^1(\Omega_\varepsilon;R^3)}+\|\mathbf{H}^2_\varepsilon\|^2_{L^2(\Omega_\varepsilon;R^3)}\Big)\ \mathrm{d}t\leq c, \tag{84}$$

$$\int_0^T \left(\|\mathbb{G}^1_\varepsilon\|^2_{\mathscr{M}^+(\Omega_\varepsilon;R^{3\times 3})} + \|\mathbb{G}^2_\varepsilon\|^2_{L^2(\Omega_\varepsilon;R^{3\times 3})} \right) \mathrm{d}t \leq c, \tag{85}$$

and

$$\operatorname*{ess\,sup}_{t\in(0,T)} \|\mathbf{G}^3_\varepsilon(t,\cdot)\|_{L^{5/3}(R^3;R^3)} \leq c, \tag{86}$$

where all constants are independent of ε.

2.5.2 Reduction to smooth solutions

With the notation introduced in the previous section, the desired relation (66) reads:

$$\left\{ t \mapsto \int_\Omega \mathbf{V}_\varepsilon(t,\cdot)\cdot\varphi \, \mathrm{d}x \right\} \text{ is precompact in } L^2(0,T) \text{ for any } \varphi \in C^\infty_c(\Omega;R^3). \tag{87}$$

In order to see (87), it is more convenient to deal with the classical (smooth) solutions of acoustic equation (78), (79). Since $\varphi \in C^\infty_c(\Omega;R^3)$ is fixed, the idea is to replace the data in (78), (79) by smooth ones in such a way that the resulting smooth solution of (78), (79) is close to $\mathbf{V}_\varepsilon$ at least on the support of φ. To this end, fixing $\varepsilon > 0$ for a moment, we consider

$$S^i_{0,\varepsilon,\delta} \in C^\infty_c(\Omega_\varepsilon),\ i=1,2,3,\ \|S^1_{0,\varepsilon,\delta}\|_{L^1(\Omega)} + \|S^2_{0,\varepsilon,\delta}\|_{L^2(\Omega)} + \|S^3_{0,\varepsilon,\delta}\|_{D^{1,2}(R^3)} \leq c, \tag{88}$$

such that

$$S^1_{0,\varepsilon,\delta} \to S^1_{0,\varepsilon} \text{ weakly-(*) in } \mathscr{M}^+(\overline{\Omega}_\varepsilon),\ S^j_{0,\varepsilon,\delta} \to S^j_{0,\varepsilon} \text{ in } L^2(\Omega_\varepsilon),\ j=2,3, \text{ for } \delta \to 0.$$

Similarly, take

$$\mathbf{V}^i_{0,\varepsilon,\delta} \in C^\infty_c(\Omega_\varepsilon;R^3),\ i=1,2,\ \|\mathbf{V}^1_{0,\varepsilon,\delta}\|_{L^1(\Omega;R^3)} + \|\mathbf{V}^2_{0,\varepsilon,\delta}\|_{L^2(\Omega;R^3)} \leq c,$$

$$\mathbf{V}^1_{0,\varepsilon,\delta} \to \mathbf{V}_{0,\varepsilon} \text{ in } L^1(\Omega_\varepsilon;R^3),\ \mathbf{V}^2_{0,\varepsilon,\delta} \to \mathbf{V}_{0,\varepsilon} \text{ in } L^2(\Omega_\varepsilon;R^3) \text{ as } \delta \to 0, \tag{89}$$

and, finally,

$$\mathbf{H}^i_{\varepsilon,\delta} \in C^\infty_c((0,T)\times\Omega_\varepsilon;R^3),\ i=1,2,$$

$$\|\mathbf{H}^1_{\varepsilon,\delta}\|_{L^2(0,T;L^1(\Omega;R^3))} + \|\mathbf{H}^2_{\varepsilon,\delta}\|_{L^2(0,T;L^2(\Omega;R^3))} \leq c,$$

$$\mathbf{H}^1_{\varepsilon,\delta} \to \mathbf{H}^1_\varepsilon \text{ in } L^2(0,T;L^1(\Omega_\varepsilon;R^3)),\ \mathbf{H}^2_{\varepsilon,\delta} \to \mathbf{H}^2_\varepsilon \text{ in } L^2(0,T;L^2(\Omega_\varepsilon;R^3)) \text{ as } \delta \to 0 \tag{90}$$

with

$$\mathbb{G}^i_{\varepsilon,\delta} \in C^\infty_c((0,T)\times\Omega_\varepsilon;R^{3\times 3}),\ i=1,2,$$

$$\|\mathbb{G}^1_{\varepsilon,\delta}\|_{L^2(0,T;L^1(\Omega;R^{3\times 3}))} + \|\mathbb{G}^2_{\varepsilon,\delta}\|_{L^2(0,T;L^2(\Omega;R^{3\times 3}))} \leq c,$$

$$\mathbb{G}^1_{\varepsilon,\delta} \to \mathbb{G}^1_\varepsilon \text{ weakly-(*) in } L^2(0,T;\mathscr{M}^+(\Omega_\varepsilon;R^{3\times 3})), \tag{91}$$

$$\mathbb{G}^2_{\varepsilon,\delta} \to \mathbb{G}^2_\varepsilon \text{ in } L^2(0,T;L^2(\Omega_\varepsilon;R^{3\times 3})) \text{ as } \delta \to 0, \tag{92}$$

$$\mathbf{G}^3_{\varepsilon,\delta} \in C^\infty_c(\Omega_\varepsilon; R^3),\ \|\mathbf{G}^3_{\varepsilon,\delta}\|_{L^{5/3}(\Omega;R^3)} \leq c,$$
$$\mathbf{G}^3_{\varepsilon,\delta} \to \mathbf{G}^3_\varepsilon \text{ in } L^{5/3}((0,T)\times\Omega_\varepsilon;R^3) \text{ as } \delta \to 0. \tag{93}$$

Assume that $S_{\varepsilon,\delta}$, $\mathbf{V}_{\varepsilon,\delta}$ is the (unique) classical solution of the acoustic equation (78), (79), with the initial data and the forcing terms replaced by their $\delta-$ approximations specified in (88 - 93). Keeping (87) in mind we will show that

$$\operatorname{ess}\sup_{t\in(0,T)} \left| \int_{\Omega_\varepsilon} \Big(\mathbf{V}_{\varepsilon,\delta}(t,\cdot) - \mathbf{V}_\varepsilon(t,\cdot) \Big) \cdot \varphi \ \mathrm{d}x \right| \leq \varepsilon \text{ whenever } \delta \text{ is small enough}, \tag{94}$$

for any fixed $\varepsilon > 0$. Consequently, it follows from (94) that it is enough to show (87) for $\mathbf{V}_{\varepsilon,\delta(\varepsilon)}$. In other words, we may assume that all the quantities appearing in the acoustic equation are smooth and all the data is compactly supported in Ω_ε.

To see (94), we fix ε and write the function φ in terms of its *Helmholtz decomposition*,

$$\varphi = \mathbf{H}[\varphi] + \mathbf{H}^\perp[\varphi],$$

where

$$\mathbf{H}^\perp[\varphi] = \nabla_x \Psi,\ \Delta\Psi = \mathrm{div}_x \varphi \text{ in } \Omega_\varepsilon,\ \nabla_x\Psi\cdot\mathbf{n}|_{\partial\Omega_\varepsilon} = 0.$$

Taking $\mathbf{H}[\varphi]$ as a test function in (79) we easily deduce that

$$\sup_{t\in(0,T)} \left| \int_{\Omega_\varepsilon} \Big(\mathbf{V}_{\varepsilon,\delta}(t,\cdot) - \mathbf{V}_\varepsilon(t,\cdot) \Big) \cdot \mathbf{H}[\varphi] \ \mathrm{d}x \right| \leq \varepsilon \tag{95}$$

whenever $\delta = \delta(\varepsilon)$ is small enough.

Now, let $\{\psi_n\}_{n=0}^\infty$ be an orthonormal system of eigenfunctions of the Laplace operator in Ω_ε endowed with the homogeneous Neumann boundary conditions, specifically,

$$-\Delta\psi_n = \lambda_n\psi_n \text{ in } \Omega_\varepsilon,\ \nabla_x\psi_n\cdot\mathbf{n}|_{\partial\Omega_\varepsilon} = 0,\ n = 0,1,\ldots.$$

Taking the quantities $\phi(t)\psi_n(x)$, $\phi(t)\nabla_x\psi_n$, $\phi \in C^\infty_c(0,T)$ as test functions in (78), (79), respectively, we obtain a system of two ordinary differential equations:

$$\varepsilon\partial_t \int_{\Omega_\varepsilon} \mathbf{V}(t,\cdot)\cdot\nabla_x\psi_n \ \mathrm{d}x - \lambda_n < S(t,\cdot),\psi_n >= \varepsilon f^1_n,$$

$$\varepsilon\partial_t < S(t,\cdot),\psi_n > + \int_{\Omega_\varepsilon} \mathbf{V}(t,\cdot)\cdot\nabla_x\psi_n \ \mathrm{d}x = \varepsilon f^2_n$$

for the unknown functions of time:

$$\left\{ t \mapsto \int_{\Omega_\varepsilon} \mathbf{V}(t,\cdot)\cdot\nabla_x\psi_n \ \mathrm{d}x \right\},\ \{t \mapsto < S(t,\cdot),\psi_n >\},$$

where the initial data as well as the forcing terms f^1_n, f^2_ε can be evaluated in terms of the (ε,δ)-quantities. Consequently, we infer that for given $\varepsilon > 0$, $N > 0$, there exists $\delta = \delta(N,\varepsilon) > 0$ such that

$$\sup_{t\in(0,T)} \left| \int_{\Omega_\varepsilon} \Big(\mathbf{V}_{\varepsilon,\delta}(t,\cdot) - \mathbf{V}_\varepsilon(t,\cdot) \Big) \cdot \nabla_x \psi_n \ \mathrm{d}x \right| \leq \varepsilon \text{ whenever } \delta \leq \delta(N,\varepsilon) \tag{96}$$

for any $n \leq N$.

Finally, since $\mathbf{V}_\varepsilon$ admits the bound (82), we have

$$\sup_{t\in(0,T)} \left| \int_{\Omega_\varepsilon} \mathbf{V}_\varepsilon \cdot \nabla_x \left(\Delta_N^{-1} \mathrm{div}_x \varphi - P_M[\Delta_N^{-1} \mathrm{div}_x \varphi] \right) \ \mathrm{d}x \right| \leq \varepsilon \text{ for all } M > M(\varphi), \tag{97}$$

where P_M denotes the orthogonal projection onto $\mathrm{span}\{\psi_1, \dots, \psi_M\}$. Moreover,

$$\int_{\Omega_\varepsilon} \mathbf{V}_{\varepsilon,\delta} \cdot \nabla_x \left(\Delta_N^{-1} \mathrm{div}_x \varphi - P_M[\Delta_N^{-1} \mathrm{div}_x \varphi] \right) \ \mathrm{d}x$$

$$= \int_{\Omega_\varepsilon} \nabla_x \Psi_{\varepsilon,\delta} \cdot \nabla_x \left(\Delta_N^{-1} \mathrm{div}_x \varphi - P_M[\Delta_N^{-1} \mathrm{div}_x \varphi] \right) \ \mathrm{d}x$$

$$= - \int_{\Omega_\varepsilon} \Psi_{\varepsilon,\delta} \Big(\mathrm{div}_x \varphi - P_M[\mathrm{div}_x \varphi] \Big) \ \mathrm{d}x,$$

where $\Psi_{\varepsilon,\delta}$ solves a wave equation

$$\varepsilon \partial_t S_{\varepsilon,\delta} + \omega \Delta \Psi_{\varepsilon,\delta} = \varepsilon \mathrm{div}_x \Big(\mathbf{H}^1_{\varepsilon,\delta} + \mathbf{H}^2_{\varepsilon,\delta} \Big), \tag{98}$$

$$\varepsilon \partial_t \Psi_{\varepsilon,\delta} + S_{\varepsilon,\delta} = \varepsilon \Delta_N^{-1} \Big[\mathrm{div}_x \mathrm{div}_x \Big(\mathbb{G}^1_{\varepsilon,\delta} + \mathbb{G}^2_{\varepsilon,\delta} \Big) \Big] + \varepsilon \Delta_N^{-1} [\mathrm{div}_x \mathbf{G}^3_{\varepsilon,\delta}], \tag{99}$$

supplemented with the boundary conditions

$$\nabla_x \Psi_{\varepsilon,\delta} \cdot \mathbf{n}|_{\partial\Omega_\varepsilon} = 0. \tag{100}$$

Thus in view of the uniform bounds (88 - 93), we can find $M = M(\varepsilon) > 0$ such that

$$\sup_{t\in(0,T)} \left| \int_{\Omega_\varepsilon} \mathbf{V}_{\varepsilon,\delta} \cdot \nabla_x \left(\Delta_N^{-1} \mathrm{div}_x \varphi - P_M[\Delta_N^{-1} \mathrm{div}_x \varphi] \right) \ \mathrm{d}x \right| < \varepsilon \text{ for all } M > M(\varepsilon),\ \delta > 0. \tag{101}$$

Combining the estimates (95 - 101) we obtain the desired conclusion (94). Consequently, we may assume that all quantities appearing in the acoustic equation are smooth, with the data compactly supported in $(0,T) \times \Omega_\varepsilon$. Accordingly, the acoustic equation reads:

$$\varepsilon \partial_t S_\varepsilon + \omega \mathrm{div}_x \mathbf{V}_\varepsilon = \varepsilon \mathrm{div}_x \Big(\mathbf{H}^1_\varepsilon + \mathbf{H}^2_\varepsilon \Big), \tag{102}$$

$$\varepsilon \partial_t \mathbf{V}_\varepsilon + \nabla_x S_\varepsilon = \varepsilon \mathrm{div}_x \Big(\mathbb{G}^1_\varepsilon + \mathbb{G}^2_\varepsilon \Big) + \varepsilon \mathbf{G}^3_\varepsilon, \tag{103}$$

supplemented with the boundary conditions

$$\mathbf{V}_\varepsilon \cdot \mathbf{n}|_{\partial\Omega_\varepsilon} = 0, \tag{104}$$

and the initial conditions

$$S_\varepsilon(0,\cdot) = S^1_{0,\varepsilon} + S^2_{0,\varepsilon} + S^3_{0,\varepsilon},\ \mathbf{V}_\varepsilon(0,\cdot) = \mathbf{V}^1_{0,\varepsilon} + \mathbf{V}^2_\varepsilon, \tag{105}$$

where

$$\|S^1_{0,\varepsilon}\|_{L^1(\Omega)} + \|S^2_{0,\varepsilon}\|_{L^2(\Omega)} + \|S^3_{0,\varepsilon}\|_{D^{1,2}(R^3)} \leq c, \tag{106}$$

$$\|\mathbf{V}^1_{0,\varepsilon}\|_{L^1(\Omega;R^3)} + \|\mathbf{V}^2_{0,\varepsilon}\|_{L^2(\Omega;R^3)} \leq c, \tag{107}$$

and

$$\|\mathbf{H}^1_\varepsilon\|_{L^2(0,T;L^1(\Omega;R^3))} + \|\mathbf{H}^2_\varepsilon\|_{L^2(0,T;L^2(\Omega;R^3))} \leq c, \tag{108}$$

$$\|\mathbb{G}^1_\varepsilon\|_{L^2(0,T;L^1(\Omega;R^{3\times 3}))} + \|\mathbb{G}^2_\varepsilon\|_{L^2(0,T;L^2(\Omega;R^{3\times 3}))} \leq c, \tag{109}$$

$$\|\mathbf{G}^3_\varepsilon\|_{L^\infty(0,T;L^{5/3}(\Omega;R^3))} \leq c. \tag{110}$$

2.5.3 Finite speed of propagation

System (102), (103) admits a finite speed of propagation proportional to $\sqrt{\omega}/\varepsilon$, specifically, if the initial data for two solutions coincide on the set

$$B_{T\sqrt{\omega}/\varepsilon} = \{x \in \Omega \mid |x| < R + T\sqrt{\omega}/\varepsilon\} \subset \Omega_\varepsilon,$$

and the forcing terms are the same on the space-time cylinder $(0,T) \times B_{T\sqrt{\omega}/\varepsilon}$, then the two solutions are the same on the cone

$$\Big\{(t,x) \mid t \in (0,T), x \in B_{T\sqrt{\omega}/\varepsilon}, \mathrm{dist}[x, \partial B_{T\sqrt{\omega}/\varepsilon}] > t\sqrt{\omega}/\varepsilon\Big\}.$$

Since we are interested only in the local behavior of solutions, specifically we want to show

$$\Big\{t \mapsto \int_\Omega \mathbf{V}_\varepsilon(t,\cdot)\cdot\varphi\ \mathrm{d}x\Big\} \text{ is precompact in } L^2(0,T) \text{ for any } \varphi \in C^\infty_c(\Omega;R^3), \tag{111}$$

we may assume that the acoustic system (102), (103) is satisfied on the whole set $(0,T)\times\Omega$ and that its solutions have compact support in $[0,T]\times\overline{\Omega}$.

2.5.4 Compactness of the solenoidal component

A short inspection of (103) implies that the family

$$\Big\{t \mapsto \int_\Omega \mathbf{V}_\varepsilon \cdot \mathbf{H}[\varphi]\ \mathrm{d}x\Big\} \text{ is precompact in } C[0,T]$$

for any $\varphi \in C^\infty_c(\Omega;R^3)$. Consequently, writing the field $\mathbf{V}_\varepsilon$ in the form of its Helmholtz decomposition in Ω:

$$\mathbf{V}_\varepsilon = \mathbf{H}[\mathbf{V}_\varepsilon] + \nabla_x \Psi_\varepsilon,$$

we can see that (87) follows as soon as we show

$$\left\{ t \mapsto \int_\Omega \nabla_x \Psi_\varepsilon \cdot \varphi \, \mathrm{d}x \right\} \to 0 \text{ in } L^2(0,T) \tag{112}$$

for any $\varphi \in C_c^\infty(\Omega; R^3)$, where Ψ_ε is the acoustic potential.

2.6 Acoustic equation - abstract formulation

In order to show (112), we introduce an abstract formulation of the acoustic equation in terms of the *Neumann Laplacean* Δ_N,

$$\Delta_N v = \Delta v \text{ in } \Omega, \ \nabla_x v \cdot \mathbf{n}|_{\partial\Omega} = 0, \ v \in C_c^\infty(\overline{\Omega}).$$

It is standard that Δ_N can be extended as a self-adjoint operator on the Hilbert space $L^2(\Omega)$. As a consequence of Rellich's theorem, the point spectrum of Δ_N is empty. Moreover, the spectrum of $-\Delta_N$ is absolutely continuous and coincides with $[0,\infty)$, see Leis [18].

Since all quantities in the acoustic equation (102), (103) are smooth, $\mathbf{V}_\varepsilon \cdot \mathbf{n}|_{\partial\Omega} = 0$, and the data $\mathbb{G}_\varepsilon^i$, $i = 1,2$, $\mathbf{G}_\varepsilon^3$ are compactly supported, we deduce that $\nabla_x S_\varepsilon \cdot \mathbf{n}|_{\partial\Omega} = 0$. In particular, system (102), (103) converts to a *wave equation*:

$$\varepsilon \partial_t S_\varepsilon + \omega \Delta_N \Psi_\varepsilon = \varepsilon \mathrm{div}_x \Big(\mathbf{H}_\varepsilon^1 + \mathbf{H}_\varepsilon^2 \Big), \tag{113}$$

$$\varepsilon \partial_t \Psi_\varepsilon + S_\varepsilon = \varepsilon \Delta_N^{-1} \mathrm{div}_x \mathrm{div}_x \Big(\mathbb{G}_\varepsilon^1 + \mathbb{G}_\varepsilon^2 \Big) + \varepsilon \Delta_N^{-1} \mathrm{div}_x \mathbf{G}_\varepsilon^3, \tag{114}$$

supplemented with the homogeneous Neumann boundary conditions

$$\nabla_x \Psi_\varepsilon \cdot \mathbf{n}|_{\partial\Omega} = 0, \tag{115}$$

and the initial conditions

$$S_\varepsilon(0,\cdot) = S_{0,\varepsilon}, \ \Psi_\varepsilon(0,\cdot) = \Delta_N^{-1} \mathrm{div}_x \mathbf{V}_{0,\varepsilon}, \tag{116}$$

where $\nabla_x \Psi_\varepsilon = \mathbf{H}^\perp[\mathbf{V}_\varepsilon]$ is the gradient component of the Helmholtz decomposition of $\mathbf{V}_\varepsilon$.

Our goal is to rewrite system (113), (114) solely in terms of the operator Δ_N and functions ranging in the Hilbert space $L^2(\Omega)$. To this end, observe first that the expression $\mathrm{div}_x \mathrm{div}_x (\mathbb{G}_\varepsilon^1 + \mathbb{G}_\varepsilon^2)(t,\cdot)$ may be viewed as a continuous linear form on $\mathscr{D}((-\Delta_N)^2) \cap \mathscr{D}((-\Delta_N)^{1/2})$ for any fixed t. Indeed it is enough to show that if

$$h \in \mathscr{D}((-\Delta_N)^2) \cap \mathscr{D}((-\Delta_N)^{1/2}),$$

then h possesses second derivatives bounded and continuous in $\overline{\Omega}$, and, in addition,

$$\nabla_x h \in L^2(\Omega;R^3),\ \nabla_x^2 h \in L^2(\Omega,R^{3\times 3}).$$

Since $\mathscr{D}((-\Delta_N)^{1/2}) = D^{1,2}(\Omega)$, we immediately get $\nabla_x h \in L^2(\Omega;R^3)$, $h \in L^6(\Omega)$. Next, taking $\psi \in C^\infty(\Omega)$, $\mathrm{supp}[\psi] \subset \Omega$, $\psi \equiv 1$ outside some ball, we get

$$\Delta(\psi h) = \psi \Delta h + 2\nabla_x \psi \cdot \nabla_x h + \Delta \psi h \text{ in } R^3,$$

where the right-hand side is bounded in $L^2(R^3)$. We conclude, by means of the well-known regularity properties of Δ on R^3, that $\nabla_x^2 h \in L^2(\Omega;R^{3\times 3})$, in particular, h is Hölder continuous and bounded in Ω. Finally, since $\Delta h \in L^2(\Omega)$, and $\Delta^2[h] \in L^2(\Omega)$, we have Δh Hölder continuous, and the standard elliptic theory provides the desired conclusion.

Estimating the remaining terms in a similar fashion, we arrive at the following system:

$$\varepsilon \partial_t S_\varepsilon + \omega \Delta_N \Psi_\varepsilon = \varepsilon \Big((-\Delta_N)^2 [h_\varepsilon^1] + h_\varepsilon^2 \Big), \tag{117}$$

$$\varepsilon \partial_t \Psi_\varepsilon + S_\varepsilon = \varepsilon \Big((-\Delta_N)[g_\varepsilon^1 + g_\varepsilon^3] + (-\Delta_N)^{-1/2}[g_\varepsilon^2 + g_\varepsilon^4] \Big), \tag{118}$$

supplemented with the initial data

$$S_\varepsilon(0) = (-\Delta_N)^2[s_{0,\varepsilon}^1] + (-\Delta_N)^{-1/2}[s_{0,\varepsilon}^2], \tag{119}$$

$$\Psi_\varepsilon(0) = \Delta_N[v_{0,\varepsilon}^1] + \Delta_N^{-1}[v_{0,\varepsilon}^2], \tag{120}$$

with

$$\{h_\varepsilon^i\}_{\varepsilon>0},\ i = 1,2, \{g_\varepsilon^j\}_{\varepsilon>0},\ j = 1,\dots,4 \text{ bounded in } L^2(0,T;L^2(\Omega)), \tag{121}$$

$$\{s_{0,\varepsilon}^i\}_{\varepsilon>0},\ i = 1,2,\ \{v_{0,\varepsilon}^j\}_{\varepsilon>0}\ j = 1,2, \text{ bounded in } L^2(\Omega). \tag{122}$$

2.6.1 Variation-of-constants formula

In accordance with (117 - 122), the acoustic potential Ψ_ε is determined through *variation-of-constants formula*, specifically,

$$\Psi_\varepsilon(t) = \frac{1}{2}\exp\left(\mathrm{i}\frac{t}{\varepsilon}\sqrt{-\omega\Delta_N}\right)\left[(-\Delta_N)[v_{0,\varepsilon}^1 + \mathrm{i}s_{0,\varepsilon}^1] + \frac{1}{(-\Delta_N)}[v_{0,\varepsilon}^2 + \mathrm{i}s_{0,\varepsilon}^2]\right] \tag{123}$$

$$+\frac{1}{2}\exp\left(-\mathrm{i}\frac{t}{\varepsilon}\sqrt{-\omega\Delta_N}\right)\left[(-\Delta_N)[v_{0,\varepsilon}^1 - \mathrm{i}s_{0,\varepsilon}^1] + \frac{1}{(-\Delta_N)}[v_{0,\varepsilon}^2 - \mathrm{i}s_{0,\varepsilon}^2]\right]$$

$$+\frac{1}{2}\int_0^t \exp\left(\mathrm{i}\frac{t-s}{\varepsilon}\sqrt{-\omega\Delta_N}\right)\left[(-\Delta_N)[g_\varepsilon^1 + g_\varepsilon^3] + \frac{1}{\sqrt{(-\Delta_N)}}[g_\varepsilon^2 + g_\varepsilon^4]\right.$$

$$+\mathrm{i}(-\Delta_N)^{3/2}[h^1_\varepsilon]+\mathrm{i}\frac{1}{\sqrt{-\Delta_N}}[h^2_\varepsilon]\Bigg]\ \mathrm{d}s$$

$$+\frac{1}{2}\int_0^t \exp\left(-\mathrm{i}\frac{t-s}{\varepsilon}\sqrt{-\omega\Delta_N}\right)\left[(-\Delta_N)[g^1_\varepsilon+g^3_\varepsilon]+\frac{1}{\sqrt{(-\Delta_N)}}[g^2_\varepsilon+g^4_\varepsilon]\right.$$

$$-\mathrm{i}(-\Delta_N)^{3/2}[h^1_\varepsilon]-\mathrm{i}\frac{1}{\sqrt{-\Delta_N}}[h^2_\varepsilon]\Bigg]\ \mathrm{d}s.$$

2.6.2 Strong convergence of velocities

We are ready to show (112), specifically,

$$\left\{t\mapsto\int_\Omega \Psi_\varepsilon(t,\cdot)\mathrm{div}_x\varphi\ \mathrm{d}x\right\}\to 0 \text{ in } L^2(0,T) \text{ as } \varepsilon\to 0 \tag{124}$$

for any fixed $\varphi\in C^\infty_c(\Omega;R^3)$.

First of all, observe that it is enough to show

$$\left\{t\mapsto\int_\Omega \chi H(-\Delta_N)[\Psi_\varepsilon(t,\cdot)]\ \mathrm{d}x\right\}\to 0 \text{ in } L^2(0,T) \tag{125}$$

for any fixed $\chi\in C^\infty_c(\Omega)$, $H\in C^\infty_c(0,\infty)$. Indeed, taking $\chi\in C^\infty_c(\Omega)$ such that $\chi|_{\mathrm{supp}[\varphi]}=1$, we have

$$\int_\Omega \Psi_\varepsilon\mathrm{div}_x\varphi\ \mathrm{d}x=\int_\Omega \chi\Psi_\varepsilon\mathrm{div}_x\varphi\ \mathrm{d}x$$

$$=\int_\Omega \chi\Big(\mathrm{Id}-H(-\Delta_N)\Big)[\Psi_\varepsilon]\mathrm{div}_x\varphi\ \mathrm{d}x+\int_\Omega \chi\mathrm{div}_x\varphi H(-\Delta_N)[\Psi_\varepsilon]\ \mathrm{d}x,$$

where, as stated in (125),

$$\left\{t\mapsto\int_\Omega \chi\mathrm{div}_x\varphi H(-\Delta_N)[\Psi_\varepsilon(t,\cdot)]\ \mathrm{d}x\right\}\to 0\ \text{ in } L^2(0,T) \text{ as } \varepsilon\to 0.$$

On the other hand,

$$\int_\Omega \chi\Big(\mathrm{Id}-H(-\Delta_N)\Big)[\Psi_\varepsilon]\mathrm{div}_x\varphi\ \mathrm{d}x=\int_\Omega \Big(\mathrm{Id}-H(-\Delta_N)\Big)[\Psi_\varepsilon]\mathrm{div}_x\varphi\ \mathrm{d}x \tag{126}$$

$$=\int_\Omega \Big(\mathrm{Id}-H(-\Delta_N)\Big)[\mathrm{div}_x\varphi]\Psi_\varepsilon\ \mathrm{d}x.$$

Taking a family of functions $H(\lambda)\nearrow 1$, in particular,

$$(H(-\Delta_N)-\mathrm{Id})[h]\to 0 \text{ for any fixed } h\in L^2(\Omega),$$

we observe that the integral (126) is small, uniformly with respect to $t\in(0,T)$ for a suitable choice of H, as soon as we can show that

$$(-\Delta_N)^{3/2}[\mathrm{div}_x \varphi],\ \frac{1}{(-\Delta_N)}[\mathrm{div}_x \varphi] \in L^2(\Omega) \tag{127}$$

since Ψ_ε is given by (123). To see (127), it is enough to observe that

$$\Delta_N[h] = \mathrm{div}_x \varphi \text{ implies } \nabla_x h \in L^q(\Omega; R^3) \text{ for any } q > 1;$$

whence, by virtue of Sobolev's theorem $h \in L^2(\Omega)$.

In view of the previous discussion, the proof of strong (a.a. pointwise) convergence of velocities reduces to showing (125). This will be done in the following section.

2.6.3 Spectral measures

Our goal in this section is to show (125). Since Ψ_ε is given by (123), it is sufficient to check that

$$\left(\int_0^T \left| \left\langle \exp\left(\mathrm{i}\sqrt{-\Delta_N}\frac{t}{\varepsilon}\right) H(-\Delta_N)[h], \varphi \right\rangle \right|^2 \ \mathrm{d}t \right)^{1/2} \leq \omega(\varepsilon, H, \varphi) \|h\|_{L^2(\Omega)} \tag{128}$$

for any $h \in L^2(\Omega)$, with

$$\omega(\varepsilon, H, \varphi) \to 0 \text{ as } \varepsilon \to 0 \text{ for any fixed } \varphi,\ H,$$

where $<,>$ denotes the standard (complex) scalar product in $L^2(\Omega)$. Uniformity with respect to h is needed when handling the time integrals in (123).

The integrand in (128) may be written by *spectral theorem* (see Reed and Simon [27, Chapter VIII]) as follows

$$\left\langle \exp\left(\mathrm{i}\sqrt{-\Delta_N}\frac{t}{\varepsilon} H(-\Delta_N)\right)[h], \varphi \right\rangle = \int_0^\infty \exp\left(\mathrm{i}\sqrt{\lambda}\frac{t}{\varepsilon}\right) H(\lambda)\tilde{h}(\lambda)\ \mathrm{d}\mu_\varphi(\lambda), \tag{129}$$

where μ_φ is the spectral measure associated to the function φ, and

$$\tilde{h} \in L^2(\Omega; \mathrm{d}\mu_\varphi),\ \|\tilde{h}\|_{L^2_{\mu_\varphi}(\Omega)} \leq \|h\|_{L^2(\Omega)}.$$

Following Last [17], we obtain

$$\int_0^T \left| \left\langle \exp\left(\mathrm{i}\sqrt{-\Delta_N}\frac{t}{\varepsilon}\right) H(-\Delta_N)[h], \varphi \right\rangle \right|^2 \ \mathrm{d}t \tag{130}$$

$$= \int_0^T \int_0^\infty \int_0^\infty \exp\left(\mathrm{i}\left(\sqrt{x} - \sqrt{y}\right) \frac{t}{\varepsilon}\right) H(x)\tilde{h}(x)\ H(y)\bar{\tilde{h}}(y)\ \mathrm{d}\mu_\varphi(x)\ \mathrm{d}\mu_\varphi(y)\ \mathrm{d}t$$

$$\leq c(H) \int_0^\infty \int_0^\infty \left(\int_{-\infty}^\infty \exp\left(-(t/T)^2\right) \exp\left(\mathrm{i}\left(\sqrt{x} - \sqrt{y}\right) \frac{t}{\varepsilon}\right)\ \mathrm{d}t \right) \times$$

$$\times H(x)\tilde{h}(x)H(y)\overline{\tilde{h}}(y)\ \mathrm{d}\mu_\varphi(x)\ \mathrm{d}\mu_\varphi(y)$$

$$\leq c(T,H)\sqrt{\pi}\int_0^\infty\int_0^\infty |\tilde{h}(x)||\tilde{h}(y)|\exp\left(-\frac{T^2|\sqrt{x}-\sqrt{y}|^2}{4\varepsilon^2}\right)\ \mathrm{d}\mu_\varphi(x)\ \mathrm{d}\mu_\varphi(y).$$

Consequently, by virtue of the Cauchy-Schwartz inequality,

$$\int_0^T \left|\left\langle \exp\left(\mathrm{i}\sqrt{-\Delta_N}\frac{t}{\varepsilon}\right)H(-\Delta)[h],\varphi\right\rangle\right|^2\ \mathrm{d}t \leq c(H)\omega^2(\varepsilon,\varphi)\|h\|^2_{L^2(\Omega)}, \tag{131}$$

where

$$\omega(\varepsilon,\varphi)=\sqrt{2}\left(\int_0^\infty\int_0^\infty \exp\left(-\frac{T^2|\sqrt{x}-\sqrt{y}|^2}{2\varepsilon^2}\right)\ \mathrm{d}\mu_\varphi(x)\ \mathrm{d}\mu_\varphi(y)\right)^{1/4}.$$

Now, it is easy to check that $\omega(\varepsilon,H,\varphi)\to 0$ as $\varepsilon\to 0$ provided the spectral measure μ_φ does not charge points in $[0,\infty)$, in other words, as long as the point spectrum of the operator Δ_N is empty. As a matter of fact, the rate of convergence is independent of the specific choice of H. Thus we have proved (125) yielding the desired conclusion

$$\mathbf{u}_\varepsilon \to \mathbf{U} \text{ in } L^2((0,T)\times K;R^3) \text{ for any compact set } K\subset\Omega. \tag{132}$$

2.7 Convergence to the limit system - part II

Since we have shown strong pointwise (a.a.) convergence of the family of the velocity fields $\{\mathbf{u}_\varepsilon\}_{\varepsilon>0}$ it is a routine matter to let $\varepsilon\to 0$ in the weak formulation of the primitive system to deduce that

$$\frac{\varrho_\varepsilon-\overline{\varrho}}{\varepsilon}\to r \text{ weakly-(*) in } L^\infty(0,T;L^{5/3}(K)) \text{ for any compact } K\subset\Omega,$$

$$\frac{\vartheta_\varepsilon-\overline{\vartheta}}{\varepsilon}\to\Theta \text{ weakly in } L^2(0,T;W^{1,2}(\Omega)),$$

$$\mathbf{u}_\varepsilon\to\mathbf{U} \text{ weakly in } L^2(0,T;W^{1,2}(\Omega;R^3)),$$

and

$$\mathbf{u}_\varepsilon\to\mathbf{U} \text{ in } L^2((0,T)\times K)\ \text{ for any compact } K\subset\Omega,$$

where $\mathbf{U}$, Θ, r is a weak solution of the Oberbeck-Boussinesq approximation (1 - 4), together with the boundary conditions

$$\mathbf{U}\cdot\mathbf{n}|_{\partial\Omega}=0,\ [\mathbb{S}(\nabla_x\mathbf{U})\mathbf{n}]\times\mathbf{n}|_{\partial\Omega}=0,\ \nabla_x\Theta\cdot\mathbf{n}|_{\partial\Omega}=0.$$

More specifically, we have

$$\mathrm{div}_x \mathbf{U} = 0 \text{ a.a. on } (0,T)\times\Omega,$$

$$\int_0^T \int_\Omega \left(\overline{\rho}(\mathbf{U}\cdot\partial_t\varphi + (\mathbf{U}\otimes\mathbf{U}):\nabla_x\varphi)\right)\,\mathrm{d}x\,\mathrm{d}t, \tag{133}$$
$$= -\int_\Omega \overline{\rho}\mathbf{U}_0\cdot\varphi\,\mathrm{d}x + \int_0^T\int_\Omega \mathbb{S}:\nabla_x\varphi - r\nabla_x G\,\mathrm{d}x\,\mathrm{d}t$$

for any test function $\varphi \in C_c^\infty([0,T)\times\overline{\Omega};R^3)$, $\mathrm{div}_x\varphi = 0$, $\varphi\cdot\mathbf{n}|_{\partial\Omega} = 0$, where we have set

$$\mathbb{S} = \mu(\overline{\vartheta})(\nabla_x\mathbf{U} + \nabla_x^t\mathbf{U}).$$

Furthermore,

$$\overline{\rho}c_p(\overline{\rho},\overline{\vartheta})\Big[\partial_t\Theta + \mathrm{div}_x(\Theta\mathbf{U})\Big] - \kappa\Delta\Theta - \overline{\rho}\overline{\vartheta}\alpha(\overline{\rho},\overline{\vartheta})\mathrm{div}_x(G\mathbf{U}) = 0 \text{ a.a. in } (0,T)\times\Omega, \tag{134}$$
$$\nabla_x\Theta\cdot\mathbf{n}|_{\partial\Omega} = 0, \Theta(0,\cdot) = \Theta_0,$$

and

$$r + \overline{\rho}\alpha(\overline{\rho},\overline{\vartheta})\Theta = 0 \text{ a.a. in } (0,T)\times\Omega.$$

We remark that the uniform bounds established above yield

$$\Theta \in L^\infty(0,T;L^2(\Omega)),$$

while

$$\mathbf{U} \in L^\infty(0,T;L^2(\Omega;R^3)),$$

in particular, the standard maximal regularity theory of the heat equation justifies validity of (134) a.a. in $(0,T)\times\Omega$.

It is interesting to note that the initial conditions for the velocity are determined through

$$\mathbf{u}_{0,\varepsilon} \to \mathbf{U}_0 \text{ weakly in } L^2(\Omega;R^3),$$

while the initial value Θ_0 reads

$$\Theta_0 = \frac{\overline{\vartheta}}{c_p(\overline{\rho},\overline{\vartheta})}\left(\frac{\partial s(\overline{\rho},\overline{\vartheta})}{\partial\rho}\rho_0^{(1)} + \frac{\partial s(\overline{\rho},\overline{\vartheta})}{\partial\vartheta}\vartheta_0^{(1)} + \alpha(\overline{\rho},\overline{\vartheta})G\right), \tag{135}$$

where

$$\rho_{0,\varepsilon}^{(1)} \to \rho_0^{(1)},\ \vartheta_{0,\varepsilon}^{(1)} \to \vartheta_0^{(1)} \text{ weakly in } L^2(\Omega).$$

Moreover, we can check if $\rho_0^{(1)}$, $\vartheta_0^{(1)}$ satisfy a compatibility condition

$$\frac{\partial p(\overline{\rho},\overline{\vartheta})}{\partial\rho}\rho_0^{(1)} + \frac{\partial p(\overline{\rho},\overline{\vartheta})}{\partial\vartheta}\vartheta_0^{(1)} = \overline{\rho}G,$$

where the expression on the left-hand side is nothing other than the linearization of the pressure at the constant state $(\overline{\rho},\overline{\vartheta})$ applied to the vector $[\rho_0^{(1)},\vartheta_0^{(1)}]$, relation

(135) reduces to

$$\Theta_0 = \vartheta_0^{(1)}.$$

The reader may consult [14, Chapter 5, Section 5.5] for other aspects of the "data adjustment" problem related to incompressible limits.

Summarizing the arguments of this section we have proved the following result:

Theorem 2.2 *Under the hypotheses of Theorem 2.1, let* $\{\rho_\varepsilon, \vartheta_\varepsilon, \mathbf{u}_\varepsilon\}_{\varepsilon>0}$ *be a family of weak solutions to the Navier-Stokes-Fourier system on the set* $(0,T)\times\Omega_\varepsilon$*, where* Ω_ε *are given by (34), and the initial* $\{\rho_{0,\varepsilon}, \vartheta_{0,\varepsilon}, \mathbf{u}_{0,\varepsilon}\}_{\varepsilon>0}$ *data satisfy (40 - 43), with*

$$\rho_{0,\varepsilon}^{(1)} \to \rho_0^{(1)},\ \vartheta_{0,\varepsilon}^{(1)} \to \vartheta_0^{(1)} \text{ weakly in } L^2(\Omega),$$

$$\mathbf{u}_{0,\varepsilon} \to \mathbf{U}_0 \text{ weakly in } L^2(\Omega;R^3).$$

Then, extracting a suitable subsequence, yields

$$\rho_\varepsilon \to \overline{\rho} \text{ in } L^\infty(0,T;L^{5/3}(K)),$$

$$\frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon} \to \Theta \text{ weakly in } L^2(0,T;W^{1,2}(K)),$$

and

$$\mathbf{u}_\varepsilon \to \mathbf{U} \text{ weakly in } L^2(0,T;W^{1,2}(K;R^3)) \text{ and, strongly in } L^2((0,T)\times K;R^3)$$

for any compact $K\subset\Omega$*, where* $\mathbf{U}$*,* Θ *is a weak solution of the Oberbeck-Boussinesq approximation in* $(0,T)\times\Omega$ *in the sense specified in (133), (134), and the initial data (135) and*

$$\mathbf{U}(0,\cdot) = \mathbf{H}[\mathbf{U}_0].$$

Note that dispersive (Strichartz' estimates) for the wave equation considered in the whole space R^3 were used by Desjardins and Grenier [8] in order to eliminate the acoustic waves in the low Mach number limit for the compressible Navier-Stokes system. Similar technique was used by Alazard [1] and Isozaki [15] in the context of Euler equations. *Weak* convergence of the convective term could be also established by a "local" method developed by Lions and Masmoudi [21](see also a nice survey by Masmoudi [23]).

3 Oberbeck-Boussinesq approximation

In the remaining part of the paper, we examine the Oberbeck-Boussinesq approximation written in the form introduced in (7 - 9), specifically,

$$\mathrm{div}_x \mathbf{U} = 0, \tag{136}$$

$$\partial_t \mathbf{U} + \mathrm{div}_x \mathbf{U} \otimes \mathbf{U} + \nabla_x P = \Delta \mathbf{U} - \theta \nabla_x G, \tag{137}$$

$$\partial_t \theta + \mathrm{div}_x(\theta \mathbf{U}) - \Delta \theta = 0, \tag{138}$$

where, for the sake of simplicity, all physical constants have been set to one. As we have seen in Section 2, the system (136 - 137), modulo an obvious change of variables specified in Section 1, can be identified as a singular limit of the full Navier-Stokes-Fourier system, where the Mach and Froude numbers tend to zero. In contrast with Section 2, where the boundary $\partial\Omega$ was supposed to be *acoustically hard* (cf. (19)), we consider the more common no-slip boundary condition

$$\mathbf{U}|_{\partial\Omega} = 0, \tag{139}$$

supplemented with a similar homogeneous Dirichlet boundary condition for the temperature deviation

$$\theta|_{\partial\Omega} = 0. \tag{140}$$

Let us remark that (139), (140) could be justified by similar arguments as in Section 2, provided (19) was replaced by more general "penalized" boundary conditions in the spirit of [10].

In addition to (136 - 140), we suppose that the (weak) solutions satisfy the *energy inequality*

$$\|\mathbf{U}(\tau)\|^2_{L^2(\Omega;R^3)} + 2\int_s^\tau \|\nabla_x \mathbf{U}\|^2_{L^2(\Omega;R^{3\times 3})}\ \mathrm{d}t \le \|\mathbf{U}(s)\|^2_{L^2(\Omega;R^3)} - \int_s^\tau \int_\Omega \theta \nabla_x G \cdot \mathbf{u}\ \mathrm{d}x\ \mathrm{d}t \tag{141}$$

for any $\tau > 0$ and a.a. $s \le \tau$ including $s = 0$. If the velocity field $\mathbf{U}$ is smooth, formula (141) follows easily by multiplying (137) by $\mathbf{U}$ and integrating by parts.

Similarly, a formal manipulation of (138) yields

$$\int_\Omega H(\theta(\tau))\ \mathrm{d}x + \int_s^\tau \int_\Omega H''(\theta)|\nabla_x \theta|^2\ \mathrm{d}x\ \mathrm{d}t \le \int_\Omega H(\theta(s))\ \mathrm{d}x \tag{142}$$

for any $\tau > 0$ and a.a. $s \le \tau$ including $s = 0$ for any smooth convex H.

The interested reader may consult [14, Section 5.5.4, Chapter 5] for a rigorous derivation of the energy inequalities (141), (142) via a singular limit process.

3.1 Suitable weak solutions

We consider the initial data for system (136 - 138) in the form

$$\mathbf{U}(0,\cdot) = \mathbf{U}_0 \in L^2(\Omega),\ \theta(0,\cdot) = \theta_0 \in L^1 \cap L^\infty(\Omega). \tag{143}$$

Motivated by the previous discussion, we shall say that $\mathbf{U}$, θ is a *suitable weak solution* to problem (136 - 140), supplemented with the initial data (143) if

$$\mathbf{U} \in C_{\rm weak}([0,T];L^2(\Omega;R^3)) \cap L^2(0,T;W_0^{1,2}(\Omega;R^3));$$

$$\theta \in C_{\rm weak}([0,T];L^2(\Omega)) \cap L^\infty(0,T;L^p(\Omega)) \text{ for any } 1 \leq p \leq \infty,$$

$$\theta \in L^2(0,T;W_0^{1,2}(\Omega));$$

$${\rm div}_x \mathbf{U} = 0 \text{ a.a. in } (0,T) \times \Omega;$$

the integral identity

$$\int_0^T \int_\Omega \Big(\mathbf{U} \cdot \partial_t \varphi + \mathbf{U} \otimes \mathbf{U} : \nabla_x \varphi \Big)\ {\rm d}x\ {\rm d}t$$

$$= \int_0^T \int_\Omega \Big(\nabla_x \mathbf{U} : \nabla_x \varphi + \theta \nabla_x G \cdot \varphi \Big)\ {\rm d}x - \int_\Omega \mathbf{U}_0 \cdot \varphi(0,\cdot)\ {\rm d}x$$

holds for any $\varphi \in C_c^\infty([0,T) \times \Omega;R^3)$, ${\rm div}_x \varphi = 0$;

$$\partial_t \theta + {\rm div}_x(\theta \mathbf{U}) - \Delta \theta = 0 \text{ a.a. in } (0,T) \times \Omega;$$

the energy inequalities (141), (142) are satisfies for a.a. $\tau \in [0,T]$.

Remark 3.1
Given the anticipated regularity of $\mathbf{U}$, θ *enforced by (141), (142), we may use the maximal regularity theory for the heat equation (138) in order to conclude that* $\partial_t \theta$, $\Delta \theta \in L^q(0,T;L^q(\Omega))$ *for a certain* $q > 1$.

Given the *a priori* bounds induced by (141), (142), the *existence* of suitable weak solutions can be proved, besides a rather complicated undirect proof in the spirit of Section 2, by means of nowadays standard methods, see the monograph by Sohr [29]. In the last part of this study, we examine the asymptotic behavior of suitable weak solutions to the Oberbeck-Boussinesq approximation for $t \to \infty$.

4 Long-time behavior of solutions to the Oberbeck-Boussinesq approximation

We conclude the present study of the Oberbeck-Boussinesq system by investigating the asymptotic behavior of solutions for large times. In contrast with [4], we show that the physically relevant choice of the forcing term $\nabla_x G$ yields strong convergence to zero of the total energy associated to system (136 - 140). Our approach is based on the available results by Miyakawa and Sohr [24] for the forced Navier-Stokes system. More precisely, we derive suitable decay estimates for the temperature deviation θ resulting from the “entropy” inequality (142) and then use the fact that, in accordance with (5), (6),

$$\nabla_x G \in L^p \cap L^\infty(\Omega; R^3) \text{ for } p > 3/2, \tag{144}$$

in particular, the forcing term in the Navier-Stokes system decays to zero sufficiently fast for $|x| \to \infty$.

4.1 Decay estimates for the temperature deviations

In this subsection we show that the solutions to the Oberbeck-Boussinesq system decay in $L^p, 1 < p \leq \infty$ at the same rate as the solutions of the underlying linear counterpart, namely the solutions to the heat equations.

Theorem 4.1 *Let* $\mathbf{U}(0,\cdot) = \mathbf{U}_0 \in L^2(\Omega),\ \theta(0,\cdot) = \theta_0 \in L^1 \cap L^\infty(\Omega)$. *Suppose* $\mathbf{U}$, θ *is a* suitable weak solution *to problem (136 - 140), with the initial data* $(\mathbf{U}_0, \theta_0)$, *then*

$$\|\theta(t,\cdot)\|_{L^p(\Omega)} \leq c(\|\theta_0\|_{L^1 \cap L^\infty(\Omega)}) t^{-\frac{3}{2}(1-1/p)},\ 1 \leq p \leq \infty,\ t > 0. \tag{145}$$

where the constant c is independent of p.

Proof

We note first that by appropriate choices of H, the estimate (142) yields

$$\|\theta(t,\cdot)\|_{L^p(\Omega)} \leq \|\theta_0\|_{L^p(\Omega)} \text{ for any } t \geq 0,\ 1 \leq p \leq \infty. \tag{146}$$

Following the well-known argument of Alikakos [2] (cf. also Cordoba, Cordoba [5]), we multiply (138) by $2j|\theta|^{2j-2}\theta$ and integrate the resulting expression over Ω, obtaining

$$\partial_t \int_\Omega |\theta^j|^2 \, \mathrm{d}x + \frac{2j(2j-1)}{j^2} \int_\Omega |\nabla_x \theta^j|^2 \, \mathrm{d}x \leq 0,$$

in particular choosing $j = 2^{k-1}$ yields

$$\partial_t \left(\frac{1}{2} \int_\Omega \left| |\theta|^{2^{k-1}} \right|^2 \, \mathrm{d}x \right) \leq -\frac{2^k - 1}{2^{k-2}} \int_\Omega \left| \nabla_x |\theta|^{2^{k-1}} \right|^2 \, \mathrm{d}x, \ k = 1, 2, \dots \tag{147}$$

By means of the standard Gagliardo-Nirenberg inequality (see Proposition 1.1), interpolating L^2 between L^1 and $\dot{H}^1$ we have

$$\left\| |\theta|^{2^{k-1}} \right\|^2_{L^2(\Omega)} \leq c \left\| \nabla_x |\theta|^{2^{k-1}} \right\|^{6/5}_{L^2(\Omega)} \left\| |\theta|^{2^{k-1}} \right\|^{4/5}_{L^1(\Omega)}. \tag{148}$$

Let $m_s = 2^s$. We proceed by induction on s. For m_0 the conclusion of the Theorem follows by (146). For $s = k-1$ we assume by induction that

$$\int_\Omega |\theta|^{2^{k-1}} \, \mathrm{d}x \leq b_{k-1} t^{-\frac{3}{2}(2^{k-1}-1)}. \tag{149}$$

Let $s = k$, define

$$\Phi_k = \int_\Omega \left| |\theta|^{2^{k-1}} \right|^2 \, \mathrm{d}x,$$

Combining (147), (148) and (149) yields

$$\partial_t \Phi_k \leq -\Phi_k^{5/3} c^{-5/3} b_{k-1}^{-4/3} t^{2^k - 2},$$

therefore, integrating in time over $[0,t]$ and a simple reordering of the terms gives

$$\Phi_k(t) \leq \left[\Phi_k(0) + \frac{2}{3} c^{-5/3} b_{k-1}^{-4/3} \frac{1}{2^k - 1} t^{2^k - 1} \right]^{-3/2}. \tag{150}$$

By virtue of (146), we can take

$$b_0 = \|\theta_0\|_{L^1(\Omega)}, \tag{151}$$

and, consequently, formula (150) yields, by induction and interpolation,

$$\|\theta(t,\cdot)\|_{L^p(\Omega)} \leq c(\|\theta_0\|_{L^1 \cap L^\infty(\Omega)}) t^{-\frac{3}{2}(1-1/p)} \text{ for any } 1 \leq p < \infty, \ t > 0. \tag{152}$$

The constant in (152) may, in principle, depend on p, however, a close inspection of (150) reveals, similarly to Alikakos [2, Theorem 3.1] that

$$b_k \leq c b_{k-1}^2, \text{ meaning, } b_k \leq C^k M^{2^k} \text{ for certain } C, M > 0,$$

It follows from (151) that

$$\Phi_k^{1/2k} \leq c(\|\theta_0\|_{L^1 \cap L^\infty(\Omega)}) t^{-\frac{3}{2}(1-1/2^k)}.$$

Taking the limit as k tends to infinity extends the decay rate to $p = \infty$, specifically,

$$\|\theta(t,\cdot)\|_{L^\infty(\Omega)} \leq c(\|\theta_0\|_{L^1\cap L^\infty(\Omega)})t^{-\frac{3}{2}},\ t > 0.$$

This concludes the proof of the theorem.

□

4.2 Decay estimates for the velocity

In view of the specific choice of the potential G (cf. (144)), and the uniform decay estimates of the temperature deviation θ established in (145), (152), the decay of the velocity field $\mathbf{U}$ follows from the results of Miyakawa and Sohr [24]. Indeed it is enough to check that (144), (152) imply that

$$\theta \nabla_x G \in L^1 \cap L^\infty(0,\infty; L^2(\Omega; R^3)),$$

therefore, by virtue of [24, Theorem 1],

$$\lim_{t\to\infty} \|\mathbf{U}(t)\|_{L^2(\Omega;R^3)} = 0. \tag{153}$$

Moreover, the velocity becomes ultimately more regular, specifically, there exists $T_0 > 0$ such that

$$\mathbf{U} \in L^2(T_0, T_0+T; W^{2,2}(\Omega;R^3)),\ \partial_t \mathbf{U} \in L^2(T_0, T_0+T, L^2(\Omega;R^3)),\ \text{for any } T > 0. \tag{154}$$

Let us summarize the results obtained in this section:

Theorem 4.2 *Let $\Omega \subset R^3$ be an unbounded (exterior) domain with compact boundary of class $C^{2+\nu}$. Let $\mathbf{U}$, θ be a suitable weak solution to the Oberbeck-Boussinesq approximation in $(0,\infty)\times\Omega$ specified in Section 3.1, emanating from the initial data*

$$\mathbf{U}_0 \in L^2(\Omega;R^3),\ \theta_0 \in L^1 \cap L^\infty(\Omega).$$

Then

$$\mathbf{U}(t,\cdot) \to 0 \textit{ in } L^2(\Omega;R^3),\ \theta(t,\cdot) \to 0 \textit{ in } L^p(\Omega) \textit{ for any } 1 < p \leq \infty \textit{ as } t \to \infty.$$

Remark 4.1

Since G is a harmonic (regular) function in Ω, the nowadays standard ultimate regularity results for the Navier-Stokes system (see e.g. the monograph by Sohr [29, Chapter V, Theorem 4.2.2]), together with a simple bootstrap argument applied to *the heat equation (138), could be used to deduce that the solution* $\mathbf{U}$, θ *becomes regular if time is large enough. Similarly, decay in stronger Sobolev norms can be shown.*

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