NONEXISTENCE OF PSEUDO-SELF-SIMILAR SOLUTIONS TO INCOMPRESSIBLE EULER EQUATIONS

MARIA SCHONBEK

ABSTRACT. In this paper we study a generalization of self-similar solutions. We show that just as for the solutions to the Navier-Stokes equations these supposedly singular solution reduce to the zero solution.

1. INTRODUCTION

Regularity, formation of turbulence and the possible construction of explicit solutions, are some of the central question for many fluid equations. In this paper we are are interested in studying a certain type of singular solutions for the Euler equations, which we call pseudo-self-similar. This family of solutions contain the usual self-similar ones. In the papers [5, 7] we investigated such solutions for the Navier-Stokes equations and we showed that they do not exist, or more precisely they just reduce to the zero solution. Here we will show that the same happens for these type of solutions corresponding to the Euler equations. Specifically we show that solutions with bounded L^2 norms and expected to satisfy the Beale-Kato Majda blow up condition for a time $T_o > 0$ reduce to the zero solutions.

In his pioneering 1934 paper [4], Jean Leray, discusses the possibility of having breakdown of regularity by constructing a nonzero singular backwards self–similar solution. Specifically the velocity and the pressure of Leray's solutions have the form

$$
\begin{aligned}
u(x,t) &= \frac{1}{\sqrt{2a(T_o - t)}} U\Big(\frac{x}{\sqrt{2a(T_o - t)}}\Big), \\
p(x,t) &= \frac{1}{2a(T_o - t)} P\Big(\frac{x}{\sqrt{2a(T_o - t)}}\Big).
\end{aligned}
\tag{1.1}
$$

Here $T \in \mathbb{R}$, $a > 0$ and $U = (U_1, U_2, U_3)$ is defined in $\mathbb{R}^3$. A simple calculation shows that U satisfies the elliptic equation

$$
\begin{aligned}
aU + ay_k \frac{\partial U}{\partial y_k} - \nu \triangle U + \nabla P + U_k \frac{\partial U}{\partial y_k} &= 0 \\
divU &= 0,
\end{aligned}
\tag{1.2}
$$

where the viscosity coefficient ν is strictly positive. An easy calculation shows that if a non zero solution of the type (1.1) existed, then it would have a bounded L^2 norm while the L^2 norm of its gradient blows up at the finite time T.

The work of M. Schonbek was partially supported by NSF Grant DMS-0900909.

The first answer to Leray's problem was given in [8] by J. Necas, M. Ruzicka, V. Sverak. They showed that zero is the only possible Leray self–similar solutions to the 3 dimensional Navier–Stokes equations for which $U \in L^3 \cap W^{1,2}_{loc}(\mathbb{R}^n)$.

It is worthwhile to note that even the "negative" result that such self–similar solutions are zero is very interesting interesting since it insures the nonexistence of a certain class of possible singular solutions.

If solutions of the above type existed (under a different set of conditions) it would solve in the positive the problem of blow-up for the 3D Navier–Stokes equations. The work of [8] work was continued under more general conditions by Tai-Peng Tsai [9]. The work in [8] was also extended in our paper [5] were we continued the work started in [8] in two directions. First we found a much simpler proof of the result in [8] using the slightly stronger, but natural, hypothesis that the U was in $W^{1,2}$. Second we studied extensions of the solutions of the form (1.1) hoping to construct a singular solution. We called these extensions pseudo-self-similar solutions and defined them as

$$\begin{aligned} u(x,t) &= \mu(t)U(\lambda(t)x), \\ p(x,t) &= \mu^2(t)P(\lambda(t)x). \end{aligned} \tag{1.3}$$

We note that as in the case of the solutions to the Navier-Stokes equations, one could have chosen P to satisfy $p(x,t) = \gamma P(\lambda(t)x)$. It is easy to show that the only choice possible for γ is $\gamma(t) = \mu(t)^2$. This follows by taking the divergence of the Euler equation and noting since $\Delta p = -\sum_{i,j=1}^{3} \partial_i u_j \partial_j u_i$ yields

$$\gamma \lambda^2 \Delta P = -\mu^2 \lambda^2 \sum_{i,j=1}^{3} \partial_i U_j \partial_j U_i.$$

In the sequel the solutions (1.3) will be referred as (λ, μ)-solutions or pseudo–self–similar solutions.

From our papers [5, 7] it follows that the only possible singular solution of the type (1.3) for the Navier-Stokes equations was the zero solution. We now want to investigate the non-existence of pseudo-self-similar (PSS) solutions for the Euler equations. Specifically we will show that all pseudo-self-similar solutions with finite L^2 energy for the Euler equations, satisfying the Beale-Kato-Majda (BKM) blow up criteria, reduce to zero. More precisely we will show that in most cases the finiteness of the L^2 norm is all what is needed, although in one particular case we will need some other Sobolev norm to be also finite. In the case of the Euler equations, the analysis of pseudo self-similar solutions is simpler then for the corresponding solutions to the Navier-Stokes equations. They reduce in general to self-similar solutions of the type studied in [2] and [3]. We note here that the conditions imposed on the self similar solutions studied in [2, 3] are different from our conditions. More details are given below.

2. Backwards pseudo-self–similar solutions for the Euler equations

The approach to study the (PSS) will be similar to the one used for the PSS solutions to the Navier–Stokes equations. The absence of diffusion will make the analysis quite different from the one for the Navier-Stokes equations. We recall the incompressible Euler equations

$$\begin{aligned} u_t + (u\cdot\nabla)u + \nabla p &= 0, \\ \mathrm{div} u &= 0. \end{aligned} \tag{2.4}$$

Local existen for the Euler equation in $\mathbb{R}^n,\ n \geq 3$ is well known. For completeness we recall the following theorem:

Theorem 2.1. *Let the initial velocity $u_0 \in H^s, s \geq 3$. Suppose $\|u_o\|_s \leq M$, with $M > 0$. Then there exist $T_* > 0$ where $T_* = T_*(M)$ depends only on M, so that the Euler equations (2.4) have a solution u satisfying*

$$u \in C([0,T]; H^s) \cap C^1([0,T]; H^{s-1})$$

for some $T \geq T_*(M)$.

Proof: see [6, 10]

□

Due to this local existence theorem it makes sense to look for solutions such that $u \in C([0,T_o); H^s) \cap C^1([0,T_o; H^{s-1})$. More precisely we can work with smooth solutions for $t < T_o$, where T_o is the first possible time where the BKM blow up criteria holds. An easy calculations shows that the pseudo-self-similar (PSS) solutions will satisfy the following nonlinear partial differential equation:

$$\begin{aligned} \frac{\mu'}{\mu^2\lambda}U + \frac{\lambda'}{\mu\lambda^2}(y\cdot\nabla)U = -(u\cdot\nabla)U + \nabla P) \\ \mathrm{div} U = 0 \end{aligned} \tag{2.5}$$

The plan is to look for solutions satisfying (2.5) with smooth functions λ and μ in the interval $[0, T_o)$, T_o the BKM possible blow up time and, show that solutions of this type with finite energy (1.3) are all zero. We first will need to exclude one class of such functions $\lambda(t)$, for which we will need and additional condition. We define the following set of functions

$$\mathcal{A} = \left\{ (\lambda(t), \mu(t)) \in C^1(-\infty, T_o), (\lambda, \mu) \neq \left(\frac{a_1}{(T_o - t)^{2/5}}, \frac{a_2}{(T_o - t)^{3/5}} \right) \right\} \tag{2.6}$$

where $a_i, i = 1, 2$ are arbitrary constants. We will first show the non existence of pseudo-self-similar solutions to Euler equations provided $(\lambda, \mu) \in \mathcal{A}$. We then will analyze the case when $(\lambda, \mu) \notin \mathcal{A}$, where we will need an additional hypothesis. The next theorem shows that for there are no nonzero pseudo self-similar solutions that satisfy the Beale-Kato-Majda blow up condition if $(\lambda, \mu) \in \mathcal{A}$.

Theorem 2.2. *There are no pseudo-self-similar solutions $u \in C([0,T_o); H^3) \cap C^1([0,T_o); H^2)$ of the incompressible 3D-Euler equations (1.3), for $t \in [0,T_o)$,with $(\lambda(t), \mu(t)) \in \mathcal{A}$, which satisfy*

$$ess\ sup_{0<t<T_o} \|u(\cdot,t)\|_{L^2(\mathbb{R}^3)} < \infty, \tag{2.7}$$

$$\lim_{t\to T_o^-} \int_0^{T_o} \|\omega(\cdot,t)\|_{L^\infty(\mathbb{R}^3)} = \infty. \tag{2.8}$$

where $\omega = curl\, u$, where T_o is the first such time.

Proof: We note that the solutions for $t < T_o$ are smooth. From the second condition above it follows that $\limsup_{t\to T_o^-} \omega(t) = \infty$. From the definition of pseudo-self-similar solutions it follows that if $\Omega = \operatorname{curl} U$

$$\omega(x,t) = \lambda(t)\mu(t)\Omega(x\lambda(t)),$$

which in turn yields that

$$\lim_{t\to T^-} \lambda(t)\mu(t) = \infty. \tag{2.9}$$

We note now that the right hand side of (2.5) is independent of time, in order for the left hand side to be also independent of time, there are two possible scenarios.

Case 1 The functions μ and λ are chosen so that the coefficients are constant. That is

$$\frac{\mu'}{\mu^2\lambda} = C_1 \tag{2.10}$$
$$\frac{\lambda'}{\mu\lambda^2} = C_2$$

In this case we have that

$$\frac{\mu'}{\mu} = C_1\mu\lambda = \frac{C_1}{C_2}\frac{\lambda'}{\lambda}$$

Let $\alpha = \frac{C_1}{C_2}$.

$$\ln\mu = \ln\lambda^\alpha + K$$

In other words

$$\mu = e^K\lambda^\alpha = K_o\lambda^\alpha.$$

We note here that by the first in the hypothesis of the theorem it follows that

$$\|u(\cdot,t)\|_{L^2(\mathbb{R}^3)} = \frac{\mu^2}{\lambda^3}\|U\|_2^2 < \infty. \tag{2.11}$$

From(2.9) and the relation between λ and μ yields that $\alpha > -1$. Suppose $\alpha \le -1$. We can immediately rule out $\alpha = -1$ since then $\lim_{t\to T_o} \lambda^{\alpha+1} = 1$, contradicting the second condition in the hypothesis. Suppose $\alpha > -1$, then if $\lim_{t\to T_o} \lambda^{\alpha+1} = \infty$ it would require that $\lim_{t\to T_o} \lambda = 0$, but this would imply that $\lim_{t\to T_o} \lambda^{2\alpha-3} = \infty$ contradicting (2.11). Hence

- Since $\alpha > -1$ we have $\lim_{t\to T_o^-} \lambda = \infty$.
- By (2.11) and the last observation it follows that $2\alpha - 3 \le 0$.

Combining (2.11) and (2.9) with the relation derived between λ and μ yields $\alpha \in (-1, 3/2]$. Recalling the definition of α, when $\alpha \in [0, 3/2]$ it turns out that either $C_i > 0$, for $i = 1,2$ or $C_i > 0$, for $i = 1,2$. We are only analyzing the first case when the constants are both positive. When the constants are negative works in similar fashion and as such is omitted . In the case that $\alpha \in (-1,0)$ either C_1 or C_2 will be strictly positive. If $C_2 > 0$ we do the analysis with the equations for λ, if $C_1 > 0$ the analysis is similar, but we have to work with equations for μ. We only will do the case for $C_2 > 0$, that is we work with the equations corresponding to λ, and we suppose $C_2 > 0$
Let $C_o = K_oC_2$, by (2.10) it follows that

$$\lambda' = C_o\lambda^{\alpha+2}.$$

Integrating over (t_o, t), for any $t < T$, yields

$$-(\alpha+1)\lambda(t)^{-(\alpha+1)} = -K_o t + M.$$

By (2.9) if we let $t \to T^-$ in the last equality the left hand side tends to zero, hence $M = K_o T$. From the last equality we have that

$$\lambda(t) = [(\alpha+1)C_o(T_o - t)]^{\frac{-1}{\alpha+1}}\,, \tag{2.12}$$

and

$$\mu(t) = K_o\left[(\alpha+1)^\alpha C_o^\alpha (T_o - t)\right]^{\frac{-\alpha}{\alpha+1}}\,. \tag{2.13}$$

The solutions corresponding to these $\lambda(t)$ are actually the usual self-similar solutions for which D. Chae [2] has obtained non existence under conditions, somewhat different from ours. Our range of "α"s is more restricted. In [2], the non existence of such self-similar solutions is established provided

- For all $t \in (0,T)$ the particle trajectory mapping $X(\cdot,t)$ generated by the classical solution $u \in C([0,T); C^1(\mathbb{R}^3;\mathbb{R}^3))$ is a C^1 diffeomorphism from $\mathbb{R}^3$ onto itself.
- The vorticity satisfies $\Omega = curl U \neq 0$, and there exists $p_1 > 0$ such that $\Omega \in L^p(\mathbb{R}^3)$ for all $p \in (0, p_1)$.

In the case of [2] Chae needs conditions on the vorticity that we are not requiring. Our results also complement the results in [3].

We return now to the proof of Theorem 2. Note first that if $\alpha = 3/2$, then $(\lambda,\mu) \notin \mathcal{A}$, since then

$$\frac{\lambda'}{\lambda^2\mu} = \frac{\lambda'}{\lambda^{7/2}}$$

hence

$$(\lambda,\mu) = (a_1(T_o - t)^{-2/5}, a_2(T_o - t)^{-3/5}) \notin \mathcal{A}.$$

This case will be dealt in the next theorem with an additional hypothesis.
Let $\alpha \in (-1, 3/2)$. Multiply equation (2.5) by U and integrate in space,

$$C_1\int |U|^2 dx + C_2\int U(y\cdot)\nabla U dx = \int UU(\cdot\nabla)U dx + \int U\nabla P dx \tag{2.14}$$

Since div $U = 0$ it follows that the right hand side of the last equation vanishes. We analyze first the case when $\alpha \in [0, 3/2)$, that is sign C_1 = sign C_2, (otherwise multiply the equation by -1), hence without loss of generality we can take both constants to be positive. Integrating by parts the left hand side and reordering yields

$$\int |U|^2 dx \left[C_1 - \frac{3}{2}C_2\right] = 0$$

It is clear that if $\alpha = \frac{C_1}{C_2} \neq 3/2$ we have $\|U\|_2^2 = 0$ and hence $U = 0$. In the second case when sign $C_1 = -$sign C_2 , it follows immediately from (2.14) that $U = 0$. This concludes Case 1.

Case 2
Now we suppose that neither $\frac{\mu'}{\mu^2\lambda}$ nor $\frac{\lambda'}{\mu\lambda^2}$ are constants. Then the sum of the first and second term on the left hand side of (2.5) needs to add up to a function which

depends solely of the variable x. To analyze this case we multiply the equation (2.5) by U

$$\frac{\mu'}{\mu^2\lambda}\int |U|^2 dx + \frac{\lambda'}{\mu\lambda^2}\int U(y\cdot)\nabla U dx = \int UU(\cdot\nabla)U dx + \int U\nabla P dx \tag{2.15}$$

Here again, the Right hand Side vanishes, due to the divergence free condition. Integrating by parts yields

$$\int |U|^2 dx \left[\frac{\mu'}{\lambda\mu^2} - \frac{3}{2}\frac{\lambda'}{\lambda^2\mu}dx\right] = 0 \tag{2.16}$$

Hence either $U = 0$ and we are done, or the functions $\lambda(t)$ and $\mu(t)$, satisfy

$$\frac{\mu'}{\lambda\mu^2} - \frac{3}{2}\frac{\lambda'}{\lambda^2\mu} = 0 \tag{2.17}$$

From where it follows that

$$\mu = K_o\Lambda^{3/2}$$

and hence as before $(\lambda,\mu) = (a_1(T_o - t)^{-2/5}, a_2(T_o - t)^{-3/5}) \notin \mathcal{A}$, where a_1, is an arbitrary constant and, $a_2 + K_o a_!^{3/2}$. As stated before for this case we need an additional hypothesis. This concludes the second case and the proof of the Theorem.

□

We now need to analyze the case we have left out in the last theorem

Theorem 2.3. *There are no pseudo-self-similar solutions $u \in C([0,T_o);H^3) \cap C^1([0,T_o);H^2)$ of the 3D-Euler equations (1.3), that satisfy (2.7) in Theorem 2 where*

$$(\lambda,\mu) = (a_1(T_o - t)^{-2/5}, a_2(T_o - t)^{-3/5}),$$

where $a_i, i = 1,2$, and one of the additional conditions

1. *ess $sup_{0<t<T_o}\|u(\cdot,t)\|_{L^p(\mathbb{R}^3)} < \infty$, for at least one $p > 5$.*
2. $\lim_{t\to T_o}\int_0^t \|\nabla u(x,t)\|_2 ds < \infty$,

holds.

Proof: We note that

$$\left(\int |u(x,t)|^p dx\right)^{1/p} = \mu(t)\lambda^{-3/p}\left(\int |U(x)|^p dx\right)^{1/p} = \frac{C}{(T_o - t)^{(3/5-3/p)}}$$

Since $p > 5$, the right hand side of the last equation tends to infinity as $T \to T_o$ contradicting the first condition in the hypothesis, hence no such solutions can exist.

We note that the second condition is a natural one for solutions to the Navier-Stokes equations. It turns out that the combination of this condition with the BKM conditions yields only zero pseudo-self-similar solutions. The second condition yields

$$\lim_{t\to T_o}\int_0^t \|\nabla u(x,t)\|_2 ds = \lim_{t\to T_o}\int_0^t \mu(t)\lambda(t)\|\nabla U\|_2 ds.$$

Since $\mu(t)\lambda(t) = K_o(T_o - t)^{-1}$, which is not integrable on $[0,T_o]$, contradicting the second condition. The conclusion of the Theorem follows.

□

In the next theorem we show that the pseudo-self-similar solutions which belong to two different L^p spaces turn out to be of self-similar solutions of the type found in [2]. They reduce to the zero solution if we suppose BKL and the second condition of last Theorem.

Theorem 2.4. *There are no pseudo-self-similar solutions* $u \in C([0,T_o);H^3) \cap C^1([0,T_o);H^2)$ *of the 3D-Euler equations (1.3), that satisfy (2.8) in Theorem 2 and the second condition in Theorem 2.3 and, which in addition belong to* $L^p \cap L^q$*, with* $p \neq q$*.*

Proof: We are going to suppose neither p or q are equal to one or two. If one of the two is equal to it follows from the two last Theorems. For $p = 1$ (or $q = 1$) it follows simply multiplying by sign u and integrating by parts and then by the same process as for Theorems 2 and 2.3. Now suppose that $u \in L^p(\mathbb{R}^3) \cap L^q(\mathbb{R}^3)$ with $p, q \neq 1, 2$. Let $C_m = (\int_{\mathbb{R}^3} |U|^{m-2} U \nabla P dx)(\|U\|_m)^{-m}, m = p, q$. Multiply the Euler equations (1.3) by $|U|^{m-2}U$ first with $m = q$, then with $m = p$. An easy computation yields

$$\frac{\mu'}{\lambda\mu^2} - \frac{3}{q}\frac{\lambda'}{\lambda^2\mu} = C_q, \tag{2.18}$$

$$\frac{\mu'}{\lambda\mu^2} - \frac{3}{p}\frac{\lambda'}{\lambda^2\mu} = C_p. \tag{2.19}$$

From the last two equationsit follows that

$$\frac{\lambda'}{\lambda^2\mu} = \frac{pq(C_q - C_p)}{3(p-q)} = K_1,$$

$$\frac{\mu'}{\lambda\mu^2} = \frac{(qC_q - pC_p)}{p-q} = K_2,$$

hence it is easy to show that

$$\mu(t) = K\lambda^\beta, \text{ with } \beta = \frac{K_2}{K_1}.$$

Replacing this μ in equation (2.18) yields

$$\frac{\lambda'}{\lambda^{\beta+2}} = C_{pq},$$

where C is a constant that depends on p and q. Integrating this, using (2.8) yields

$$\lambda = C_{p,q}(T-t)^{-(\beta+1)}.$$

Hence it follows that

$$\mu(t) = \tilde{C}_{p.q}(T-t)^{\frac{-\beta}{\beta+1}}.$$

Since $\mu(t)\lambda(t) = C(T-t)^{-1}$, it follows that the second condition of Theorem 2.3 does not hold. Hence the solutions reduce to the zero solution.

□

Remark 2.5. The case were we had to add an extra hypothesis is considered by the results in [2], where the extra hypothesis needed is that the curl U belongs to all $L^p(\mathbb{R}^3)$ spaces for $p \in (0, p_1)$ some $p_1 > 0$. It would be interesting to show that these self-similar solutions do not exist with just the conditions supposed in Theorem 2. Or show that they do not exist using similar methods as the one used in [2], but requiring the condition on the vorticity for $p \in [1, p_1)$ with $p_1 > 1$.

References

[1] J. T. Beale, T. Kato and A. Majda, *Remarks on the breakdown of smooth solutions for the 3-D Euler equations*, Comm. Math. Phys., 94, (1984), pp. 61-66.

[2] D. Chae *Nonexistence of self-similar singularities for the 3D incompressible Euler equations* Comm. Math. Phys. 273 (2007), no. 1, 203215

[3] Xinju He *Self-Similar Singularities of the 3D Euler Equations* Applied Mathematics Letters 13 (2000) 41-46

[4] J. Leray,*Essai sur le mouvement dun fiuide visqueux emplissant lespace*, Acta Math. 63 (1934), pp. 193-248.

[5] J. Malek, J. Necas, M. Pocorny, M. Schonbek *On the possible singular solutions to the Navier-Stokes equations, Mathematische Nachrichten*, 199, (1999) 97-114.

[6] Marsden, J., Ebin, D., Fischer, A. E.*Diffeomorphism groups, hydrodynamics and relativity.* Proceedings of the thirteenth biennial seminar of the Canadian Math. Congress, Vanstone, J. R. (ed.,) Montreal, 1972

[7] J. Miller, M. OLeary, M.Schonbek *Nonexistence of Singular Pseudo-self-similar Solutions of the Navier-Stokes System*, Mathematiche Annalen, 319, 809- 815, (2001).

[8] J. Nečas, M. Ružička and V. Šverák, *On Lerays self-similar solutions of the Navier-Stokes equations*, Acta Math., 176, (1996), pp. 283-294.

[9] T-P. Tsai, *On Lerays self-similar solutions of the Navier-Stokes equations satisfying local energy estimates*, Arch. Rat. Mech. Anal.,443, no. 1, (1998), pp. 29-51.

[10] R. Temam *Local existence of C^∞ solutions of the Euler equations of incompressible perfect fluids.* Lecture Notes in Mathematics, Vol. 565, Berlin, Heidelberg, New York: Springer 184-194 (1976)

Department of MAthematics, UC Santa Cruz, CA 95064, USA
E-mail address: schonbek@ucsc.edu