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Asymptotic Behavior of Solutions to the Three-Dimensional Navier-Stokes Equations

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1. Introduction. We consider the asymptotic behavior of solutions to the Navier-Stokes equations in three space dimensions

$$\text{(N–S)} \qquad \begin{aligned} u_t + u \cdot \nabla u + \nabla p &= \Delta u, \\ \operatorname{div} u &= 0. \end{aligned}$$

This paper is a continuation of work started in [9] where we established lower bounds to the solutions in two spatial dimensions and non-uniform lower bounds for solutions in three spatial dimensions to the Navier-Stokes equations. The purpose of this paper is to show that solutions to the three-dimensional Navier-Stokes equations also have uniform lower bounds of rates of decay.

More precisely, it is shown that if $u(x,t)$ is a Leray-Hopf solution to the three-dimensional Navier-Stokes equations in the sense of Cafarelli, Kohn and Nirenberg [1] with zero average initial data outside a class of functions of radially equidistributed energy, then there exist constants C_0 and C_1 depending only on norms of the initial data such that

$$C_0(t+1)^{-5/2} \le \|u(\cdot,t)\|_{L^2}^2 \le C_1(t+1)^{-5/2}.$$

We recall that in [7, 8] we showed that for non-zero average initial data there exist constants C_2, C_3 depending only on the L^2 and L^1 norms of the initial data such that

$$C_2(t+1)^{-3/2} \le \|u(\cdot,t)\|_{L^2}^2 \le C_3(t+1)^{-3/2}.$$

For initial data which have radially equidistributed energy, an extension of an example suggested by A. Majda for the two-dimensional Navier-Stokes shows that solutions can be constructed which decay exponentially, showing that the condition of radially equidistributed energy on the data is necessary.

In Section 2 standard notation is recalled and some theorems which were proved in [9] are stated. In Section 3 it is shown that the solutions to the thre-dimensional Navier-Stokes equations admit an algebraic uniform lower bound rate of decay.

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It is interesting to note that this uniform phenomenon of slow rate decay is not present at the level of the solutions to the heat equations. The presence of the nonlinear terms seems to produce some mixing of the Fourier modes creating long waves which eventually will introduce some extra dissipation slowing down the decay process. As a result, even highly oscillating data will produce solutions decaying at most at algebraic rate.

We expect that the proofs presented here can be extended to solutions in n space dimensions with $n > 3$, using the techniques introduced by Kayikisa and Miyakawa [4] and Wiegner [10].

2. Notation and some earlier results. In this section several theorems which will be needed in the remainder of the paper are stated. The proof of these theorems can be found in [9] and will be omitted. First some notation is given:

$$V(\mathbf{R}^n) = C_0^\infty(\mathbf{R}^n) \cap \{u : \nabla \cdot u = 0\}$$

$$H = H(\mathbf{R}^n) = \text{closure of } V \text{ in } L^2.$$

The following weighted spaces will be used:

$$W_1 = \left\{u : \int_{\mathbf{R}^n} |x|^2 \, |u| \, dx < \infty\right\}, \quad W_2 \left\{u : \int_{\mathbf{R}^2} |x| \, |u|^2 \, dx < \infty\right\},$$

$$|u|_{w_1} = \int_{\mathbf{R}^n} |x|^2 \, |u| \, dx, \quad |u|_{w_2} \left(\int_{\mathbf{R}^n} |x| \, |u|^2 \, dx\right)^{1/2}.$$

Note that if $u \in W_1 \cap W_2 \cap L^2$, then $\int |x| \, |u| \, dx < \infty$, since

$$\begin{aligned}
\int_{\mathbf{R}^n} |x| \, |u| \, dx &= \int_{|x|\le 1} |x| \, |u| \, dx + \int_{|x|\ge 1} |x| \, |u| \, dx \\
&\le \int_{|x|\le 1} \left(|x|^2 + |u|^2\right) \, dx + \int_{|x|\ge 1} |x|^2 \, |u| \, dx \\
&\le \infty.
\end{aligned}$$

The choice of weighted spaces ensures that the data has at least two Fourier derivatives in L^∞. Let $u \in \mathbf{R}^n$, $m_{ij} \in \int_{\mathbf{R}^n} u_i u_j \, dx$, define $M = \{u :$ matrix $(m_{ij}$ is scalar$\}$,

$$\alpha_i^j(to, u) = \int_0^{t_0} m_{ii} - m_{jj} \, ds,$$

$$\beta_i^j(to, u) = \int_0^{t_0} m_{ij} \, ds, \qquad i \ne j.$$

The next theorem and corollary give estimates on decay rate for solutions to the heat equations. Specifically, these results describe a class of initial data D for which solutions to the heat equation admit an algebraic lower bound on the L^2 decay rate.

Theorem 2.1. *Let $v_0 \in L^2(\mathbf{R}^n)$. Let v be a solution to the heat equation with data v_0. Suppose that there exist functions ℓ and h, such that the Fourier transform of v_0 for $|\xi| \le \delta$, $\delta > 0$, admits the representation*

$$\hat{v}_0(\xi) = \xi \cdot \ell(\xi) + h(\xi), \qquad \ell = (\ell_1, \ldots, \ell_n),$$

where ℓ and h satisfy the following conditions:

(i) $|h(\xi)| \le M_0|\xi|^2$, *for some $M_0 > 0$;*
(ii) *ℓ is homogeneous of degree zero;*
(iii) $\alpha_1 = \int_{|\omega|=1} |\omega \cdot \ell(\omega)|^2 \, d\omega > 0$.

Let

$$M_1 = \sup_{|y|=1} |\ell(y)|,$$
$$M_2 = \sup_{\delta/2 \le |y| \le 1} |\nabla \ell(y)|,$$
$$K = \max(M_0, M_1, M_2).$$

Then there exists constants C_0 and C_1 such that

$$C_0(t+1)^{-(n/2+1)} \le |v(\cdot, t)|_{L^2}^2 \le C_1(t+1)^{-(n/2+1)},$$

where C_0 and C_1 both depend on n, M_0, M_1, δ, and $|v_0|_{L^2}$ and C_0 also depends on K and α.

Proof. See [9]. □

Corollary 2.2. *Let v be a solution to the heat equation with data $v_0 \in L^2(\mathbf{R}^n)$ where v_0 has the Fourier representation described in Thereom 2.1 and ℓ and h satisfy* (i), (ii). *If, in addition, ℓ satisfies*

(1) $\omega_0 \cdot \ell(\omega_0) = \alpha \ne 0$, *for some $\omega_0 \in S^{n-1}$,*
(2) $\xi \cdot \ell(\xi) \in C^1(\mathbf{R}^n \setminus \{0\})$,

then the conclusion of Theorem 2.1 holds.

Proof. See [9]. □

The next theorems show that for initial data in some weighted spaces and outside a set of radially equidistributed energy, the Fourier transform of the corresponding solution to the Navier-Stokes equations will take the form described in Theorem 2.1 after a short time $t = t_0 \ge 0$. Hence the solution of the heat equation, starting with data $u(x, t_0)$, will have a lower bound of rate of decay. More precisely, there are two cases:

(i) the Fourier transform of initial data has a zero at the origin of order one,
(ii) the zero at the origin is of order greater than one.

Theorem 2.3. *Let $g \in H \cap W_1 \cap W_2(\mathbf{R}^n)$, $n = 2,3$. If $\hat{g}$ has a zero order at the origin, then there exists $\delta > 0$ such that for $|\xi| \leq \delta$,*

$$\hat{g}(\xi) = \xi \cdot \ell(\xi) + \ell(\xi),$$

where ℓ and h satisfy the hypothesis of Theorem 2.1, with $M_0 = \sup_{|x| \leq \delta} |\widehat{\nabla^2 g}(\xi)|$ and α depending only on $\widehat{\nabla g}(0)$.

Proof. See [9]. □

The case when the data g has a zero of order greater than one in Fourier space will be treated in Section 3.

The next theorem gives a comparison between decay rates of solutions to the Navier-Stokes equations and solutions to the heat equation.

Theorem 2.4. *Let $u_0 \in L^1 \cap W_2 \cap H(\mathbf{R}^n)$, $n = 2,3$. Let v be a solution to the heat equation with data u_0. Suppose*

$$C_0(1+t)^{-(n/2+1)} \leq \left|v(\cdot,t)\right|_{L^2}^2 \leq C_1(1+t)^{-(n/2+1)}.$$

For $n = 2$, let $u(\cdot,t)$ be a solution to the Navier-Stokes equations with data u_0. Then there exists constants M_0 and M_1 such that

$$M_0(1+t)^{-(n/2+1)} \leq \left|u(\cdot,t)\right|_{L^2}^2 \leq M_1(1+t)^{-(n/2+1)} \tag{1}$$

where M_0 and M_1 depend on C_1, n and the L^1 and L^2 norms of u_0 and M_0 depends also on C_0 and W_2 norm of u_0.

For $n = 3$; $u(x,t)$ is a Leray-Hopf solution in the sense of Caffarelli, Kohn and Nirenberg.

Proof. See [9]. □

We note that in [9] we only got the lower bound for almost all t. The reason being that the proof was applied to the approximating solution constructed by Caffarelli, Kohn and Nirenberg [1] and then passing to the limit and from the construction in [1] it is only apparent that the approximating solutions converge strongly to a Leray [3], Galdi [2], and Wiegner [11].

3. Radially equidistributed solutions and lower bounds. Here we show that solutions with smooth data g which are not radially equidistributed, i.e., $g \notin M$, will be such that $\alpha_i^j(t,u) \neq 0$ or $\beta_i^j(t,u) \neq 0$ at least for a short time t.

In order to show this, recall that if $g \in H^1$, then for a short time $u(x,t)$ will belong to H^1.

Theorem 3.1. *Let $g \in H^1$. Let $u(x,t)$ be a solution of the Navier-Stokes equations with data g. Then there exists a $t_0 > 0$ such that*

$$\left\|\nabla u(\cdot,t)\right\|_{L^2}^2 \leq C.$$

Proof. The proof can be found in several textbooks; see [5, 6]. For completeness we include a well-known formal outline, which can be made rigorous by applying it to approximating solutions and passing to the limit. This version of the proof was mentioned to be by E. Titi.

Multiply the Navier-Stokes equations Δu and integrate in space. After some integration by parts it follows that

$$\frac{d}{dt}\int_{\mathbf{R}^3} |\nabla u|^2\, dx + \int_{\mathbf{R}^3} |\Delta u|^2\, dx = \int_{\mathbf{R}^3} (u\nabla u)\Delta u\, dx;$$

hence

$$\frac{d}{dt}|\nabla u|^2\, dx + \int_{\mathbf{R}^3} |\Delta u|^2\, dx \leq \left(\int_{\mathbf{R}^3} (u\nabla u)\Delta u\, dx\right)^{1/2} \left(\int_{\mathbf{R}^3} |\Delta u|^2\, dx\right)^{1/2}.$$

Recall that by Agmon's inequality

$$\left|u(\cdot,t)\right|_{L^\infty} \leq C\left\|u\right\|_{L^2}^{1/4} \left\|\Delta u\right\|_{L^2}^{3/4}.$$

Hence

$$\begin{aligned}\frac{d}{dt}\int_{\mathbf{R}^3} |\nabla u|^2\, dx + \int_{\mathbf{R}^3} |\nabla u|^2\, dx &\leq C|u|_\infty \left(\int_{\mathbf{R}^3} |\nabla u|^2\, dx\right)^{1/2} \left(\int_{\mathbf{R}^3} |\nabla u|^2\, dx\right)^{1/2} \\ &\leq C\left\|u\right\|_{L^2}^{1/4} \left\|\Delta u\right\|_{L^2}^{7/4} \|\nabla u\|_{L^2}.\end{aligned}$$

By Young's inequality for a, $b > 0$

$$ab \leq \frac{(\varepsilon a)^p}{p} + \left(\frac{b}{\varepsilon}\right)^p \frac{1}{p'} \qquad\qquad \frac{1}{p} + \frac{1}{p'} = 1.$$

Let $\alpha = \left\|\Delta u\right\|_{L^2}^{7/4}$, $b = C\left\|u\right\|_{L^2}^{1/4} \|\nabla u\|_{L^2}$. Let $p = \frac{8}{7}$, $p' = 8$, $K = C^4\|u_0\|_{L^2/14}$. Here we used that $\|u\|_{L^2} \leq \|u_0\|_{L^2}$. Then the two inequalities yield

$$\frac{d}{dt}\int_{\mathbf{R}^3} |\nabla u|^2\, dx + \int_{\mathbf{R}^3} |\nabla u|^2\, dx \leq \frac{1}{2}\int_{\mathbf{R}^3} |\nabla u|^2\, dx + K\left(\int_{\mathbf{R}^3} |\nabla u|^2\, dx\right)^4.$$

Let $\omega(t) = \int |\nabla u|^2$. Hence

$$\frac{d}{dt}\omega \leq K\omega^4.$$

Standard ODE results imply that there exist t_0 such that for $t < t_0$

$$\omega(t) = \int_{\mathbf{R}^3} |\nabla u|^2 \leq M;$$

moreover, t_0 can be chosen to be $\left[K\omega(0)^3 2\right]^{-1}$, then $M = 2\omega(0)^3$. □

Lemma 3.2. *Let $u_0^1 = (u_0^1, u_0^2, u_0^3) \in M^C \cap H^1(\mathbf{R}^3) \cap H$. Let $u(x,t)$ be a Leray-Hopf solution to the Navier-Stokes equations in the sense of Caffarelli, Kohn and Nirenberg with data u_0. Then:*

(i) *if for some i, j $\alpha_i^j = \int_{\mathbf{R}^3} |u_0^i|^2 - |u_0^j|^2 \, dx \neq 0$, then there exists t_0 such that*

$$\left| \int_0^T \int_{\mathbf{R}^3} |u_i|^2 - |u_j|^2 \, dx dt \right| \geq \left(\frac{\alpha_i^j}{2} \right) T$$

for all $T \leq t_0$, t_0 depending only on the H^1 norm of the gradient of u_0.

(ii) *if for some i, j, $\beta_i^j = \int_{\mathbf{R}^3} u_i^0 u_j^0 \, dx \neq 0$, then there exists t_0 such that*

$$\left| \int_0^T \int_{\mathbf{R}^3} u_i u_j \, dx dt \right| \geq \left(\frac{\alpha}{2} \right) T$$

for all $T \leq t_0$, t_0 depending only on the H^1 norm of the data.

Proof. Without loss of generality suppose $\alpha_1^2 > 0$. Let $\alpha = \alpha_1^2$. Let $F(t) = \int_{\mathbf{R}^3} |u_1|^2 - |u_2|^2 \, dx$.

$$\left| \frac{d}{dt} F(t) \right| \leq C \int_{\mathbf{R}^3} |\nabla u|^2 \, dx.$$

Multiply by u_1 the equation of the first component of the Navier-Stokes equation and the second by u_2. Subtract and integrate in space. Hence

$$\begin{aligned} \left| \frac{d}{dt} F(t) \right| \leq & \left| \int_{\mathbf{R}^3} u_1 \sum_{i=1}^3 u_i \partial_i u_1 - u_i \partial_1 p + u_1 \Delta u_1 \, dx \right. \\ & \left. - \int_{\mathbf{R}^3} u_2 \sum_{i=1}^3 u_i \partial_1 u_2 - u_2 \partial_2 p + u_2 \Delta u_2 \, dx \right| \\ \leq & \, C \left(\int_{\mathbf{R}^3} |\nabla u|^2 \, dx + \int_{\mathbf{R}^3} |u|^4 \, dx + \int_{\mathbf{R}^3} |p|^2 \, dx \right). \end{aligned} \tag{3.1}$$

Note that since $u_0 \in H^1$ this estimate makes sense for $t < t_0 = \left(\frac{1}{6} \int |\nabla u_0|^2 \, dx \right)$ as shown in Theorem 3.1.

To estimate the L^4 norm recall the following estimate.

Lemma 3.3. *If $n = 3$ for any space set $\Omega \subset \mathbf{R}^3$*

$$\|u\|_{L^4(\Omega)} \leq 2^{1/2} \|u\|_{L^2(\Omega)}^{1/4} \, \|\nabla u\|_{L^2(\Omega)}^{3/4}$$

for all $u \in H_0^1(\Omega)$.

Proof. See Teman [6], page 297. □

Remark. The last lemma can be trivially extended to the case $\Omega = \mathbf{R}^3$ for all $u \in H^1$ with $|u(x)| \to 0$ as $|x| \to \infty$.

To estimate the pressure term p recall that p satisfies an elliptic equation which is obtained by taking the divergence of the Navier-Stokes equations

$$\Delta p = -\sum_{ij} \frac{\partial^2}{\partial x_i \partial x_j} u_i u_j,$$

hence

$$\hat{p} = -\sum_{ij} \frac{\xi_i \xi_j}{|\xi|^2} \widehat{u_i u_j},$$

and by Lemma 3.3

$$\begin{aligned}\int_{\mathbf{R}^3} |p|^2\, dx = \int_{\mathbf{R}^3} |\hat{p}|^2\, d\xi &\leq \int_{\mathbf{R}^3} \sum_{ij} \frac{\xi_i \xi_j}{|\xi|^2} \widehat{u_i u_j}\, d\xi \\ &\leq 4 \int_{\mathbf{R}^3} |u|^4\, dx \\ &\leq \left(8 \int_{\mathbf{R}^3} |\nabla u|^2 dx\right)^{3/2}.\end{aligned}$$

Hence the right side of (3.1) can be estimated as follows for

$$t \leq t_0 = (2K)^{-1} \|\nabla u_0\|_{L^2}^{-6},$$

with K defined in Theorem 3.1:

$$\left|\frac{d}{dt} F(t)\right| \leq C\left(\left(\int_{\mathbf{R}^3} |\nabla u|^2\, dx\right)^{3/2} + \int_{\mathbf{R}^3} |\nabla u|^2\right) \tag{3.2}$$

the constant C depending on the L^2 norm of the data. Moreover, by Lemma 3.1, for $t < (2K)^{-1}$, $\|\nabla u_0\|_{L^2}^{-6} = t_0$; it follows that integrals on the right-hand side of (3.2) are bounded by $C_0 = 4\left[\left(\int |\nabla u_0|^2\right)^3 + \left(\int \nabla u_0|^2\right)^{9/2}\right]$. Hence for $t < t_0$

$$\left|\frac{d}{dt} F(t)\right| \leq C C_0 = C_1.$$

By the mean value theorem for some $\bar{s} \in [0, t]$

$$|F(t) - F(0)| \leq |F'(\bar{s})| t = C_1 t.$$

Thus $F(t) \geq F(0) - C_1 t$. Integrating over $[0,T]$ yields

$$\int_0^T F(t)\,dt \geq F(0)T - C_1\frac{T^2}{2}$$

hence for any $T \leq \min\left(t_0, \frac{F(0)}{2C_1}\right)$ where t_0 was obtained in Theorem 3.1 as $t_0 - \left(2K\left\|\nabla u_0\right\|_{L^2}^3\right)^{-1}$ and K depends only on the constant in Agmon's inequality and the L^2 norm of the data. Thus,

$$\left|\int_0^T F(t)\,dt\right| \geq \frac{F(0)}{2}T.$$

Let $T_0 = \min\left(t_0, \frac{F(0)}{2C_1}\right)_0$ and Part (i) of the lemma follows.

Part (ii). Without loss of generality let $\beta = \beta_2^1 > 0$. Let $A = A(\vartheta)$ be the rotation by $\frac{\pi}{4}$, i.e.,

$$A = \begin{bmatrix} \cos\vartheta & -\mathrm{sen}\vartheta & 0 \\ \mathrm{sen}\vartheta & \cos\vartheta & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \text{with } \vartheta = \tfrac{\pi}{4}.$$

Then if $v^t - (v_1, v_2, v_3)^t = A^{-1}u^t$, it follows that

$$\int_{\mathbf{R}^3} u_1 u_2\,dx = \int_{\mathbf{R}^3} v_1^2 - v_2^2\,dx$$

and hence Part (ii) follows from Part (i) applied to the rotated coordinates v_1, v_2, v_3.

Lemma 3.2 implies that if the data u_0 does not have radially equidistributed energy then for a short time, i.e., $\leq t_0$, $\alpha_i^j(t,u) \neq 0$ or $\beta_i^j(t,u) \neq 0$ for some i, $j = 1,2,3$. Hence the Fourier transform of the solutions to Navier-Stokes equations with data u_0 has the form

$$\hat{u}_k(\xi, t_0) = \xi \cdot \ell_k(\xi, t_0) + h_k(\xi, t_0)$$

where ℓ_k and h_k satisfy the conditions of Theorem 2.1 and t_0 is given by Lemma 3.2. More precisely:

Proposition 3.4. *Let $g \in H \cap H^1 \cap W_2 \cap M^c(\mathbf{R}^3)$. Let $u(x,t)$ be a suitable Leray-Hopf solution in the sense of Caffarelli, Kohn, and Nirenberg. Let $\hat{g}$ hav a zero at the origin of order greater than one. Then there exists δ such that for $|\xi| \leq \delta$*

$$\hat{u}_k(\xi, t_0) = \xi \cdot \ell_k(\xi, t_0) + h_k(\xi, t_0)$$

where t_0 is given by Lemma 3.2 and ℓ_k and h_k satisfy

(i) $\left|h_k(\xi)\right| \leq M_0|\xi|^2$;
(ii) *ℓ_k is homogeneous of degree zero.*

(iii) $\omega_0 \cdot \ell_k(\omega_0) = \alpha \neq 0$ *for some* $\omega_0 \in S^{n-1}$ *and at least one of the* ℓ_k,

(iv) $\xi \cdot \ell_k(\xi) \in C^1(\mathbf{R}^n) \setminus \{0\}$. *The constant* M_0 *depends only on* $|g|_{L^2}$, $|g|_{w_2}$ *and* t_0.

Proof. We start the proof as in the first part of Theorem 3.3 [9]. Recall that a weak solution with data g satisfies

$$\begin{aligned}(3.3)\qquad \langle u(t), \varphi(t)\rangle - \int_0^2 \Big\{ \Big\langle u(s), \frac{\partial}{\partial s}\varphi(s)\Big\rangle + \langle \nabla u(s), \nabla\varphi(s)\rangle \\ + \Big\langle \big(u(s)\cdot\nabla\big)u(s), \varphi(s)\Big\rangle \Big\}\, ds - \langle g, \varphi(0)\rangle = 0\end{aligned}$$

for all smooth vectors φ with compact support and div$\varphi = 0$. Following Wiegner's argument [10], choose φ to be a solution to the heat equation with data $\varphi_0 \in C_0^\infty(\mathbf{R}^3)$ with div$\varphi_0 = 0$. This φ is smooth and bounded in L^∞ and (3.3) holds for φ by approximations. Let $t_0 > 0$ fixed and $t^* > t_0$. For $0 \le s \le t$ let

$$\varphi(s) = \mathcal{F}^{-1}\left(\mathcal{F}(\varphi_0)\exp\big(-|\xi|^2(t^* - s)\big)\right)$$

which is the solution to the homogenous heat system with data φ_0 at time $t^* - s$. For that choice of φ_0, (3.3) yields

$$\begin{aligned}(3.4)\qquad \hat{u}_k(\xi, t_0) = \sum_{j=1}^{3}\left(\delta_{jk} - \xi_k\xi_j|\xi|^{-2}\right) \\ \left[\hat{g}_j e^{-|\xi|^2 t_0} - \int_0^{t_0} (u\cdot\widehat{\nabla)}u_j(s)e^{-|\xi|^2(t_0 - s)}\, ds\right].\end{aligned}$$

For more details we refer the reader to [10]. By hypothesis

$$\hat{g}(\xi) = \hat{g}_j(0) + \widehat{\nabla g_j}(0)\cdot\xi + \widehat{\nabla^2 g_j}(\eta)(\xi,\xi) = \widehat{\nabla^2 g_j}(\eta)(\xi,\xi).$$

Hence we only have to consider the terms in

$$(3.5)\qquad \sum_{j=1}^{3}\left(\delta_{kj} - \xi_k\xi_j|\xi|^{-2}\right)\int_0^{t_0} v\Delta u_j e^{-|\xi|^2(t_0 - S)}\, ds.$$

Let $a_{ij} = \widehat{u_i u_j}$, $a_{ij}^0(t) = \widehat{u_i u_j}(0,t)$. Then (3.5) can be rewritten as

$$\sum_{j=1}^{3}\left(\delta_{kj} - \xi_k\xi_j|\xi|^2\right)\int_0^{t_0}\sum_{i=1}^{3} a_{ij}e^{-|\xi|^2(t_0 - s)}\, ds.$$

By Lemma 8.2 of [2] there is a set A with Lebesgue measure zero such that if $t \notin A$ then

$$a_{ij}i(\xi,t) = a_{ij}^0(t) + \xi\nabla_\xi a_{ij}(\xi,t).$$

Hence (3.5) can be expanded as

$$-\sum_{j=1}^{3}(\delta_{kj}-\xi_k\xi_j|\xi|^{-2})\int_0^{t_0}\xi_i a_{ij}^0(s)\,ds+k(\xi). \tag{3.6}$$

where $|k(\xi)|\leq M|\xi|^2$, M depending only on L^2, W_2 norms of the data and t_0. Without loss of generality let $k=1$. The first term in (3.6) can be rewritten as

$$-i\xi\cdot\ell_1(\xi,t_0)=-i\sum_{i=1}^{3}\xi_i\ell_1^i(\xi,t_0),$$

where

$$\ell_1^1(\xi,t_0)=\frac{|\xi_2|^2}{|\xi|^2}\int_0^{t_0}a_{11}^0-a_{22}^0\,ds+\left(\frac{|\xi_3|^2}{|\xi|^2}\right)\int_0^{t_0}a_{11}^0-a_{33}^3\,ds, \tag{3.7}$$

$$\ell_1^2(\xi,t_0)=\left[1-\frac{(2|\xi|^2)}{|\xi|^2}\right]\int_0^{t_0}a_{21}^0\,ds-\frac{\xi_1\xi_3}{|\xi|^2}\int_0^{t_0}a_{32}^0\,ds,$$

$$\ell_1^3(\xi,t_0)=\left[\frac{1-(2|\xi_1|^2)}{|\xi|^2}\right]\int_0^{t_0}a_{31}^0\,ds-\frac{\xi_1\xi_2}{|\xi|^2}\int_0^{t_0}a_{32}^0\,ds.$$

From (3.4), (3.5), (3.6), and (3.7) it follows that

$$\hat{u}_k(\xi,t_0)=\xi\cdot\ell_k(\xi,t_0)+h_k(\xi,t_0)$$

with $|h_k(\xi,t_0)|\leq M_0|\xi|^2$ and M_0 depending only on $\sup_{|\xi|\leq\delta}|\widehat{\nabla^2 g}(\xi)|$, the L^2, W_2 norms of g and t_0. Conditions (i), (ii), and (iv) follow trivially. For (iii) we only need to choose ω_0 appropriately. Since $g\in M^c\cap H^1$ by Lemma 3.2 it follows that for $i\neq j$, $\alpha_i^j(t_0)=\int_0^{t_0}a_{ii}^0-a_{jj}^0\,ds\neq 0$ or $\beta_i^j(t_0)=\int_0^{t_0}a_{ij}^0\,ds\neq 0$ for some i, $j=1,2,3$. We only analyze $\omega_0\cdot\ell_k(\omega_0)$, $k=1$; for either k it follows similarly. If $\omega_0\cdot\ell(\omega_0)\neq 0$, then simple continuity arguments show that $\alpha_1=\int_{|\omega|=1}\omega\cdot\ell(\omega)\,ds>\delta>0$. Let e_j with the jth element of the canonical basis in $\mathbf{R}^3$.

(i) If $\alpha_i^j\neq 0$ choose $\omega_0=(e_i+e_j)/\sqrt{2}$.

(ii) If $\alpha_1^j=0$ and $\beta_i^j\neq 0$. Without loss of generality, suppose i is one. Then let $\omega_0=e_j$.

(iii) If $\alpha_i^j=0$, $\beta_1^j=0$, and $\beta_i^j\neq 0$, i and j not one. Let $\omega_0=(e_1+e_j-e_r)1/\sqrt{3}$. Note that multiplying the appropriate element of the canonical basis by sign α_i^j or sign β_i^j, one can always show that $\omega_0\cdot\ell_k(\omega_0)>0$. □

We recall that if the initial data $u_0 \in H \cap L^1$ have nonzero average, we establish upper and lower bounds for the rates of decay of the solutions to the Navier-Stokes equations in three dimensions in [8]. More precisely, it is shown that

$$C_0(t+1)^{-3/2} \leq |u(\cdot,t)_{L^2}^2| \leq C_1(t+1)^{-3/2}$$

with C_0, C_1 depending only on the L^2 norm of the data.

If the average of the initial data $u_0 \in H \cap L^1 \cap W_1 \cap W_2$ is zero and $\hat{u}_0(\xi,t)$ has a zero of order one, in [9] we showed that for $n = 2$ there exist constants C_2 and C_3 depending on the L^2, W_1 and W_2 norms of the data such that

$$C_2(t+1)^{-n/2} \leq |u(\cdot,t)_{L^2}^2| \leq C_1(t+1)^{-n/2}. \tag{3.8}$$

In [9] we gave an outline of the proof for $n \geq 3$ of (3.8). The main step in the proof of (3.8) was a comparison theorem between the solutions to the Navier-Stokes equations and the solutions to the heat equations which satisfies an inequality of type (3.8). More precisely the following theorem is the essential step leading to a lower bound.

Theorem 3.5. *Let $u_0 \in L^2 \cap W_2 \cap H(\mathbf{R}^3)$. Let v be a solution to the heat equation with data u_0. Suppose*

$$C_0(1+t)^{-5/2} \leq \left\|v(\cdot,t)\right\|_{L^2}^2 \leq C_1(1+t)^{-5/2}.$$

Let $u(x,t)$ be a solution to the Navier-Stokes equations with data u_0, then there exist constants M_0 and M_1 such that

$$M_0(1+t)^{-5/2} \leq \left\|v(\cdot,t)\right\|_{L^2}^2 \leq M_1(1+t)^{-5/2}.$$

where M_0 and M_1 depend on C_1, n, the L^1 and the L^2 norm of u_0 and M_- also depends on the W_2 norm of u_0.

Note 1. The proof is based on the proof presented in [9], where the 2–dimensional case was established and the n–dimensional was outlined. We give only the changes necessary to complete the proof in [8].

Note 2. The outline of the proof in [8] is formal. To make it rigorous apply it to approximating sequences and pass to the limit.

Proof. There are two cases to be considered. Let $i \neq j$. Let $\mathcal{A}_{ij}(t)$ and $\mathcal{B}_{ij}(t)$ be defined by

$$\mathcal{A}_{ij}(t) = \left|\int_0^t \alpha_{ij}(x,s)\,ds\right| \geq \frac{t}{2}\alpha_{ij}^0,$$

$$\mathcal{B}_{ij}(t) = \left|\int_0^t \beta_{ij}(x,s)\,ds\right| \geq \frac{t}{2}\beta_{ij}^0.$$

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