ASYMPTOTIC BEHAVIOR OF SOLUTIONS TO THE LIQUID CRYSTALS SYSTEM IN $\mathbb{R}^3$

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Abstract. In this paper we study the large time behavior of solutions to a nematic liquid crystals system in the whole space $\mathbb{R}^3$. The fluid under consideration has constant density and small initial data.

1. Introduction

In this paper we consider the asymptotic behavior of solutions to the simplified model of nematic liquid crystals (LCD) with constant density:

$$\begin{aligned} u_t + u \cdot \nabla u + \nabla p &= \nu \triangle u - \nabla \cdot (\nabla d \otimes \nabla d), \\ d_t + u \cdot \nabla d &= \triangle d - f(d), \\ \nabla \cdot u &= 0. \end{aligned} \tag{1.1}$$

The equations are considered in $\mathbb{R}^3 \times (0, T)$. Here $p : \mathbb{R}^3 \times [0, T] \to \mathbb{R}$ is the fluid pressure, $u : \mathbb{R}^3 \times [0, T] \to \mathbb{R}^3$ is the fluid velocity and $d : \mathbb{R}^3 \times [0, T] \to \mathbb{R}^3$ is the direction field representing the alignment of the molecules. The constant $\nu > 0$ stands for the viscosity coefficient. Without loss of generality, by scaling, we can set $\nu = 1$. The force term $\nabla d \otimes \nabla d$ in the equation of the conservation of momentum denotes the 3×3 matrix whose ij-th entry is given by "$\nabla_i d \cdot \nabla_j d$" for $1 \le i, j \le 3$. This force $\nabla d \otimes \nabla d$ is the stress tensor of the energy about the direction field d, where the energy is given by:

$$\frac{1}{2} \int_{\mathbb{R}^3} |\nabla d|^2 dx + \int_{\mathbb{R}^3} F(d) dx$$

where

$$F(d) = \frac{1}{4\eta^2} (|d|^2 - 1)^2, \quad f(d) = \nabla F(d) = \frac{1}{\eta^2} (|d|^2 - 1) d,$$

for a constant η in this paper. We note that $F(d)$ is the penalty term of the Ginzburg-Landau approximation of the original free energy of the direction field with unit length.

In this paper we consider the following initial conditions:

$$u(x, 0) = u_0(x), \quad \nabla \cdot u_0 = 0, \tag{1.2}$$

$$d(x, 0) = d_0(x), \quad |d_0(x)| = 1, \tag{1.3}$$

and

$$u_0 \in H^1(\mathbb{R}^3), \quad d_0 - w_0 \in H^2(\mathbb{R}^3), \tag{1.4}$$

with a fixed vector $w_0 \in S^2$, i.e., $|w_0| = 1$.

The flow of nematic liquid crystals can be treated as slow moving particles where the fluid velocity and the alignment of the particles influence each other. The hydrodynamic theory of liquid crystals was established by Ericksen [7, 8] and Leslie [16, 17] in the 1960's. As Leslie points out in his 1968 paper: "liquid crystals are states of matter which are capable of flow, and in which the molecular arrangements give rise to a preferred direction". There is a vast literature on the hydrodynamic of liquid crystal systems. For background we list a few, with no intention to be complete: [9, 13, 14, 19, 20, 18, 1, 2, 3, 4, 28, 22, 12]. In particular, the asymptotic behavior of solutions to the flow of nematic liquid crystals was studied for bounded domains in [19, 28]. It was shown in [28] that, with suitable initial conditions, the velocity converges to zero and the direction field converges to the steady solution to the following equation

$$\begin{cases} -\Delta d + f(d) = 0, x \in \Omega \\ d(x) = d_0(x), x \in \partial\Omega. \end{cases} \tag{1.5}$$

In [28], Lemma 2.1 the Łojasiewicz-Simon inequality is used to derive the convergence when Ω is a bounded domain. Lack of compactness considerations do not allow us to use similar arguments in the whole space $\mathbb{R}^3$.

In this paper we consider the asymptotic behavior of the solutions to (1.1) in the whole space $\mathbb{R}^3$. subject to the additional condition on the direction field which insures that the direction tends to a constant unit vector w_0, as the space variable tends to infinity:

$$\lim_{|x|\to\infty} d_0(x) = w_0. \tag{1.6}$$

This simplifies the situation and allows us to obtain the stability without needing the Liapunov reduction and Łojasiewicz-Simon inequality, since w_0 is a nondegenerate steady solution to (1.5).

We start from the basic energy estimates (2.16) and Ladyzhenskaya estimates (2.17) [15, 6] (see the extension to the whole space in appendix of this paper) for the system (1.1). We then establish the convergence of the direction field d to the constant steady solution w_0 based on Gagliardo-Nirenberg interpolation techniques. More precisely, the convergence obtained is in $L^p(\mathbb{R}^3)$ for any $p > 1$, with an algebraic decay rate of $(1+t)^{-\frac{3}{2}(1-\frac{1}{p})}$. We then focus on the conservation of momentum equation in (1.1). We apply the Fourier splitting technique [23, 24, 27] to obtain the decay of the velocity u with an algebraic decay rate of $(1+t)^{-\frac{1}{4}}$ in $L^2(\mathbb{R}^3)$ norm. This rate coincides with the decay rate of solutions to Navier-Stokes equations with a force decaying at a rate of $(1+t)^{-\frac{3}{4}}$ [23].

The existence of global regular solutions of (1.1) with the initial and bounadry conditions has been established in [19] (in [6] for nonconstant density) provided that the viscosity constant is large enough or initial data are small enough. Based on the arguments in [6] the existence of global regular solutions of (1.1), for small initial data, is established in the appendix as follows:

Theorem 1.1. *Let u_0 and d_0 satisfy (1.2)-(1.4). Assume that $u_0 \in H^1(\mathbb{R}^3)$ and $d_0 - w_0 \in H^2(\mathbb{R}^3) \cap L^1(\mathbb{R}^3)$ for a unit vector w_0. There is a positive small number ϵ_0 such that if*

$$\|u_0\|^2_{H^1(\mathbb{R}^3)} + \|d_0 - w_0\|^2_{H^2(\mathbb{R}^3)} \le \epsilon_0, \tag{1.7}$$

then the system (1.1) has a classical solution (u,p,d) *in the time period* $(0,T)$*, for all* $T>0$*. That is, for some* $\alpha\in(0,1)$

$$\begin{aligned} u &\in C^{1+\alpha/2,2+\alpha}((0,T)\times\mathbb{R}^3) \\ \nabla p &\in C^{\alpha/2,\alpha}((0,T)\times\mathbb{R}^3) \\ d &\in C^{1+\alpha/2,2+\alpha}((0,T)\times\mathbb{R}^3). \end{aligned} \tag{1.8}$$

And the solution (u,p,d) *satisfies the following basic energy estimate and higher order energy estimate (also called Ladyzhenskaya energy estimate in* [6] *and* [19]*)*

$$\begin{aligned} &\int_{\mathbb{R}^3}|u|^2+|\nabla d|^2+2F(d)dx+2\int_0^T\int_{\mathbb{R}^3}|\nabla u|^2+|\Delta d-f(d)|^2dxdt \\ &\le \|u_0\|^2_{L^2(\mathbb{R}^3)}+\|\nabla d_0\|^2_{L^2(\mathbb{R}^3)} \end{aligned} \tag{1.9}$$

$$\begin{aligned} &\int_{\mathbb{R}^3}|\nabla u|^2+|\Delta d|^2dx+\int_0^T\int_{\mathbb{R}^3}|\Delta u|^2+|\nabla\Delta d|^2dxdt \\ &\le C(\|u_0\|^2_{H^1(\mathbb{R}^3)}+\|d_0-w_0\|^2_{H^2(\mathbb{R}^3)}). \end{aligned} \tag{1.10}$$

Furthermore, the solution d *satisfies*

$$\int_{\mathbb{R}^3}|d(t)-w_0|dx\le (C_0t+\int_{\mathbb{R}^3}|d_0-w_0|dx)e^{Ct} \tag{1.11}$$

with the constants C_0 *and* C *depending only on initial data and on* η*, respectively.*

For the smooth solution obtained in Theorem 1.1, our main asymptotic result is:

Theorem 1.2. *Let* (u,p,d) *be smooth solution obtained in Theorem 1.1. Assume additionally* $u_0\in L^1(\mathbb{R}^3)$ *and* $d_0-w_0\in L^p(\mathbb{R}^3)$*, for any* $p\ge 1$ *and a unit vector* w_0*. There exists a small number* $\epsilon_0>0$ *such that if*

$$\|u_0\|^2_{H^1(\mathbb{R}^3)}+\|d_0-w_0\|^2_{H^2(\mathbb{R}^3)}\le\epsilon_0, \tag{1.12}$$

then

$$\|d(\cdot,t)-w_0\|_{L^p(\mathbb{R}^3)}\le C\|d_0-w_0\|_{L^p(\mathbb{R}^3)}(1+t)^{-\frac{3}{2}(1-\frac{1}{p})}, \tag{1.13}$$

$$\|\nabla(d(\cdot,t)-w_0)\|^2_{L^2(\mathbb{R}^3)}\le C(1+t)^{-\frac{3}{4}}, \tag{1.14}$$

$$\|u(\cdot,t)\|^2_{L^2(\mathbb{R}^3)}\le C(1+t)^{-\frac{1}{2}}, \tag{1.15}$$

where the various constants C *only depend on initial data.*

The paper is organized as follows: in Section 2 we establish the decay for the difference $d-w_0$, using the basic energy estimate (1.9) and the Ladyzhenskaya energy estimate (1.10). Combining the decay of $d-w_0$ and Fourier splitting technique [23], in Section 3 we obtain an algebraic decay for the velocity u in $L^2(\mathbb{R}^3)$. In the appendix, we sketch a proof for the existence theorem 1.1.

2. Convergence of the direction field

In this section we study the L^p decay of the direction field $d-w_0$ and the decay for the first derivative. The first step is to derive a uniform estimate of $d-w_0$ in $L^2(\mathbb{R}^3)$. This yields a uniform estimate for $d-w_0$ in $L^p(\mathbb{R}^3)$ for any $p\geq 1$. This L^p estimate, is the basis to establish the decay results. For ease of reading we state the basic energy estimate and Ladyzhenskaya energy estimate satisfied by the smooth solution (u,p,d) (see appendix for details),

$$\begin{aligned}(2.16)\qquad &\|u\|_{L^2}^2+\|\nabla d\|_{L^2}^2+\|F(d)\|_{L^1}+2\int_0^T\|\nabla u\|_{L^2}^2+\|\Delta d-f(d)\|_{L^2}^2dt\\ &\leq\|u_0\|_{L^2}^2+\|\nabla d_0\|_{L^2}^2\end{aligned}$$

$$(2.17)\quad \|\nabla u\|_{L^2}^2+\|\Delta d\|_{L^2}^2+\int_0^T\|\Delta u\|_{L^2}^2+\|\nabla\Delta d\|_{L^2}^2dt\leq C(\|u_0\|_{H^1}^2+\|d_0-w_0\|_{H^2}^2).$$

In the sequel we need to use a Gagliardo-Nirenberg interpolation inequality. For completeness we recall from [11] the inequality here

Proposition 2.1. [11] *Let $w\in W^{m,p}(\mathbb{R}^n)\cap L^q(\mathbb{R}^n)$, for $1\leq p\leq\infty$ and $1\leq q\leq\infty$. Then*

$$(2.18)\qquad \|D^kw\|_{L^r(\mathbb{R}^n)}\leq C\|D^mw\|_{L^p(\mathbb{R}^n)}^a\|w\|_{L^q(\mathbb{R}^n)}^{1-a}$$

for any integer $k\in[0,m-1]$, where

$$(2.19)\qquad \frac{1}{r}=\frac{k}{n}+a(\frac{1}{p}-\frac{m}{n})+(1-a)\frac{1}{q}$$

with $a\in[\frac{k}{m},1]$, either if $p=1$ or $p>1$ and $m-k-\frac{n}{p}\notin\mathcal{N}\cup\{0\}$, while $a\in[\frac{k}{m},1)$, if $p>1$ and $m-k-\frac{n}{p}\in\mathcal{N}\cup\{0\}$.

2.1. Uniform estimate of $d-w_0$ in $L^2(\mathbb{R}^3)$.

In this part, we show that the integrals

$$\int_{\mathbb{R}^3}|d(x,t)-w_0|^2dx \text{ and } \int_0^T\int_{\mathbb{R}^3}|\nabla(d(x,t)-w_0)|^2dxdt$$

are uniformly bounded by the initial data, applying estimates for the solutions obtained in Theorem 1.1. We have the following lemma,

Lemma 2.2. *Let d be the solution obtained in Theorem 1.1. There exists ϵ_0 sufficiently small so that if $\|u_0\|_{L^2(\mathbb{R}^3)}+\|\nabla d_0\|_{L^2(\mathbb{R}^3)}\leq\epsilon_0$. Then*

$$(2.20)\qquad \int_{\mathbb{R}^3}|d(x,t)-w_0|^2dx+\int_0^T\int_{\mathbb{R}^3}|\nabla(d(x,t)-w_0)|^2dxdt\leq C,$$

where the constant C depends on the initial data and the norm $\|d_0-w_0\|_{L^2(\mathbb{R}^3)}$.

Proof: Since $|w_0|=1$ and $f(w_0)=\frac{1}{\eta^2}(|w_0|^2-1)w_0=0$, the second equation in (1.1) can be expressed as

$$(2.21)\qquad (d-w_0)_t+u\cdot\nabla(d-w_0)=\Delta(d-w_0)-f(d)+f(w_0).$$

Applying the mean value theorem for vector valued functions, we have

$$f(d) - f(w_0) = \left(\int_0^1 Df(w_0 + s(d - w_0))ds \right) \cdot (d - w_0), \tag{2.22}$$

where Df denotes the Jacobian matrix of f. Multiplying (2.21) by $d - w_0$ yields

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3} |d - w_0|^2 dx \\ &= -\int_{\mathbb{R}^3} [u \cdot \nabla(d - w_0)](d - w_0)dx \\ &+ \int_{\mathbb{R}^3} \Delta(d - w_0) \cdot (d - w_0)dx \\ &- \int_{\mathbb{R}^3} (d - w_0)^T \left(\int_0^1 Df(w_0 + s(d - w_0))ds \right) (d - w_0)dx, \\ &\equiv -I_4 + I_5 + I_6. \end{aligned} \tag{2.23}$$

The three terms I_4, I_5 and I_6 are estimated as follows:

$$\begin{aligned} |I_4| &= |\lim_{R\to\infty} \int_{B_R} [u \cdot \nabla(d - w_0)](d - w_0)dx| \\ &= |\lim_{R\to\infty} \frac{1}{2} \int_{\partial B_R} |d - w_0|^2 u \cdot n d\sigma| \\ &\leq C \lim_{R\to\infty} \left(\int_{\partial B_R} |d - w_0|^4 d\sigma \right)^{1/2} \left(\int_{\partial B_R} |u|^2 d\sigma \right)^{1/2} \\ &\leq C \lim_{R\to\infty} \left(\int_{\partial B_R} |d - w_0| d\sigma \right)^{1/2} \left(\int_{\partial B_R} |u|^2 d\sigma \right)^{1/2}. \end{aligned} \tag{2.24}$$

Denote the set $A = \left\{ R : \int_{\partial B_R} |d - w_0| d\sigma \geq M \right\}$, for a certain constant $M > 0$. A is closed and its complement A^c is open. Write

$$\int_{\mathbb{R}^3} |d(t) - w_0| dx = \int_0^\infty \int_{\partial B_R} |d(t) - w_0| d\sigma dR.$$

Recall that by Theorem (1.1) the last expression is bounded by the initial data, for any fixed time $t > 0$. Thus by Chebyshev's inequality we have

$$\mu\{A\} \leq \frac{C}{M} \tag{2.25}$$

where μ denotes the measure of a set, and C denotes a constant depending only on the initial data. Since the energy estimate (2.16) implies that

$$\int_{\mathbb{R}^3} |u|^2 dx = \int_0^\infty \int_{\partial B_R} |u|^2 d\sigma dR$$

is bounded by initial data, there exists a sequence $\{R_i\}_{i=1}^\infty \subset A^c$ with $R_i \to \infty$ such that

$$\int_{\partial B_{R_i}} |u|^2 d\sigma \to 0 \tag{2.26}$$

Combining the inequalities (2.24), (2.25) and (2.26), yields for all $t > 0$,

$$I_4 = 0. \tag{2.27}$$

For I_5, we have

$$I_5 = \lim_{R\to\infty}[\int_{\partial B_R} \frac{\partial(d-w_0)}{\partial n}\cdot(d-w_0)d\sigma - \int_{B_R}|\nabla(d-w_0)|^2dx] \tag{2.28}$$

The boundary term is estimated as follows:

$$\begin{aligned}&\int_{\partial B_R}\frac{\partial(d-w_0)}{\partial n}\cdot(d-w_0)d\sigma \\ &\le \left(\int_{\partial B_R}|\nabla(d-w_0)|^2d\sigma\right)^{1/2}\left(\int_{\partial B_R}|d-w_0|^2d\sigma\right)^{1/2} \\ &\le C\left(\int_{\partial B_R}|\nabla(d-w_0)|^2d\sigma\right)^{1/2}\left(\int_{\partial B_R}|d-w_0|d\sigma\right)^{1/2}\end{aligned} \tag{2.29}$$

From Theorem 1.1, we have that $\|\nabla(d-w_0)\|_{L^2(\mathbb{R}^3)}$ is uniformly bounded for any fixed $t>0$, $\|d(t)-w_0\|_{L^1(\mathbb{R}^3)}$ has a time dependent bound. For the inequality (2.29), we apply a similar argument used to derive (2.27) and obtain the existence of a sequence R_i approaching infinity satisfying

$$\lim_{R_i\to\infty}\int_{\partial B_{R_i}}\frac{\partial(d-w_0)}{\partial n}\cdot(d-w_0)d\sigma \to 0$$

It follows then that

$$I_5 = -\int_{\mathbb{R}^3}|\nabla(d-w_0)|^2dx. \tag{2.30}$$

With respect to I_6, let $\tilde{d} = w_0 + s(d-w_0)$ and $z = d - w_0$. Using the definition of $f(d) = \frac{1}{\eta^2}(|d|^2-1)d$, a straightforward calculation yields

$$\begin{aligned}&(d-w_0)^T\left(\int_0^1 Df(w_0+s(d-w_0))ds\right)(d-w_0) = z^T\int_0^1 Df(\tilde{d})dsz \\ &= \int_0^1[(2\tilde{d}_1^2+|\tilde{d}|^2-1)z_1^2+(2\tilde{d}_2^2+|\tilde{d}|^2-1)z_2^2+(2\tilde{d}_3^2+|\tilde{d}|^2-1)z_3^2 \\ &+4\tilde{d}_1\tilde{d}_2z_1z_2+4\tilde{d}_1\tilde{d}_3z_1z_3+4\tilde{d}_2\tilde{d}_3z_2z_3]ds \\ &= \int_0^1[2\tilde{d}_1^2z_1^2+2\tilde{d}_2^2z_2^2+2\tilde{d}_3^2z_3^2+4\tilde{d}_1\tilde{d}_2z_1z_2+4\tilde{d}_1\tilde{d}_3z_1z_3 \\ &+4\tilde{d}_2\tilde{d}_3z_2z_3]+(|\tilde{d}|^2-1)(z_1^2+z_2^2+z_3^2)ds \\ &= \int_0^1 2[\tilde{d}_1z_1+\tilde{d}_2z_2+\tilde{d}_3z_3]^2+(|\tilde{d}|^2-1)|z|^2ds \\ &= \int_0^1 2[\tilde{d}\cdot z]^2+(|\tilde{d}|^2-1)|z|^2ds.\end{aligned}$$

In the above equation, the third and forth equality comes from regrouping terms and completing a perfect square. Thus, I_6 can be written as

$$\begin{aligned}
I_6 &= -\frac{1}{\eta^2}\int_{\mathbb{R}^3}\int_0^1 2[(w_0+s(d-w_0))\cdot(d-w_0)]^2 \\
&+(|w_0+s(d-w_0)|^2-1)|d-w_0|^2 dsdx \\
&= -\frac{1}{\eta^2}\int_{\mathbb{R}^3} 2[w_0\cdot(d-w_0)]^2+3w_0\cdot(d-w_0)|d-w_0|^2+|d-w_0|^4dx \\
&= -\frac{1}{\eta^2}\int_{\mathbb{R}^3} 2[w_0\cdot(d-w_0)+\frac{3}{4}|d-w_0|^2]^2-\frac{1}{8}|d-w_0|^4dx \\
&\le \frac{1}{8\eta^2}\int_{\mathbb{R}^3}|d-w_0|^4dx.
\end{aligned} \tag{2.31}$$

Combining (2.23) and the inequalities (2.27), (2.30), and (2.31) gives

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3}|d-w_0|^2dx+\int_{\mathbb{R}^3}|\nabla(d-w_0)|^2dx \le \frac{1}{8\eta^2}\int_{\mathbb{R}^3}|d-w_0|^4dx. \tag{2.32}$$

The right hand side of the inequality (2.32) can be estimated as

$$\begin{aligned}
\int_{\mathbb{R}^3}|d-w_0|^4dx &\le (\int_{\mathbb{R}^3}|d-w_0|^2dx)^{1/2}(\int_{\mathbb{R}^3}|d-w_0|^6dx)^{1/2} \\
&\le C(\int_{\mathbb{R}^3}|d-w_0|^2dx)^{1/2}(\int_{\mathbb{R}^3}|\nabla(d-w_0)|^2dx)^{3/2} \\
&\le C\int_{\mathbb{R}^3}|d-w_0|^2dx\left(\int_{\mathbb{R}^3}|\nabla(d-w_0)|^2dx\right)^2+\frac{1}{2}\int_{\mathbb{R}^3}|\nabla(d-w_0)|^2dx.
\end{aligned}$$

Gagliardo-Nirenberg interpolation inequality (Proposition 2.1) yields

$$\|\nabla(d-w_0)\|^2_{L^2(\mathbb{R}^3)} \le C\|d-w_0\|_{L^2(\mathbb{R}^3)}\|\Delta(d-w_0)\|_{L^2(\mathbb{R}^3)}.$$

Combining the last two inequalities with (2.32) gives

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^3}|d-w_0|^2dx+\int_{\mathbb{R}^3}|\nabla(d-w_0)|^2dx \\
&\le C\left(\int_{\mathbb{R}^3}|d-w_0|^2dx\right)^2\int_{\mathbb{R}^3}|\Delta(d-w_0)|^2dx.
\end{aligned} \tag{2.33}$$

Denote $\phi(t)=\int_{\mathbb{R}^3}|d(t)-w_0|^2dx$. Then

$$\frac{d\phi}{\phi^2}\le C\int_{\mathbb{R}^3}|\Delta(d-w_0)|^2dxdt.$$

Integrating the last inequality over $[0,t]$ yields

$$-\frac{1}{\phi(t)}+\frac{1}{\phi(0)}\le C\int_0^t\int_{\mathbb{R}^3}|\Delta(d-w_0)|^2dxdt.$$

Thus,

$$\phi(t)\le\frac{\phi(0)}{1-C\phi(0)\int_0^t\int_{\mathbb{R}^3}|\Delta(d-w_0)|^2dxdt}.$$

From the basic energy estimate (2.16), and the hypothesis we have

$$\int_0^t \int_{\mathbb{R}^3} |\Delta(d-w_0)|^2 dxdt \le \|u_0\|_{L^2}^2 + \|\nabla d_0\|_{L^2}^2 \le \epsilon_0.$$

Assume that ϵ is so small that $C\epsilon_0\phi(0) < 1/2$, then

$$C\phi(0)\int_0^t \int_{\mathbb{R}^3} |\Delta(d-w_0)|^2 dxdt < \frac{1}{2}.$$

Hence for any $t > 0$,

$$\phi(t) \le 2\phi(0),$$

that is,

$$\int_{\mathbb{R}^3} |d(t)-w_0|^2 dx \le 2\int_{\mathbb{R}^3} |d_0-w_0|^2 dx. \tag{2.34}$$

Due to the estimates (2.34) and (2.33), we have

$$\frac{d}{dt}\int_{\mathbb{R}^3} |d-w_0|^2 dx + \int_{\mathbb{R}^3} |\nabla(d-w_0)|^2 dx \le C\int_{\mathbb{R}^3} |\Delta(d-w_0)|^2 dx,$$

where the constant C only depends on the initial data. Integrating over $[0,t]$, by the basic energy inequality (2.16) it follows that

$$\begin{aligned}
&\int_{\mathbb{R}^3} |d(t)-w_0|^2 dx + \int_0^t \int_{\mathbb{R}^3} |\nabla(d-w_0)|^2 dxdt \\
&\le \int_{\mathbb{R}^3} |d_0-w_0|^2 dx + C\int_0^t \int_{\mathbb{R}^3} |\Delta(d-w_0)|^2 dxdt \\
&\le C.
\end{aligned}$$

Thus,

$$\int_0^t \int_{\mathbb{R}^3} |\nabla(d-w_0)|^2 dxdt \le C, \tag{2.35}$$

where the constant C only depends on initial data. This completes the proof of the lemma.

□

The following auxiliary estimate shows that, provided the initial data is small enough, the norm $\|d(\cdot,t)-w_0\|_{L^\infty(\mathbb{R}^3)}$ will be as small as necessary. This smallness yields that $|d|$ will be close to 1, for all time.

Lemma 2.3. *Let d be the solution obtained in Theorem 1.1. Then*

$$\|d(\cdot,t)-w_0\|_{L^\infty(\mathbb{R}^3)} \le C\|\nabla d(t)\|_{L^2(\mathbb{R}^3)}^{1/2}\|\Delta d(t)\|_{L^2(\mathbb{R}^3)}^{1/2}, \tag{2.36}$$

where C is an absolute constant.

Proof: Let $\mathcal{F}$ denote the Fourier transform. By Lemma 2.2, we can take the Fourier transform of $d - w_0$

(2.37)

$$\begin{aligned}
&\|d(\cdot,t) - w_0\|_{L^\infty(\mathbb{R}^3)} \le \int_{\mathbb{R}^3} |\mathcal{F}(d-w_0)| d\xi \\
&= \int_{|\xi|\le\lambda} \mathcal{F}(d-w_0)|d\xi + \int_{|\xi|\ge\lambda} |F(d-w_0)|d\xi \\
&= \int_{|\xi|\le\lambda} \frac{1}{|\xi|}\cdot|\xi||F(d-w_0)|d\xi + \int_{|\xi|\ge\lambda} \frac{1}{|\xi|^2}\cdot|\xi^2||\mathcal{F}(d-w_0)|d\xi \\
&\le (\int_{|\xi|\le\lambda} \frac{1}{|\xi|^2} d\xi)^{\frac{1}{2}} (\int_{|\xi|\le\lambda} |\xi||\mathcal{F}(d-w_0)|^2 d\xi)^{\frac{1}{2}} + (\int_{|\xi|\ge\lambda} \frac{1}{|\xi|^4} d\xi)^{\frac{1}{2}} (\int_{|\xi|\ge\lambda} |\xi^2\mathcal{F}(d-w_0)|^2 d\xi)^{\frac{1}{2}} \\
&\le C(\int_{r\le\lambda} \frac{r^2}{r^2} dr)^{\frac{1}{2}} (\int_{\mathbb{R}^3} |\nabla(d-w_0)|^2 dx)^{\frac{1}{2}} + C(\int_{r\ge\lambda} \frac{r^2}{r^4} dr)^{\frac{1}{2}} (\int_{\mathbb{R}^3} |\Delta(d-w_0)|^2 dx)^{\frac{1}{2}} \\
&\le C\lambda^{\frac{1}{2}} \|\nabla d(t)\|_{L^2(\mathbb{R}^3)} + C\lambda^{-\frac{1}{2}} \|\Delta d(t)\|_{L^2(\mathbb{R}^3)}.
\end{aligned}$$

To find a λ that minimizes the right hand side of the last inequality, take the derivative in λ and set the right hand side equal to zero:

$$\lambda^{\frac{1}{2}} \|\nabla d(t)\|_{L^2(\mathbb{R}^3)} = \lambda^{-\frac{1}{2}} \|\Delta d(t)\|_{L^2(\mathbb{R}^3)},$$

yielding

$$\lambda = \|\Delta d(t)\|_{L^2(\mathbb{R}^3)} / \|\nabla d(t)\|_{L^2(\mathbb{R}^3)}.$$

Thus using this λ in (2.37) gives inequality (2.36), and the proof of the Lemma is complete.

□

Corollary 2.4. Suppose the initial data $\|u_0\|_{H^1} + \|d_0 - w_0\|_{H^2}$ are small enough. Then, $|d(x,t)| \ge \frac{1}{2}$.

Proof: It follows combining (2.16), (2.17), (2.36) since $|w_0| = 1$.

□

2.2. **Uniform estimate of $d - w_0$ in L^p with any $p \ge 1$.** Here we show provided the data is small enough, all the L^p norms of $d - w_0$ are bounded.

Lemma 2.5. *Let d be the solution obtained in Theorem 1.1. There exist λ_p, depending on p, so that if $\|u_0\|^2_{H^1(\mathbb{R}^3)} + \|d_0 - w_0\|^2_{H^2(\mathbb{R}^3)} \le \lambda_p$, then for $p > 1$*

$$\begin{aligned}
\frac{1}{p}\int_{\mathbb{R}^3} |d(x,t) - w_0|^p dx + \frac{2(p-1)}{p^2} \int_0^T \int_{\mathbb{R}^3} |\nabla |d(x,t) - w_0|^{p/2}|^2 dx dt \\
\le C_p \int_{\mathbb{R}^3} |d_0 - w_0|^p dx,
\end{aligned} \tag{2.38}$$

where the constant C_p depends on p and λ_p. And for $p = 1$ we have

$$\begin{aligned}
\int_{\mathbb{R}^3} |d - w_0| dx &\le \int_{\mathbb{R}^3} |d_0 - w_0| dx + \int_0^T \int_{\mathbb{R}^3} |\nabla(d-w_0)|^2 + |\Delta(d-w_0)|^2 dx dt \\
&\le C.
\end{aligned}$$

Proof: Recall that since $f(w_0)=0$, we have that the direction equation can be rewritten as

$$(d-w_0)_t + u\cdot\nabla(d-w_0) = \Delta(d-w_0) - f(d) + f(w_0).$$

Multiplying the last equation by $(d-w_0)|d-w_0|^{p-2}$, for any $p\geq 2$ (or alternatively by $(d-w_0)/(|d-w_0|+\epsilon)^{2-p}$, when $p\in[1,2)$, and letting $\epsilon\to 0$) yields

$$\begin{aligned}
\frac{1}{p}\frac{d}{dt}\int_{\mathbb{R}^3}&|d-w_0|^p dx \\
&= -\int_{\mathbb{R}^3}[u\cdot\nabla(d-w_0)](d-w_0)|d-w_0|^{p-2}dx \\
&+ \int_{\mathbb{R}^3}\Delta(d-w_0)(d-w_0)|d-w_0|^{p-2}dx \\
&- \int_{\mathbb{R}^3} 2[w_0\cdot(d-w_0)+\frac{3}{4}|d-w_0|^2]^2|d-w_0|^{p-2} - \frac{1}{8}|d-w_0|^{p+2}dx \\
&\equiv I_7 + I_8 + I_9,
\end{aligned} \tag{2.39}$$

where the I_9 was obtained similarly as in the previous calculation for I_6. We estimate I_7, I_8 and I_9 as follows:
Integrating by parts over ball B_R gives

$$I_7 = \lim_{R\to\infty}[\int_{\partial B_R}|d-w_0|^p u\cdot n d\sigma - (p-1)\int_{B_R}[u\cdot\nabla(d-w_0)](d-w_0)|d-w_0|^{p-2}dx].$$

It implies that

$$\begin{aligned}
pI_7 &= \lim_{R\to\infty}\int_{\partial B_R}|d-w_0|^p u\cdot n d\sigma \\
&\leq \lim_{R\to\infty}\left(\int_{\partial B_R}|d-w_0|^2 d\sigma\right)^{1/2}\left(\int_{\partial B_R}|u|^2 d\sigma\right)^{1/2}
\end{aligned} \tag{2.40}$$

for any $p\geq 1$, where we used that $|d-w_0|\leq C$.
By Lemma 2.2 we know that $\int_{\mathbb{R}^3}|d-w_0|^2dx$ is bounded, and $\int_{\mathbb{R}^3}|u|^2dx$ is bounded from the energy estimate (2.16). Thus, for the inequality (2.40), using arguments similar to the ones applied to derive the convergence (2.27), will yield

$$\int_{\partial B_{R_i}}|d-w_0|^p u\cdot n d\sigma \to 0$$

for an appropriate sequence $R_i\to\infty$, for any $p\geq 1$. Thus,

$$I_7 = 0. \tag{2.41}$$

For I_8, integrating by parts over B_R yields

$$\begin{aligned}
I_8 &= \lim_{R\to\infty}[\int_{\partial B_R}\frac{\partial(d-w_0)}{\partial n}\cdot(d-w_0)|d-w_0|^{p-2}d\sigma \\
&- \int_{B_R}|\nabla(d-w_0)|^2|d-w_0|^{p-2}dx \\
&- (p-2)\int_{B_R}|\nabla(d-w_0)\cdot(d-w_0)|^2|d-w_0|^{p-4}dx] \\
&\equiv \lim_{R\to\infty}(K_1-K_2-K_3).
\end{aligned} \tag{2.42}$$

The boundary term K_1 is estimated as

$$K_1 = \int_{\partial B_R} \frac{\partial(d-w_0)}{\partial n} \cdot (d-w_0)|d-w_0|^{p-2} d\sigma \tag{2.43}$$
$$\leq \int_{\partial B_R} |\nabla(d-w_0)||d-w_0|^{p-1} d\sigma$$
$$\leq \int_{\partial B_R} |\nabla(d-w_0)| d\sigma$$

where we used that $|d-w_0|^{p-1} \leq C$ for any $p \geq 1$.
In Proposition (2.1), let $k=1$, $m=2$, $r=1$, $p=2$ and $q=1$. For $a=\frac{2}{7}$ the inequality (2.18) yields

$$\int_{\mathbb{R}^3} |\nabla(d(t)-w_0)| dx$$
$$\leq C \left(\int_{\mathbb{R}^3} |\Delta(d(t)-w_0)| dx \right)^{1/7} \left(\int_{\mathbb{R}^3} |d(t)-w_0| dx \right)^{5/7}$$
$$\leq C_0 (C(t))^{5/7},$$

where we used Ladyzhenskaya estimate (1.10) and the estimate (1.11), C_0 is a constant depending on initial data and $C(t)$ is the time dependent function in (1.11). Thus, for any fixed $t>0$,

$$\int_{\partial B_{R_i}} |\nabla(d-w_0)| d\sigma \to 0$$

for a sequence $R_i \to \infty$. Hence, from (2.43), we have

$$\lim_{R\to\infty} K_1 = 0. \tag{2.44}$$

Since

$$-|\nabla(d-w_0)|^2 |d-w_0|^2 \leq -|\nabla(d-w_0)\cdot(d-w_0)|^2,$$

we have

$$-|\nabla(d-w_0)|^2 |d-w_0|^{p-2} \leq -|\nabla(d-w_0)\cdot(d-w_0)|^2 |d-w_0|^{p-4},$$

which implies from (2.42) that

$$-K_2 - K_3 \leq (p-1) \int_{B_R} -|\nabla(d-w_0)\cdot(d-w_0)|^2 |d-w_0|^{p-4} dx \tag{2.45}$$
$$= -\frac{4(p-1)}{p^2} \int_{B_R} |\nabla |d-w_0|^{\frac{p}{2}}|^2 dx.$$

Combining (2.42), (2.44) and (2.45) yields

$$I_8 \leq -\frac{4(p-1)}{p^2} \int_{\mathbb{R}^3} |\nabla |d-w_0|^{\frac{p}{2}}|^2 dx \tag{2.46}$$

for any $p \geq 1$. Slightly modifying the process to estimate I_6 gives

$$I_9 \leq \frac{1}{8\eta^2} \int_{\mathbb{R}^3} |d-w_0|^{p+2} dx. \tag{2.47}$$

Combining (2.39) with inequalities (2.41), (2.46), and (2.47) yields

$$\begin{aligned}\frac{1}{p}\frac{d}{dt}\int_{\mathbb{R}^3}|d-w_0|^p dx+(p-1)\int_{\mathbb{R}^3}|\nabla(d-w_0)|^2|d-w_0|^{p-2}dx\\ \leq C\int_{\mathbb{R}^3}|d-w_0|^{p+2}dx.\end{aligned}\tag{2.48}$$

Denote $v=|d-w_0|^{p/2}$. Then (2.48) can be rewritten as

$$\frac{1}{p}\frac{d}{dt}\int_{\mathbb{R}^3}|v|^2dx+\frac{4(p-1)}{p^2}\int_{\mathbb{R}^3}|\nabla v|^2dx\leq C\int_{\mathbb{R}^3}|v|^2|d-w_0|^2dx.\tag{2.49}$$

Applying Hölder inequality and Sobolev inequality to the right hand side yields

$$\begin{aligned}\int_{\mathbb{R}^3}|v|^2|d-w_0|^2dx&\leq\left(\int_{\mathbb{R}^3}|v|^6dx\right)^{1/3}\left(\int_{\mathbb{R}^3}|d-w_0|^3dx\right)^{2/3}\\ &\leq C\int_{\mathbb{R}^3}|\nabla v|^2dx\left(\int_{\mathbb{R}^3}|d-w_0|^2dx\right)^{2/3}\|d-w_0\|^{2/3}_{L^\infty(\mathbb{R}^3)}\end{aligned}$$

From Lemma 2.2 and Lemma 2.3, when $p>1$, we can choose initial data small enough so that

$$C\left(\int_{\mathbb{R}^3}|d-w_0|^2dx\right)^{2/3}\|d-w_0\|^{2/3}_{L^\infty(\mathbb{R}^3)}\leq\frac{2(p-1)}{p^2}.\tag{2.50}$$

It follows from (2.49) that

$$\frac{1}{p}\frac{d}{dt}\int_{\mathbb{R}^3}|v|^2dx+\frac{2(p-1)}{p^2}\int_{\mathbb{R}^3}|\nabla v|^2dx\leq 0.\tag{2.51}$$

The inequality (2.38), for $p>1$, can be obtained integrating (2.51) over $[0,T]$.

When $p=1$, we have from (2.48)

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}^3}|d-w_0|dx&\leq C\int_{\mathbb{R}^3}|d-w_0|^3dx\\ &\leq C\int_{\mathbb{R}^3}|d-w_0|^2dx\|d-w_0\|_{L^\infty(\mathbb{R}^3)}\\ &\leq C\int_{\mathbb{R}^3}|\nabla d|^2+|\Delta d|^2dx\end{aligned}$$

where we used Lemma 2.2 and Lemma 2.3. Integrating over time $[0,T]$ yields

$$\begin{aligned}&\int_{\mathbb{R}^3}|d-w_0|dx\\ &\leq\int_{\mathbb{R}^3}|d_0-w_0|dx+C\int_0^T\int_{\mathbb{R}^3}|\nabla d|^2+|\Delta d|^2dxdt\leq C\end{aligned}$$

where we used the energy estimate (2.16). This completes the proof of the lemma. □

2.3. **Decay of $d - w_0$ and $\nabla(d - w_0)$.** We first establish that $d - w_0$ decays in $L^p(\mathbb{R}^3)$, for $p > 1$ at the rate $(1+t)^{-\frac{3}{2}(1-\frac{1}{p})}$.

Theorem 2.6. *Let d be the solution obtained in Theorem 1.1. Assume $d - w_0 \in L^p(\mathbb{R}^3)$, $p \geq 1$. Assume the initial data satisfies the conditions of Lemma 2.5. Then for any $1 < p < \infty$, $t > 0$*

$$\|d(\cdot,t) - w_0\|_{L^p(\mathbb{R}^3)} \leq C(1+t)^{-\frac{3}{2}(1-\frac{1}{p})}, \tag{2.52}$$

where the constant C depends on λ_p as defined in Lemma 2.5.

Proof: Note that as $p \to \infty$ the constants λ_p in Lemma 2.5 will tend to zero, hence we cannot pass to the limit as $p \to \infty$. Therefore this result does not give the decay for the L^∞ norm.
We proceed by induction for k with $p = 2^k$. The other powers p follow by interpolation. When $k = 0$ the theorem follows by Lemma 2.5. Suppose it holds for $s = k$, then we have

$$\|d(\cdot,t) - w_0\|_{L^{2^k}(\mathbb{R}^3)} \leq C(1+t)^{-\frac{3}{2}(1-\frac{1}{2^k})}, \tag{2.53}$$

Let $v = |d(\cdot,t) - w_0|^{2^k}$.

Recall the inequality (2.51) (which holds provided the data satisfies (2.50))

$$\frac{d}{dt}\int_{\mathbb{R}^3} |v|^2 dx + C_p \frac{2(p-1)}{p} \int_{\mathbb{R}^3} |\nabla v|^2 dx \leq 0, \tag{2.54}$$

By Gagliardo-Nirenberg we have

$$\int_{\mathbb{R}^3} |v|^2 dx \leq C \left(\int |\nabla v|^2 dx \right)^{3/5} \left(\int |v| dx \right)^{4/5}$$

Hence using the inductive hypothesis on the last integral on the right hand side we have

$$-\int |\nabla v|^2 dx \leq -C \left(\int_{\mathbb{R}^3} |v|^2 dx \right)^{5/3} (1+t)^{2(2^k-1)}$$

Combining the last inequality with (2.54) yields

$$\frac{d}{dt} \frac{\int_{\mathbb{R}^3} |v|^2 dx}{\left(\int_{\mathbb{R}^3} |v|^2 dx\right)^{5/3}} \leq -(1+t)^{2(2^k-1)}$$

Integrating and reordering terms yields

$$\int_{\mathbb{R}^3} |v|^2 dx \leq \left[\frac{v_0}{(1+(1+t))^{2(2^k-1)+1}} \right]^{3/2}$$

Since $\frac{3}{2}(2(2^k-1)+1) = \frac{3}{2}(2^{k+1}-1)$ the induction step is obtained, establishing the conclusion of the theorem.

□

As a consequence of the last theorem, we derive the decay of $\nabla(d - w_0)$.

Corollary 2.7. Let d be the solution to system (1.1) obtained in Theorem 1.1. Then

$$\|\nabla(d - w_0)\|_{L^2}^2 \leq C(1+t)^{-\frac{3}{4}},$$

where C depends on initial data.

Proof: Take $p = 2$ in Theorem 2.6,

$$\|d(\cdot,t) - w_0\|_{L^2(\mathbb{R}^3)} \leq C\|d_0 - w_0\|_{L^2(\mathbb{R}^3)}(1+t)^{-\frac{3}{4}}. \tag{2.55}$$

Gagliardo-Nirenberg inequality (2.18) yields

$$\begin{aligned}\int_{\mathbb{R}^3} |\nabla(d-w_0)|^2 dx &\leq C\left(\int_{\mathbb{R}^3} |d-w_0|^2 dx\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^3} |\Delta(d-w_0)|^2 dx\right)^{\frac{1}{2}} \\ &\leq C\left(\int_{\mathbb{R}^3} |d-w_0|^2 dx\right)^{\frac{1}{2}} \\ &\leq C(1+t)^{-\frac{3}{4}},\end{aligned}$$

where in the last two steps we used Ladyzhenskaya energy estimate (2.17) and (2.55), respectively. The constant C depends on initial data. It completes the proof.

□

3. Decay of Velocity

An application of the Fourier Splitting method [23] is used to establish L^2 decay of velocity u.

Theorem 3.1. *Let u be the solution obtained in Theorem 1.1. If additionally $u_0 \in L^1(\mathbb{R}^3)$, then*

$$\|u(\cdot,t)\|_{L^2}^2 \leq C(1+t)^{-\frac{1}{2}},$$

where C depends on initial data, the L^1 and L^2 norm of u_0.

Proof: Multiplying the Navier-Stokes equation in system (1.1) by u and integrating by parts yields

$$\frac{d}{dt}\int_{\mathbb{R}^3} |u|^2 + 2\int_{\mathbb{R}^3} |\nabla u|^2 dx = 2\int_{\mathbb{R}^3} \nabla u(\nabla d \otimes \nabla d) dx. \tag{3.56}$$

Hölder and Cauchy Schwartz inequalities yield

$$2\int_{\mathbb{R}^3} \nabla u(\nabla d \otimes \nabla d) dx \leq \int_{\mathbb{R}^3} |\nabla u|^2 dx + C\int_{\mathbb{R}^3} |\nabla d \otimes \nabla d|^2 dx.$$

Thus, we derive from (3.56)

$$\frac{d}{dt}\int_{\mathbb{R}^3} |u|^2 + \int_{\mathbb{R}^3} |\nabla u|^2 dx \leq C\int_{\mathbb{R}^3} |\nabla d \otimes \nabla d|^2 dx.$$

The right hand side of above inequality can be estimated as

$$\begin{aligned}\int_{\mathbb{R}^3} |\nabla d \otimes \nabla d|^2 dx &= \int_{\mathbb{R}^3} (\nabla d \otimes \nabla d)(\nabla d \otimes \nabla d) dx \\ &= -3\int_{\mathbb{R}^3} (d-w_0)\, \Delta d\, \nabla d \otimes \nabla d \\ &\leq \frac{1}{2}\int_{\mathbb{R}^3} |\nabla d \otimes \nabla d|^2 dx + C\int_{\mathbb{R}^3} |d-w_0|^2 |\Delta d|^2 dx,\end{aligned}$$

from which it follows

$$\begin{aligned}\int_{\mathbb{R}^3} |\nabla d \otimes \nabla d|^2 dx &\le C \int_{\mathbb{R}^3} |d - w_0|^2 |\Delta d|^2 dx \\ &\le C \left(\int_{\mathbb{R}^3} |d - w_0|^p dx \right)^{\frac{2}{p}} \left(\int_{\mathbb{R}^3} |\Delta d|^{\frac{2p}{p-2}} dx \right)^{\frac{p-2}{p}} \\ &= C \|d - w_0\|^2_{L^p(\mathbb{R}^3)} \left(\int_{\mathbb{R}^3} |\Delta d|^{2+\frac{4}{p-2}} dx \right)^{\frac{p-2}{p}} \\ &\le C \|d - w_0\|^2_{L^p(\mathbb{R}^3)},\end{aligned}$$

for $p \ge 2$, where the last step followed from Ladyzhenskaya estimate (2.17) and the fact that $\|\Delta d\|_{L^\infty(\mathbb{R}^3 \times [0,T])}$ is bounded since d is regular in the sense stated in Theorem 1.1. Thus, it follows from Theorem 2.6 that

$$\int_{\mathbb{R}^3} |\nabla d \otimes \nabla d|^2 dx \le C(1+t)^{-3(1-\frac{1}{p})},$$

for any $p \ge 2$. Therefore,

$$\frac{d}{dt} \int_{\mathbb{R}^3} |u|^2 dx + \int_{\mathbb{R}^3} |\nabla u|^2 dx \le C(1+t)^{-3(1-\frac{1}{p})}. \tag{3.57}$$

Applying Plancherel's theorem to (3.57) gives

$$\frac{d}{dt} \int_{\mathbb{R}^3} |\hat{u}|^2 d\xi + \int_{\mathbb{R}^3} |\xi|^2 |\hat{u}|^2 d\xi \le C(1+t)^{-3(1-\frac{1}{p})}. \tag{3.58}$$

The idea is to decompose the frequency domain $\mathbb{R}^3$ in integral $\int_{\mathbb{R}^3} |\xi|^2 |\hat{u}|^2 d\xi$ into two time-dependent subdomains. The time dependent subdomains are a 3-dimensional sphere, $S(t)$, centered at the origin with an appropriate time dependent radius and its complement. For this we rewrite (3.58) as

$$\frac{d}{dt} \int_{\mathbb{R}^3} |\hat{u}|^2 d\xi \le - \int_{S(t)^c} |\xi|^2 |\hat{u}|^2 d\xi - \int_{S(t)} |\xi|^2 |\hat{u}|^2 d\xi + C(1+t)^{-3(1-\frac{1}{p})},$$

where $S(t)$ is the ball

$$S(t) = \{\xi \in \mathbb{R}^3 : |\xi| \le r(t) = (\frac{k}{1+t})^{1/2}\}$$

for a certain k, which will be determined below. Hence

$$\begin{aligned}\frac{d}{dt} \int_{\mathbb{R}^3} |\hat{u}|^2 d\xi &\le - \frac{k}{1+t} \int_{S(t)^c} |\hat{u}|^2 d\xi - \int_{S(t)} |\xi|^2 |\hat{u}|^2 d\xi + C(1+t)^{-3(1-\frac{1}{p})} \\ &= - \frac{k}{1+t} \int_{\mathbb{R}^3} |\hat{u}|^2 d\xi + \int_{S(t)} (\frac{k}{1+t} - |\xi|^2) |\hat{u}|^2 d\xi + C(1+t)^{-3(1-\frac{1}{p})}\end{aligned}$$

and

$$\frac{d}{dt} \int_{\mathbb{R}^3} |\hat{u}|^2 d\xi + \frac{k}{1+t} \int_{\mathbb{R}^3} |\hat{u}|^2 d\xi \le \frac{k}{1+t} \int_{S(t)} |\hat{u}|^2 d\xi + C(1+t)^{-3(1-\frac{1}{p})}. \tag{3.59}$$

The following estimate, which will be established later, is needed

$$|\hat{u}(\xi, t)| \le C|\xi|^{-1} \tag{3.60}$$

for $\xi \in S(t)$, where C is a constant only depending on the initial data. Combining the inequalities (3.59) and (3.60) yields

$$\frac{d}{dt}\int_{\mathbb{R}^3}|\hat{u}|^2 d\xi + \frac{k}{1+t}\int_{\mathbb{R}^3}|\hat{u}|^2 d\xi \le \frac{C}{1+t}\int_{S(t)}|\xi|^{-2}d\xi + C(1+t)^{-3(1-\frac{1}{p})}.$$

Multiplying by the integrating factor $(1+t)^k$ yields

$$\frac{d}{dt}\left[(1+t)^k\int_{\mathbb{R}^3}|\hat{u}|^2 d\xi\right] \le C(1+t)^{k-1}\int_{S(t)}|\xi|^{-2}d\xi + C(1+t)^{k-3(1-\frac{1}{p})}.$$

Since $p \ge 2$ and

$$\int_{S(t)}|\xi|^{-2}d\xi \le C\int_0^{r(t)} r^2 r^{-2}dr \le C(1+t)^{-1/2},$$

it follows that

$$\frac{d}{dt}\left[(1+t)^k\int_{\mathbb{R}^3}|\hat{u}|^2 d\xi\right] \le C(1+t)^{k-\frac{3}{2}}.$$

Integrating in time yields

$$(1+t)^k\int_{\mathbb{R}^3}|\hat{u}|^2 d\xi \le \int_{\mathbb{R}^3}|\hat{u}(\xi,0)|^2 d\xi + C[(1+t)^{k-\frac{1}{2}} - 1].$$

Thus,

$$\int_{\mathbb{R}^3}|\hat{u}|^2 d\xi \le (1+t)^{-k}\int_{\mathbb{R}^3}|\hat{u}(\xi,0)|^2 d\xi + C[(1+t)^{-\frac{1}{2}} - (1+t)^{-k}].$$

Since $u_0 \in L^2$, it follows that $\hat{u}(0) \in L^2$ from Plancherel's theorem. Hence

$$\int_{\mathbb{R}^3}|\hat{u}|^2 d\xi \le C(1+t)^{-\frac{1}{2}},$$

Hence

$$\int_{\mathbb{R}^3}|u|^2 dx \le C(1+t)^{-\frac{1}{2}}.$$

To complete the proof we need to establish the inequality (3.60). Taking the Fourier transform of Navier-Stokes equation in system (1.1) yields

$$\hat{u}_t + |\xi|^2\hat{u} = G(\xi,t) \tag{3.61}$$

where

$$G(\xi,t) = -\mathcal{F}(u\cdot\nabla u) - \mathcal{F}(\nabla p) - \mathcal{F}(\nabla\cdot(\nabla d\otimes\nabla d)),$$

and $\mathcal{F}$ indicates the Fourier transform. Multiplying (3.61) by the integrating factor $e^{|\xi|^2 t}$ yields

$$\frac{d}{dt}[e^{|\xi|^2 t}\hat{u}] = e^{|\xi|^2 t}G(\xi,t).$$

Integrating in time gives

$$\hat{u}(\xi,t) = e^{-|\xi|^2 t}\hat{u}_0 + \int_0^t e^{-|\xi|^2(t-s)}G(\xi,s)ds. \tag{3.62}$$

We assume for the moment the following auxiliary estimate, which we will prove below,

$$|G(\xi,t)| \le C|\xi|. \tag{3.63}$$

Combining (3.62) and (3.63) yields

$$|\hat{u}(\xi,t)| \le e^{-|\xi|^2 t}|\hat{u}_0| + \int_0^t e^{-|\xi|^2(t-s)}|\xi|ds. \tag{3.64}$$

Since $u_0 \in L^1$, we have $|\hat{u}_0| \le C$ for all ξ and some constant C.Performing integration in (3.64) gives

$$|\hat{u}(\xi,t)| \le Ce^{-|\xi|^2 t} + \frac{C}{|\xi|}(1 - e^{-|\xi|^2 t}) \le C|\xi|^{-1}$$

for $\xi \in S(t)$. To finish the proof we need to establish (3.63). For this purpose we analyze each term in $G(\xi,t)$ separately. We have

$$|\mathcal{F}(u\cdot\nabla u)| = |\mathcal{F}(\nabla\cdot(u\otimes u))| \le \sum_{i,j}\int_{\mathbb{R}^3}|u^i u^j||\xi_j|dx.$$

Since $u \in L^\infty(L^2)$ by the basic energy estimate, we have

$$|\mathcal{F}(u\cdot\nabla u)| \le C|\xi|.$$

By the basic energy inequalities (2.16, 2.17) we have $\nabla d \in L^\infty(L^2)$ proceeding similarly as for the last inequality we have

$$|\mathcal{F}(\nabla\cdot(\nabla d\otimes\nabla d))| \le C|\xi|.$$

Taking divergence of Navier-Stokes equation in system (1.1) gives that

$$\Delta p = -\sum_{i,j}\frac{\partial^2}{\partial x_i\partial x_j}(u^i u^j) - \sum_{i,j}\frac{\partial^2}{\partial x_i\partial x_j}(\nabla d^i\nabla d^j).$$

Taking the Fourier transform then yields

$$|\xi|^2\mathcal{F}(p) = -\sum_{i,j}\xi_i\xi_j\mathcal{F}(u^i u^j) - \sum_{i,j}\xi_i\xi_j\mathcal{F}(\nabla d^i\nabla d^j).$$

Since $\mathcal{F}(u^i u^j) \in L^\infty$ and $\mathcal{F}(\nabla d^i\nabla d^j) \in L^\infty$, it follows that

$$\mathcal{F}(p) \le C,$$

and thus $\mathcal{F}(\nabla p) \le C|\xi|$. It completes the proof of (3.63) and hence completes the proof of theorem.

□

Combining Theorem 2.6, Corollary 2.7 and Theorem 3.1 yields the proof of the main Theorem 1.2.

□

Remark 3.2. The decay rate for the velocity u in L^2 obtained in [28], for the bounded domain case, is $(1+t)^{-\frac{\theta}{1-2\theta}}$ where $\theta \in (0, \frac{1}{2})$. When θ is close to 0, then $-\frac{\theta}{1-2\theta}$ would be very small, meaning the decay is very slow. In this paper, we obtained the decay rate for velocity u in L^2 with $(1+t)^{-\frac{1}{4}}$, a fixed constant algebraic rate. The advantage comes from the fact that we work on the whole space $\mathbb{R}^3$ where we can apply the Fourier splitting method.

Remark 3.3. It was pointed out in the first section that there is an essential difficulty to apply Łojasiewicz-Simon approach in whole space $\mathbb{R}^3$. However, in weighted Sobolev spaces of $\mathbb{R}^3$, the compactness is recovered. Thus, we expect there is hope to construct certain Łojasiewicz-Simon type inequality in weighted Sobolev spaces and proceed with the method in [28] to derive the decay of solutions to the LCD system in weighted Sobolev spaces.

Appendix A. Existence of Classical Solutions in $\mathbb{R}^3$

In this section we sketch a brief proof of the existence Theorem 1.1, Section 1. As mentioned in the introduction, for bounded domains in $\mathbb{R}^3$, the existence of global regular solutions to the flow of nematic liquid crystals with constant density has been established in [19] provided the viscosity is large enough. The existence of global regular solutions to the flow of nematic liquid crystals with non-constant density has been established in [6] provided the initial data is small enough. In both of the above papers, a Ladyzhenskaya energy estimate (higher order derivative estimate) was derived and hence a relatively standard bootstrapping argument yielded a regular solution.

The proof of Theorem 1.1 will be given through four steps. In the first step, on a sequence of balls B_{R_n} with radius R_n, centered at the origin, we obtain the existence of a Galerkin approximated solution $(u^{n,m}, d^{n,m})$ for the system (1.1) with modified initial data, for each $m = 1, 2, 3, ...$ In the second step, we establish an estimate of $d^{n,m} - w_0$ in $L^1(B_{R_n})$ for any fixed time $t > 0$. In the third step, we take the limit $m \to \infty$. In the forth step, we take the limit $R_n \to \infty$. In fact, we are able to show that all the estimates in Ladyzhenskaya energy method in step one are independent of the domain size. Thus we can take a subsequence of solutions on balls B_{R_n} which converge to a limit in $\mathbb{R}^3$ when R_n goes to infinity.

Lemma A.1. *Assume $u_0 \in H^1(\mathbb{R}^3)$ and $\tilde{d}_0 \equiv d_0(x) - w_0 \in H^2(\mathbb{R}^3) \cap L^1(\mathbb{R}^3)$ with $|w_0| = 1$. There exists a sequence of functions $\{(u_0^n, d_0^n)\}_{n=1}^\infty$ and a sequence of real numbers $\{R_n\}_{n=1}^\infty$ with $R_n \to \infty$ as $n \to \infty$ such that,*

$$u_0^n \in H_0^1(B_{R_n}), \text{ with } u_0^n \to u_0 \text{ in } H^1(\mathbb{R}^3) \text{ as } n \to \infty \tag{A.65}$$

$$\tilde{d}_0^n \in H_0^2(B_{R_n}), \text{ with } \tilde{d}_0^n \to \tilde{d}_0 \text{ in } H^2(\mathbb{R}^3) \text{ as } n \to \infty \tag{A.66}$$

where $\tilde{d}_0^n = d_0^n - w_0$. Moreover,

$$\|\tilde{d}_0^n\|_{L^1(B_{R_n})} \le C\|\tilde{d}_0\|_{L^1(\mathbb{R}^3)}, \tag{A.67}$$

and $|d_0^n| \le 1$.

Proof: Such a sequence of functions can be constructed easily as follows. Let ζ_n be a sequence of smooth functions such that

$$\zeta_n(x) = \begin{cases} 1, & \text{in } B_{R_n} \\ 0, & \text{in } B_{2R_n} \end{cases}$$

and $|\zeta_n(x)| \le 1$ for all $x \in \mathbb{R}^3$.
Define $u_0^n = \zeta_n u_0$ and $\tilde{d}_0^n = \zeta_n \tilde{d}_0$. Let $d_0^n = \tilde{d}_0^n + w_0$. These ζ_n can be chosen such that (A.65), (A.66) and (A.67) are satisfied. In addition, we have

$$|d_0^n| = |\zeta_n d_0 + (1-\zeta_n) w_0| \le \zeta_n + (1-\zeta_n) = 1.$$

□

For the sequel we assume as initial conditions

$$\begin{cases} u(x,0) = u_0^n(x), \\ d(x,0) = d_0^n(x), \end{cases}, \text{ in } B_{R_n} \times [0,T), \tag{A.68}$$

where u_0^n and d_0^n are as obtained in Lemma A.1, and the boundary conditions

$$\begin{cases} u(x,t) = 0, \\ d(x,t) - w_0 = 0, \end{cases}, \text{ on } \partial B_{R_n} \times [0,T). \tag{A.69}$$

We have the following existence of solutions to system (1.1) with (A.68) and (A.69) on each ball B_{R_n}.

Theorem A.2. *Let B_{R_n} be the ball centered at origin with radius R_n in $\mathbb{R}^3$. Assume that $u_0 \in H^1(\mathbb{R}^3)$ and $d_0 - w_0 \in H^2(\mathbb{R}^3)$. The system (1.1) with initial and boundary conditions (A.68) and (A.69) has a smooth solution $(u^{n,m}, p^{n,m}, d^{n,m})$ for each $m = 1,2,3,...$ satisfying, for any $T > 0$*

$$u^{n,m} \in L^2(0,T; H_0^1(B_{R_n})) \cap L^\infty(0,T; L^2(B_{R_n}))$$
$$\tilde{d}^{n,m} \in L^2(0,T; H_0^2(B_{R_n})) \cap L^\infty(0,T; H_0^1(B_{R_n}))$$

with $\tilde{d}^{n,m} = d^{n,m} - w_0$ and $|d^{n,m}| \le 1$. Moreover, it satisfies the energy inequality

$$\begin{aligned} &\frac{d}{dt}\int_{B_{R_n}} \frac{1}{2}|u^{n,m}|^2 + \frac{1}{2}|\nabla d^{n,m}|^2 + F(d^{n,m})dx \\ &+ \int_{B_{R_n}} |\nabla u^{n,m}|^2 + |\Delta d^{n,m} - f(d^{n,m})|^2 dx \le 0. \end{aligned} \tag{A.70}$$

Proof: The existence proof is obtained through the standard Galerkin approximation method, see [19] and [6]. We only need to give a brief explanation on the claim $|d^{n,m}| \le 1$ by applying a maximum principle argument. Notice that the approximated initial data d_0^n satisfies $|d_0^n(x)| \le 1$ for all $x \in \mathbb{R}^3$. Suppose there exists a point (x_0, t_0) in the interior of the domain $B_{R_n} \times [0,T)$, such that $|d^{n,m}|^2$ attains a maximum value at this point. Multiplying the equation

$$d_t^{n,m} + u^{n,m} \cdot \nabla d^{n,m} = \Delta d^{n,m} - \frac{1}{\eta^2}(|d^{n,m}|^2 - 1) d^{n,m}$$

by $d^{n,m}$ yields

$$\begin{aligned} &\frac{d}{dt}|d^{n,m}|^2 + u^{n,m} \cdot \nabla |d^{n,m}|^2 \cdot d^{n,m} \\ &= \triangle |d^{n,m}|^2 - 2|\nabla d^{n,m}|^2 - \frac{2}{\eta^2}(|d^{n,m}|^2 - 1)|d^{n,m}|^2. \end{aligned} \tag{A.71}$$

At the maximum point (x_0, t_0), we have $\frac{d}{dt}|d^{n,m}|^2 = \nabla |d^{n,m}|^2 = 0$ and $\triangle |d^{n,m}|^2 \le 0$. Thus, it follows from the equation (A.71) that, at the point (x_0, t_0)

$$(|d^{n,m}|^2 - 1)|d^{n,m}|^2 \le 0.$$

This insures that $|d^{n,m}| \leq 1$ at any interior maximum point (x_0, t_0). Therefore, $|d^{n,m}| \leq 1$ for all points in $B_{R_n} \times [0, T)$.

□

For the solution $(u^{n,m}, d^{n,m})$ obtained in the above theorem on $B_{R_n} \times [0, T)$, we define energy quantity

$$\Phi^2_{n,m}(t) = \|\nabla u^{n,m}\|^2_{L^2(B_{R_n})} + \|\triangle(d^{n,m} - w_0)\|^2_{L^2(B_{R_n})}.$$

With a slight modification of the proof of Theorem 3.1 in [6], we are able to show that

Theorem A.3. *Assume that $u_0 \in H^1(B_{R_n})$ and $d_0 \in H^2(B_{R_n})$, and $\|u_0\|^2_{H^1(\mathbb{R}^3)} + \|\tilde{d}_0\|^2_{H^2(\mathbb{R}^3)} < \infty$. Let $(u^{n,m}, d^{n,m})$ be solutions obtained in Theorem A.2. There is a positive small number ϵ_0 such that if*

$$\|u_0\|^2_{H^1(\mathbb{R}^3)} + \|d_0 - w_0\|^2_{H^2(\mathbb{R}^3)} \leq \epsilon_0, \tag{A.72}$$

then

$$\begin{aligned} &\int_{B_{R_n}} |\nabla u^{n,m}|^2 + |\Delta(d^{n,m} - w_0)|^2 dx \\ &+ \int_0^T \int_{B_{R_n}} |\Delta u^{n,m}|^2 + |\nabla\Delta(d^{n,m} - w_0)|^2 dx \\ &\leq C(\|u_0\|^2_{H^1(B_{R_n})} + \|d_0 - w_0\|^2_{H^2(B_{R_n})}), \end{aligned} \tag{A.73}$$

for any $T > 0$, where the constant C is independent of domain size R_n and m.

There is no need to prove the theorem except that we need a brief explanation on the last claim that constant C is independent of R_n. In the proof of Ladyzhenskaya energy estimate in [6], we only use the Gagliardo-Nirenberg interpolation inequalities and standard elliptic inequalities. That is we use

$$\|u^{n,m}\|^4_{L^4} \leq C\|u^{n,m}\|_{L^2}\|\nabla u^{n,m}\|^3_{L^2}$$

$$\|\nabla(d^{n,m} - w_0)\|^4_{L^4} \leq C\|\nabla(d^{n,m} - w_0)\|_{L^2}\|\Delta(d^{n,m} - w_0)\|^3_{L^2}$$

and the elliptic estimate

$$\|D^2 u^{n,m}\|_{L^2} \leq C\|\Delta u^{n,m}\|_{L^2}$$

for $u^{n,m}$ and $d^{n,m} - w_0$ vanishing on the boundary. In the above inequalities, the various constants C are independent of the size of the domain. Thus the constant C in (A.73) is independent of R_n.

For the second step, we derive a time dependent estimate of $d^{n,m}(t) - w_0$ in $L^1(B_{R_n})$.

Lemma A.4. *Let $d^{n,m}$ be the solution obtained in Theorem A.2. In addition, assume $d_0 - w_0 \in L^1(\mathbb{R}^3)$. Then*

$$\int_{B_{R_n}} |d^{n,m}(t) - w_0| dx \leq (C_0 t + \int_{\mathbb{R}^3} |d_0 - w_0| dx) e^{Ct} \tag{A.74}$$

where the constant C_0 only depends on initial data and constant C only depends on η.

Proof: By the second equation in (1.1) we have,

(A.75)

$$\begin{aligned}
&\frac{d}{dt}\int_{B_{R_n}} |d^{n,m}(t)-w_0|dx \\
&= \int_{B_{R_n}} \frac{(d^{n,m}(t)-w_0)\cdot d_t^{n,m}}{|d^{n,m}(t)-w_0|}dx \\
&= \int_{B_{R_n}} \frac{(d^{n,m}(t)-w_0)\cdot \Delta d^{n,m}(t)}{|d^{n,m}(t)-w_0|}dx - \int_{B_{R_n}} \frac{(d^{n,m}(t)-w_0)\cdot [u^{n,m}\cdot\nabla d^{n,m}(t)]}{|d^{n,m}(t)-w_0|}dx \\
&- \int_{B_{R_n}} \frac{(d^{n,m}(t)-w_0)\cdot f(d^{n,m}(t))}{|d^{n,m}(t)-w_0|}dx \\
&\equiv I_1 + I_2 + I_3.
\end{aligned}$$

There is no need to worry about the singular points of $(d^{n,m}-w_0)^{-1}$ in the above equation, since each term on the right hand side contains $\frac{d^{n,m}-w_0}{|d^{n,m}-w_0|}$. We deal with the three terms I_1, I_2 and I_3 in the following way.
Since $d^{n,m}(t)-w_0=0$ on the boundary ∂B_{R_n}. Replacing d_t by its value in the direction equation and integration by parts yields

$$\begin{aligned}
I_1 &= -\int_{B_{R_n}} \frac{|\nabla(d^{n,m}(t)-w_0)|^2}{|d^{n,m}(t)-w_0|}dx \\
&+ \int_{B_{R_n}} |(d^{n,m}(t)-w_0)\cdot\nabla(d^{n,m}(t)-w_0)|^2|d^{n,m}(t)-w_0|^{-3}dx \\
&\le 0,
\end{aligned} \tag{A.76}$$

since $|(d^{n,m}(t)-w_0)\cdot\nabla(d^{n,m}(t)-w_0)|^2 \le |(d^{n,m}(t)-w_0)|^2|\nabla(d^{n,m}(t)-w_0)|^2$.
By Hö lder inequality we have

$$\begin{aligned}
|I_2| &\le \int_{B_{R_n}} |u^{n,m}||\nabla(d^{n,m}-w_0)|dx \\
&\le C\left(\int_{B_{R_n}} |u^{n,m}|^2dx\right)^{1/2}\left(\int_{B_{R_n}} |\nabla d^{n,m}|^2dx\right)^{1/2} \le C_0,
\end{aligned} \tag{A.77}$$

where we used the energy estimate (A.70), and the constant C_0 only depends on the initial data.
Recall that by definition $f(d^{n,m}) = \frac{1}{\eta^2}(|d^{n,m}|^2-1)d^{n,m}$, and $|d^{n,m}| \le 1$ from Theorem A.2 and $|w_0|=1$, hence

$$\begin{aligned}
|I_3| &\le \frac{1}{\eta^2}\int_{B_{R_n}} |f(d^{n,m})|dx \\
&\le C\int_{B_{R_n}} |d^{n,m}-w_0||d^{n,m}+w_0||d^{n,m}|dx \le C\int_{B_{R_n}} |d^{n,m}-w_0|dx,
\end{aligned} \tag{A.78}$$

where the constant C depends on η. Combining the inequalities (A.75), (A.76), (A.77) and (A.78) yields

$$\frac{d}{dt}\int_{B_{R_n}} |d^{n,m}(t)-w_0|dx \le C\int_{B_{R_n}} |d^{n,m}(t)-w_0|dx + C_0.$$

Integrating over $[0,t]$, and Gronwall's inequality (see [10]) gives,

$$\int_{B_{R_n}} |d^{n,m}(t) - w_0| dx \le (C_0 t + \int_{B_{R_n}} |d_0^n - w_0| dx) e^{Ct},$$

for any $t > 0$. Inequality (A.74) now follows from the last estimate in Lemma A.1. This concludes the second step.

In the third step, we take the limit of the Galerkin approximating solutions $(u^{n,m}, d^{n,m})$ as $m \to \infty$. By the estimates (A.70) and (A.73), there exists (u^n, d^n) for each $n = 1, 2, 3, ...$ such that, taking subsequence if necessary,

$$\begin{gathered} u^{n,m} \rightharpoonup u^n \ \text{ weakly in } L^2(0,T; H_0^1(B_{R_n})), \\ u^{n,m} \to u^n \ \text{ strongly in } L^\infty(0,T; L^2(B_{R_n})), \\ d^{n,m} \rightharpoonup d^n \ \text{ weakly in } L^2(0,T; H^2(B_{R_n})), \\ d^{n,m} \to d^n \ \text{ strongly in } L^2(0,T; H^1(B_{R_n})) \text{ with } d^n = w_0 \text{ on } \partial B_{R_n}. \end{gathered}$$

It follows easily from the above convergence that (u^n, d^n) is a weak solution to the system (1.1) with initial condition (A.68) and boundary condition (A.69) on $B_{R_n} \times [0,T)$. Moreover, the solutions (u^n, d^n) satisfy the basic energy inequality

$$\begin{aligned} \frac{d}{dt} \int_{B_{R_n}} & \frac{1}{2}|u^n|^2 + \frac{1}{2}|\nabla d^n|^2 + F(d^n) dx \\ & + \int_{B_{R_n}} |\nabla u^n|^2 + |\Delta d^n - f(d^n)|^2 dx \le 0, \end{aligned} \tag{A.79}$$

and the higher order energy inequality

$$\begin{aligned} & \int_{B_{R_n}} |\nabla u^n|^2 + |\Delta (d^n - w_0)|^2 dx \\ & + \int_0^t \int_{B_{R_n}} |\Delta u^n|^2 + |\nabla \Delta (d^n - w_0)|^2 dx \\ & \le C(\|u_0\|^2_{H^1(B_{R_n})} + \|d_0 - w_0\|^2_{H^2(B_{R_n})}), \end{aligned} \tag{A.80}$$

for any $t > 0$, where the constant C is independent of domain size R_n. In addition, from Lemma A.4 it follows that d^n satisfies the estimate

$$\int_{B_{R_n}} |d^n(t) - w_0| dx \le (C_0 t + \int_{\mathbb{R}^3} |d_0 - w_0| dx) e^{Ct} \tag{A.81}$$

where the constant C_0 only depends on initial data and constant C only depends on η.

In the forth step, we extend the solutions (u^n, d^n) on B_{R_n} to the whole space $\mathbb{R}^3$ by taking limit $R_n \to \infty$. With the estimates (A.79) and (A.80) we can extract a subsequence $\left\{(u^{1k}, d^{1k})\right\}_{k=1}^{\infty}$ from (u^n, d^n) for $n \ge 1$ such that

$$\begin{gathered} u^{1k} \rightharpoonup u_{(1)} \text{ in } L^2(0,T; H^1(B_{R_1})) \\ u^{1k} \to u_{(1)} \text{ in } L^\infty(0,T; L^2(B_{R_1})) \\ d^{1k} \rightharpoonup d_{(1)} \text{ in } L^2(0,T; H^2(B_{R_1})) \\ d^{1k} \to d_{(1)} \text{ in } L^2(0,T; H^1(B_{R_1})) \end{gathered}$$

and the limit $(u_{(1)}, d_{(1)})$ satisfies system (1.1) in distribution sense and the estimates (A.79) (A.80) on $B_{R_1} \times [0,T)$.

On $B_{R_2} \times [0,T)$, we take subsequence $\{(u^{2k}, d^{2k})\}_{k=1}^{\infty}$ from $\{(u^{1k}, d^{1k})\}_{k=1}^{\infty}$ such that $\{u^{2k}\}_{k=1}^{\infty}$, $\{p^{2k}\}_{k=1}^{\infty}$ and $\{d^{2k}\}_{k=1}^{\infty}$ converge to $u_{(2)}$ and $d_{(2)}$ respectively in the same convergence sense as above. And we have that

$$u_{(2)}|_{B_1} = u_{(1)}, \quad d_{(2)}|_{B_1} = d_{(1)}.$$

Repeating the process on each $B_{R_n} \times [0,T)$, we can take subsequence $\{(u^{nk}, d^{nk})\}_{k=1}^{\infty}$ from the sequence $\{(u^{(n-1)k}, d^{(n-1)k})\}_{k=1}^{\infty}$, such that $\{u^{nk}\}_{k=1}^{\infty}$ and $\{d^{nk}\}_{k=1}^{\infty}$ converge to $u_{(n)}$ and $d_{(n)}$ respectively. And we have that

$$u_{(n)}|_{B_{n-1}} = u_{(n-1)}, \quad d_{(n)}|_{B_{n-1}} = d_{(n-1)}.$$

Then we take the diagonal sequence $\{(u^{kk}, d^{kk})\}_{k=1}^{\infty}$ and let $k \to \infty$. This sequence (if necessary, take a subsequence of it) converges to (u,d), in $\mathbb{R}^3 \times [0,T)$. The limit (u,d) satisfies the system (1.1) in the sense of distributions and satisfies the energy estimates

$$\begin{aligned} &\int_{\mathbb{R}^3} |u|^2 + |\nabla d|^2 + 2F(d)dx + 2\int_0^T \int_{\mathbb{R}^3} |\nabla u|^2 + |\Delta d - f(d)|^2 dxdt \\ &\leq \int_{\mathbb{R}^3} |u_0|^2 + |\nabla d_0|^2 dx \end{aligned} \tag{A.82}$$

$$\begin{aligned} &\int_{\mathbb{R}^3} |\nabla u|^2 + |\Delta d|^2 dx + \int_0^T \int_{\mathbb{R}^3} |\Delta u|^2 + |\nabla \Delta d|^2 dxdt \\ &\leq C(\|u_0\|^2_{H^1(\mathbb{R}^3)} + \|d_0 - w_0\|^2_{H^2(\mathbb{R}^3)}). \end{aligned} \tag{A.83}$$

In addition, the solution d satisfies the estimate

$$\int_{\mathbb{R}^3} |d(t) - w_0| dx \leq (C_0 t + \int_{\mathbb{R}^3} |d_0 - w_0| dx) e^{Ct}$$

with constant C_0 only depending on initial data and constant C only depending on η.

Estimates (A.82) and (A.83) allow us to apply the "bootstrapping argument" as used in [6] and [19], and prove that the limit (u,p,d) is a classical solution to system (1.1) satisfying the desired estimates in Theorem 1.1. This completes the proof of Theorem 1.1.

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