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Some results on the asymptotic behaviour of solutions to the Navier-Stokes equations

Maria E. Schonbek*
Department of Mathematics
University of California
Santa Cruz, CA 95064

§1. Introduction

We consider the asymptotic behaviour of solutions to the Navier-Stokes equations in $n \geq 2$ spatial dimensions

$$u_t + u \cdot \nabla u + \nabla p = \Delta u$$

$$\operatorname{div} u = 0$$

In earlier papers we discussed the upper bounds of rates of decay in three space dimensions with data $u_0 \in L^2 \cap L^p$, $1 \leq p \leq 2$ [6, 7]. There have been several extensions and improvements on these results [2, 11]. Here we first present a survey of results on lower bound of the L^2 rates of decay in two- and three-dimensions [8, 9] and then extend these results to $n \geq 3$ dimensions.

The study of the lower bounds is a much more subtle problem than the one corresponding to the upper bounds. The solutions to Navier-Stokes, unlike the solutions to the heat equation, do not decay at arbitrarily large algebraic rates or even exponentially depending on how oscillatory the initial data is. More precisely, solutions to Navier-Stokes outside a set M of radially equidistributed data have an algebraic lower bound of decay rate which is independent of the oscillations of the data and depends only on the number of dimensions of the space. The algebraic lower bound is a consequence of the nonlinear structure of the equations. The inertial term $\operatorname{div}(u \otimes u)$ in the Navier-Stokes equations appears to convert short-waves into long-waves reducing the decay rate. For data in M an example suggested by A. Majda shows that there are exponentially decaying solutions.

There are two cases to consider. First case: the average of the initial data is nonzero, i.e., the initial data has long waves. Here the argument relies on a comparison argument with solutions to the heat equation with the same data. Second case: the average of the data is zero. The

* This paper was partially funded by NSF grant No. DMS-9020941.

approach now is to find conditions on the data so that the corresponding solutions to the heat equations decay at a specific algebraic rate. These conditions will be met by the solution to the Navier-Stokes at some time $t_0 \geq 0$. The lower bound in the solution to the heat equation will be an essential tool to establish the corresponding estimate for solutions to Navier-Stokes.

§2. Main Results

The following notation will be used.

$$V(\mathbb{R}^n) = C_0^\infty(\mathbb{R}^n) \cap \{u: \nabla \cdot u = 0\},$$

$$H(\mathbb{R}^n) = H = \text{closure of } V \text{ in } L^2,$$

$$W_1 = \left\{u: \int_{\mathbb{R}^n} |x|^2 |u| \, dx < \infty\right\}, \quad W_2 = \left\{u: \int_{\mathbb{R}^n} |u|^2 |x| \, dx < \infty\right\},$$

$$|u|_{W_1} = \int_{\mathbb{R}^n} |x|^2 |u| \, dx, \quad |u|_{W_2} = \int_{\mathbb{R}^n} |x| |u|^2 dx.$$

Let $u \in \mathbb{R}^n$, $m_{ij} = \int_{\mathbb{R}^n} u_i u_j \, dx$, define $M = \{u: u \in \mathbb{R}^n, \text{matrix } (m_{ij}) \text{ is scalar}\}$,

$$\alpha_i^j(t_0, u) = \int_0^{t_0} m_{ii} - m_{jj} ds, \quad \beta_i^j(t_0, u) = \int_0^{t_0} m_{ij} \, ds, \; i \neq j.$$

We recall that if the average of the initial data u_0 is nonzero, i.e., the initial data has long waves, the corresponding solutions to the heat equation have a lower bound of rate of decay of $(t+1)^{-n/2}$ and the difference between the solution to the heat equation and solutions to Navier-Stokes decay at most like $(t+1)^{-n/2-1}$. Hence, a straightforward comparison argument shows that.

Theorem 2.1: Let $u_0 \in H \cap L^1$ and $\widehat{u}(0, t) = \int_{\mathbb{R}^n} u(x, t) \, dt \neq 0$ then there exist constants C_0 and C_1 such that

$$C_o(t+1)^{-n/2} \leq |u(\cdot, t)|_{L^2}^2 \leq C_1(t+1)^{-n/2}$$

with C_0, C_1 depending only on the L^2 and L^1 norms of the data.

Proof: See [8]. □

The case when $\hat{u}(0, t) = \int u(x, t)dx = 0$ is more subtle. The reason being that the mass $\int u(x, t)dx$ is invariant with time and hence stays equal to zero for all time. Hence comparison with the heat equation cannot be expected to work in a straightforward way. There are two preliminary steps to be carried out. First find conditions on the data so that the corresponding solution to the heat equation decays at a slow rate. Second show that solutions to the Navier-Stokes equation, with zero average data lying in M^c, satisfy these conditions at some time $t_0 \geq 0$. In other words long waves will develop eventually.

Once this has been achieved a comparison argument will be used. We note that in this case the lower bound of the rate of decay for solutions to the heat equation and the upper bound of the rate of decay of the difference between solutions to the heat equation and Navier-Stokes equations is the same. Hence the comparison between these two equations is much more difficult.

The data theorem for lower bounds for the solutions to the heat equation can be stated as follows.

Theorem 2.2: Let $v_0 \in L^2(\mathbb{R}^n)$. Let v be a solution to the heat equation with data v_0. Suppose there exists function ℓ and h such that the Fourier transform of δ_0 for $|\xi| \leq \delta$, $\delta > 0$ admits the representation

$$\hat{v}_0(\xi) = \xi \cdot \ell(\xi) + h(\xi) \qquad \ell = (\ell_1, \ldots, \ell_n)$$

where ℓ and h satisfy

i. $|h(\xi)| \leq M_0|\xi|^2$, since $M_0 > 0$,

ii. ℓ is homogeneous of degree zero,

iii. $\alpha_1 = \int_{|w|=1} |w \cdot \ell(w)|^2 dw > 0$.

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Let $M_1 = \sup_{|y|=1} |\ell(y)|$, $M_2 = \sup|\nabla \ell(y)|$, $K = \max(M_0, M_1, M_2)$ then there exists constants C_0 and C_1 such that

$$C_0(t+1)^{-(n/2+1)} \leq |v(\cdot, t)|_{L^2}^2 \leq C_1(t+1)^{-(n/2+1)},$$

where C_0 and C_1 depend only on $M_0, M_1, \delta, |v_0|_{L^2}$ and C_0 also depends on K and α_1.

Proof: We give only the idea of the proof. Note that by the form of the initial data $|\widehat{v_0}(\xi)|^2 = O(|\xi|^2)$ and hence by Parseval

$$\int_{\mathbb{R}^n} |v|^2 dx = \int_{\mathbb{R}^n} |\widehat{v_0}(\xi)|^2 e^{-|\xi|^2 t} d\xi = \int_{\mathbb{R}^n} O(|\xi|^2) e^{-|\xi|^2 t} d\xi$$

which when made rigorous and after change of variables implies that $\int_{\mathbb{R}^n} |v|^2\, dx$ is of order $(t+1)^{-(n/2+1)}$.

For a detailed proof see [8]. □

Corollary 2.3 The conclusion of theorem 2.2 holds if i and ii of theorem 2.2 hold and iii is replaced by the following conditions.

iii. $\omega_0 \cdot \ell(\omega_0) = \alpha \neq 0$, for some $\omega_0 \in S^{n-1}$,

iii'. $\xi \cdot \ell(\xi) \in C^1(\mathbb{R}^n \setminus 0)$.

Proof: See [8]. □

For the second preliminary step there are two cases. 1. The zero of the data is of order one. 2. The zero of the data is of order greater than zero. In the first case we use the following.

Theorem 2.4: Let $g \in H \cap W_1 \cap W_2(\mathbb{R}^n)$, $n = 2, 3$. If g has a zero of order one, then there exists $\delta > 0$ such that for $|\xi| \leq \delta$

$$\widehat{g}(\xi) = \xi \cdot \ell(\xi) + h(\xi)$$

where ℓ and h satisfy the hypothesis of theorem 2.2 with $M_0 = \sup_{|\xi|\le\delta} |\nabla^2 g(\xi)|$ and α_1 depending only on $\widehat{\nabla} g(0)$.

Proof: See [7]. The proof presented in [8] is valid for all n. □

For the second case the data $u_0 = u(x, 0)$ has to be in M^c. Before treating this case we give several auxiliary lemmas.

Lemma 2.4: Let $u_0 \in H^1(\mathbb{R}^n) \cap H(\mathbb{R}^n)$. Let $u(x, t)$ be a Leray-Hopf solution of the Navier-Stokes equations with data u_0. Then there exists $t_0 > 0$ such that

$$\|\nabla u(\cdot, t)\|_{L^2}^2 \le C, \text{ for } t \le t_0.$$

Proof: The proof is standard. In [9] we give a proof for $n = 3$. The main idea is to first multiply the equation by Δu and integrate in space. Using Agmon's inequality one derives easily an ordinary differential inequality for $|\nabla u|_{L^2}^2$. The solution of this inequality exists for $t \le t_0$ where $t_0 = t_0(|\nabla u_0|_{L^2}, |u_0|_{L^2}, n)$. □

Lemma 2.5: Let $u_0 \in H^1(\mathbb{R}^n) \cap H(\mathbb{R}^n)$. Let $u(x, t)$ be a Leray-Hopf solution to the Navier-Stokes equation with data u_0, then there exist $t_0 > 0$ such that for $t \le t_0$

$$|u(\cdot, t)|_{L^\infty} \le C.$$

where C depends only on the L^2 norms of the data u_0 and the gradient of the data ∇u_0.

Proof: Follows by Agmon's standard inequality and the last lemma. □

The proof of the next lemma is formal for $n \ge 3$. In order to make it rigorous it should be applied to approximating solutions as the ones constructed by Caffarelli, Kohn and Nirenberg [1] for $n = 3$ or for $n \ge 3$ by Kayikiya and Miyikawa [2] or by von Wahl [13].

Lemma 2.6: Let $u_0 \in H \cap H^s(\mathbb{R}^n) \cap W_1 \cap W_2$ where $s = \left[\frac{n}{2}\right] + 2$. Let u be a weak solution to the Navier-Stokes equations with data u_0. Then there exists $t_0 > 0$ such that for $t \le t_0$

$$a_{ij} = a_{ij}^0 + \xi \nabla_\xi a_{ij}(\bar{\xi}),$$

where $a_{ij} = a_{ij}(\xi, t) = \widehat{u_i u_j(\xi, t)}$, $a_{ij}^0 = (\xi, 0)$.

Note for $n = 3$ it suffices to have $s = 1$ and the result is valid for almost all t.

Proof: For $n = 3$ the theorem is an immediate consequence of Lemma (8.1) in [1]. Actually this lemma establishes the following bound for almost all t:

$$\frac{1}{2}\int_{\mathbb{R}^3} |u(x, t)|^2 |x| \, dx \le A(t)$$

with $A(t)$ depending only on the L^2 and W_2 norms of the data. Hence letting

$$A = \left\{ t : |\partial_\xi \widehat{u_i u_j}(\xi, t)| \le A(t) \right\},$$

hence for all $t \in A^0$ the conclusion of the theorem follows in the case $n = 3$. for higher dimensions we use the well-known fact that solutions are regular for a short period of time (this argument can also be used for three dimensions). There are several ways of establishing short time regularity in particular for 3 dimensions, see Kato [4]. For higher dimensions, see Temam [10]. A simple way of obtaining short time regularity is to bound the Dirichlet norm for a short time. This can be done formally by multiplying the Navier-Stokes equations by Laplacian and integrating in space. Agmon's inequality for the L^∞ norm will yield an ODE for the Dirichlet norm from where the bound for short time follows. From here Temam's methods will give rigorous short time regularity.

To obtain the conclusion of the theorem for $n \ge 4$, let t_0 be such that the solutions are regular for $t \le t_0$. It will be necessary to show that $\nabla_{\xi_k} a_{ij}$ is well defined or equivalently that for $t \le t_0$

$$\int_{\mathbb{R}^n} |u(x, t)|^2 |x| \, dx \le \infty.$$

Since this is an auxiliary result we will give a proof in an appendix at the end of the paper. □

We will use the notation for $i \neq j$

$$\alpha_{ij} = \alpha_{ij}(t) = \int_{\mathbb{R}^n} u_i u_j \, dx, \qquad \alpha_{ij}^0 = \alpha_{ij}(0), \quad i, j = 1,\ldots,n \tag{2.1}$$

$$\beta_{ij} = \beta_{ij}(t) = \int_{\mathbb{R}^n} u_i u_j \, dx, \qquad \beta_{ij}^0 = \beta_{ij}(0), \quad i, j = 1,\ldots,n \tag{2.2}$$

Lemma 2.7: Let $u_0 \in H \cap H^s(\mathbb{R}^n) \cap M^c$. Let $u(x, t)$ be a weak solution to the NS with data u_0. Then

i. If $\alpha_{ij}^0 \neq 0$ for some i, j then there exists t_0 such that

$$\left| \mathrm{A}_{ij}(t) \right| = \left| \int_0^t \alpha_{ij}(x, s) \, ds \right| \geq \frac{t}{2} \alpha_{ij}^0, \tag{2.3}$$

for all $t \leq t_0$, t_0 depending only on the H^s norm of the data.

ii. If $\beta_{ij}^0 \neq 0$ for some i, j then there exists t_0 such that

$$\left| \mathrm{B}_{ij}(t) \right| = \left| \int_0^t \beta_{ij}(x, s) \, ds \right| \geq \frac{t}{2} \beta_{ij}^0, \tag{2.4}$$

for all $t \leq t_0$, t_0 depending only on the H^s norm of the data.

Proof: Let t_0 be such that the solution is regular for $t < t_0$. The proof follows the same lines as in [8, 9]. The proof in [8] can be used for $n \geq 3$ since we do have short time regularity. Note that ii is a consequence of i since β_{ij} is a rotation of α_{ij} by an angle of $\pi/4$. □

Theorem 2.8: Let $u_0 \in H \cap H^m \cap W_2 \cap M^c(\mathbb{R}^n)$, $n \geq 2$, $m \geq \left[\frac{n}{2}\right] + 2$ (if $n = 2$ it suffices if $m = 1$). Let $u(x, t)$ be a solution to the NS equations with data u_0. If $\hat{u}_0$ has a zero greater than one at the origin there exists $t_0 > 0$ and $\delta > 0$ such that for $|\xi| \leq \delta$

$$\hat{u}_k(\xi, t_0) = \xi \cdot \ell_k(\xi, t_0) + h_k(\xi, t_0),$$

where t_0 depends on the H^m and W_2 norms of the data, ℓ_k and h_k satisfy the conditions of corollary 2.3.

$$\alpha_{ij}^0 = \alpha_{ij}(0), \qquad i,j = 1,\ldots,n$$

$$\beta_{ij}^0 = \beta_{ij}(0), \qquad i,j = 1,\ldots,n$$

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Proof: For the proof we work in Fourier space. We will give the main outline. More details can be found in [8]. The proof follows the general lines of [8].

Note that the solution satisfies

$$\widehat{u}_t + |\xi|^2\widehat{u} = -\widehat{H}, \quad \widehat{u}_0(\xi) = \widehat{u}_0(\xi, 0),$$

where $\widehat{H} = \widehat{u\cdot\nabla u} + \widehat{\nabla p}$. Arguments of Wiegner [12] show that

$$\widehat{u}_k(\xi, t_0) = \sum_{j=1}^{n} (\delta_{jk} - \xi_k\xi_j|\xi|^{-2})\left[\widehat{u}_{0,j}e^{-|\xi|^2 t_0} - \int_0^{t_0} \widehat{u\cdot\nabla u_j}(\xi, s)e^{-|\xi|^2(t_0-s)}ds\right], \tag{2.5}$$

where $\widehat{u}_{0,j}$ is the j-th component of $\widehat{u}_0$ and t_0 is given by lemma 2.6. By hypothesis, $\widehat{u}_{0,j}$ has a zero of order greater than one, hence we only have to consider the terms in

$$\sum_{j=1}^{n} (\delta_{kj} - \xi_k\xi_j|\xi|^{-2})\left[\int_0^{t_0} \widehat{u\cdot\nabla u_j}(\xi, s)e^{-|\xi|^2(t_0-s)}ds\right]. \tag{2.6}$$

Since $\widehat{u\cdot\nabla u_j} = \sum_{i=1}^{3} \xi_i\widehat{u_iu_j}$ and $e^{-|\xi|^2(t_0-s)} = 1 + O(|\xi|^2)$ it follows by Lemma 2.5 that (2.6) can be rewritten as

$$-i\sum_{j,i=1}^{n} (\delta_{kj} - \xi_k\xi_j|\xi|^{-2})\left[\int_0^{t_0} \xi_i a_{ij}^0(s)\, ds\right] + K_j^k(\xi), \tag{2.7}$$

where $|K_j^k(\xi)| \le M|\xi|^2$, $M = M(\|u_0\|_{L^2}, \|u_0\|_{W^2}, t_0)$. Without loss of generality let $k = 1$. The sum in (2.7) will be rewritten as

$$-i\xi\cdot \ell_1(\xi, t_0) = -i\sum_{j=1}^{n} \xi_j\ell_1^j(\xi, t_0), \qquad \ell_i = (\ell_i^1,\cdots,\ell_i^n) \tag{2.8}$$

For this we first subdivide the sum in (2.7) into three parts.

a. When $i = j = 1$ the corresponding terms of the sum are

$$\xi_1\left[1 - \frac{|\xi_1|^2}{|\xi|^2}\right]\int_0^{t_0} a_{11}^0 ds = \xi_1\sum_{i=2}^{n}\frac{|\xi_i|^2}{|\xi|^2}\int_0^{t_0} a_{11}^0 ds.$$

b. When $j = 1$ and $i \geq 2$ or $i = 1, j \geq 2$, the terms of the sum are

$$\sum_{r=1}^{n} \xi_r \left[1 - 2\frac{|\xi_1|^2}{|\xi|^2}\right] \int_0^{t_0} a_{r1}^0 ds.$$

c. When $i \geq 1$ and $j \geq 1$, the terms of the sum are

$$-\sum_{i \neq j > 1}^{n} \frac{\xi_1 \xi_i \xi_j}{|\xi|^2} \int_0^{t_0} a_{ij}^0 ds - \sum_{i=2}^{n} \frac{\xi_1 |\xi_i|^2}{|\xi|^2} \int_0^{t_0} a_{ii}^0 ds .$$

Hence, $\ell_i(\xi, t) = (\ell_1^1, \cdots, \ell_1^n)$ can be defined as follows

$$\ell_1^i = \ell_r(\xi, t_0) = \sum_{i=2}^{n} \frac{|\xi_i|^2}{|\xi|^2} \int_0^{t_0} a_{11}^0 - a_{ii}^0 \, ds.$$

$$\ell_1^r = \ell_1^r(\xi, t_0) = \left[1 - 2\frac{|\xi_1|^2}{|\xi|^2}\right] \int_0^{t_0} a_{r1}^0 ds - \sum_{r \neq j > 1}^{n} \frac{\xi_1 \xi_j}{|\xi|^2} \int_0^{t_0} a_{rj}^0 ds,$$

for $r \geq 2$. Hence the sum (2.7) can be expressed as described by (2.6). Combining (2.5), (2.6) and (2.8) yields

$$\widehat{u}_k(\xi, t_0) = \xi \cdot \ell_k(\xi, t_0) + h_k(\xi, t_0),$$

where ℓ_k and $h_k = \sum_{j=1}^{n} k_j^k(\xi)$ satisfy trivially conditions i, ii and iii' of corollary (2.3). To establish iii', let e_j be the canonical basis of $\mathbb{R}^n$. Let α_{ij}^0 and β_{ij}^0 be defined by (2.1) and (2.2), then by hypothesis either $\alpha_{ij}^0 \neq 0$ or $\beta_{ij}^0 \neq 0$. There are various cases to analyze.

i. For some i, j, $\alpha_{ij}^0 \neq 0$. Without loss of generality, let $i = 1$. Choose $\omega_0 = \dfrac{e_1 + e_j}{\sqrt{2}}$, then $\omega_0 \cdot \ell(\omega_0) = \mathcal{A}_{ij}(t_0) \neq 0$, where $\mathcal{A}_{ij}(t_0)$ was defined by (2.3) (see lemma (2.6)).

ii. For all i, j, $\alpha_{ij}^0 = 0$. Suppose $\beta_{ij}^0 \neq 0$ for some i, j. Suppose either i or j is one. Without loss of generality, let $i = 1$. Choose $\omega_0 = e_j$, then $\omega_0 \cdot \ell(\omega_0) = \mathcal{B}_{ij}(t_0) \neq 0$, where $\mathcal{B}_{ij}(t_0)$ was defined by (2.4) (see lemma (2.6)).

iii. For all i, j $\alpha_{ij}^0 = 0$ and $\beta_{ij}^0 = 0$. Suppose $\beta_{ij}^0 \neq 0$ for some i, j. Suppose i and j are not one. Choose $\omega_0 = \frac{1}{\sqrt{3}}(e_1 + e_j - e_j)$. Hence $\omega_0 \cdot \ell(\omega_0) \neq 0$. Note that multiplying by appropriate signs of $\mathcal{A}_{ij}$ or $\mathcal{B}_{ij}$, $\omega_0 \cdot \ell(\omega_0) > 0$. □

The next theorem was the essential step in establishing the lower bound of rate of decay for the L^2 norm in [8].

Theorem 2.9: Let $u_0 \in L^2 \cap W_2 \cap H(\mathbb{R}^n)$. Let v be a solution to the heat equation with data u_0. Suppose

$$C_0(1+t)^{-(n/2+1)} \leq \|v(\cdot, t)\|_{L^2}^2 \, C_1(1+t)^{-(n/2+1)}.$$

Let $u(x, t)$ be a solution to the NS equations with data u_0, then there exist constants M_0 and M_1 such that

$$M_0(1+t)^{-(n/2+1)} \leq \|u(\cdot, t)\|_{L^2}^2 \, M_1(1+t)^{-(n/2+1)}.$$

where M_0 and M_1 depend on C_1, n, the L^1 and the L^2 norm of u_0 and M_{-} also depends on the W_2 norm of u_0.

Note 1: The proof is based on the proof presented in [8], where the 2-dimensional case was established and the n-dimensional was outlined. We give only the changes necessary to complete the proof in [8].

Note 2: The outline of the proof in [8] is formal. To make it rigorous apply it to approximating sequences and pass to the limit.

Proof: There are two cases to be considered. Let $i \neq j$.

Case 1: Given t there exists $T > t$, such that for all pairs i, j

$$\left| A_{ij}(T) \right| < \beta \sqrt{C_0} \quad \text{or} \quad \left| B_{ij}(T) \right| < \beta \sqrt{C_0}.$$

Case 2: There exists T_0 such that for all $t \geq T_0$ and for at least one pair $1 \leq i, j \leq n$

$$\left| A_{ij}(t) \right| \geq \beta \sqrt{C_0} \quad \text{and} \quad \left| B_{ij}(t) \right| \geq \beta \sqrt{C_0}.$$

Here β is such that $16\beta^2 A_n = 1/16$ where $A_n = 2\sigma_i(2n)^{n/2+1}$ and σ_i = measure of the $n-1$ sphere of radius one. See [8] for reason of this choice.

The proof of case 1 is the same as the one presented in [9] if we replace $\widehat{H}$ by $\widehat{\mathcal{H}}$ where

$$\widehat{H}(\xi, t) = i\xi \cdot \left(\frac{|\xi_2|^2}{|\xi|^2} [a_{11}^0 - a_{22}^0], \left[1 - 2\frac{|\xi_1|^2}{|\xi|^2} \right] a_{12}^0 \right) + O(|\xi|^2),$$

$$\widehat{\mathcal{H}}(\xi, t) = i\xi \cdot \Gamma, \ \Gamma = (\Gamma_1, \ldots, \Gamma_n), \text{ where}$$

$$\Gamma_1 = \Gamma_1(\xi) = \sum_{i=2}^{n} \frac{|\xi_i|^2}{|\xi|^2} (a_{11}^0 - a_{ii}^0),$$

$$\Gamma_r = \Gamma_r(\xi) = \left[1 - 2\frac{|\xi_1|^2}{|\xi|^2} \right] a_{r1}^0 - \sum_{r \neq j > 1}^{n} \frac{\xi_1 \xi_j}{|\xi|^2} a_{jr}^0 \, ds, \ r \geq 2.$$

The proof of case 2 is the same as the one given in [8] if we replace $\widehat{H}$ by $\widehat{\mathcal{H}}$ and note that the auxiliary lower bound for $\alpha_i = \int_{|\omega|=i} |\omega \cdot \ell_i(\omega)|^2 d\sigma$ can be given easily using that we can find a $\omega_0 \cdot \ell(\omega_0) = \alpha > 0$, as shown above and hence the smoothness of $\ell(\omega)$ ensures that in some neighborhood of ω_0, $\omega \cdot \ell(\omega) > \alpha/2$.

Combining theorems 2.2, 2.4, 2.7, 2.8, lemma 2.5, 2.6 and corollary 2.3 yields

Theorem 2.9: Let $u_0 \in L^1 \cap H(\mathbb{R}^n)$ be such that $\widehat{u}(0, t) = \int_{\mathbb{R}^n} u(x, t)dx = 0$.

i. If $\widehat{u}_0(\xi)$ has a zero of order one at the origin then also let $u_0 \in W_1 \cap W_2$.

ii. If $\widehat{u}_0(\xi)$ has a zero of order greater than one at the origin then also let

$u_0 \in H^m \cap M^c \cap W_2$, $m = \left[\frac{n}{2}\right] + 2$ (if $n = 3$ then $m = \left[\frac{n}{2}\right] + 1$ suffices). If u is a solution to the NS equations then there exist constants C_0 and C_1 such that

$$C_0(t+1)^{-\left(\frac{n}{2}+1\right)} \le \|u(\cdot, t)\|_{L^2(\mathbb{R}^n)} \le C_1(t+1)^{-\left(\frac{n}{2}+1\right)},$$

where in case i C_0 and C_1 depend only on the L^2, W_1 and W_2 norms of u_0 and in case ii C_0 and C_1 depend on the L^1 and L^2 norms of u_0 and C_0 also depends on the H^m norms of the data.

Proof: For upper bounds see [2, 7, 11]. The lower bounds are immediate consequences of theorems 2.2, 2.4, 2.8, 2.9, corollary 2.3, lemma 2.4 and 2.5. To obtain a rigorous proof it will be necessary to apply the above theorems and lemmas to approximating sequences and pass to the limit. See [1, 2, 3, 5, 12] for construction of such sequences. Note that the lower bounds will be first shown to be valid only a.e. in t. To obtain them for all t we use the following lemma.

Lemma 2.10: Let $A \subset \mathbb{R}$ be a set such that the Lebesgue measure $\mu(A^c) = 0$. If $u(x, t)$ is a solution to the Navier-Stokes equations and for $t \in A$

$$C_0(t+1)^{-\alpha} \le \|u(\cdot, t)\|_2^2 \le C_1(t+1)^{-\alpha}$$

then for all $t \in \mathbb{R}$

$$\left(\frac{1}{2}\right)^{\alpha} C_0(t+1)^{-\alpha} \le \|u(\cdot, t)\|_2^2 \le C_1(t+1)^{-\alpha} 2^{\alpha}.$$

Proof: Let $t_0 \in A^c$, $t_1, t_2 \in A$ such that $t_0 \in [t, t_2]$, $(t_0+1)(t_1+1)^{-1} \le 2$. and $(t_0+1)(t_2+1)^{-1} \ge 1/2$. Then since $\|u(\cdot, t)\|$ is a decreasing function

$$\left(\frac{1}{2}\right)^{\alpha} (t_0+1)^{-\alpha} \le \|u(\cdot, t)\|_2^2 \le 2^{\alpha} C_1(t_0+1)^{-\alpha}. \qquad \square$$

Having data in M^c is essential since the example presented in [8, 9] of exponentially decaying vorticity can be easily extended to n dimensions (see [9]). This example was suggested by A. Majda for solutions in two spatial dimensions.

3. Appendix

Here we establish lemma 2.7 for $n \geq 3$.

Lemma A: Let $u_0 \in H \cap H^s(\mathbb{R}^n) \cap W_1 \cap W_2$ where $s = \left[\frac{n}{2}\right] + 2$. Let $u = (u_1, u_2, \ldots, u_n)$ be a Leray-Hopf solution to the Navier-Stokes solutions. Then there exists $t_0 > 0$, such that for $t \leq t_0$

$$a_{ij} = a_{ij}^0 + \xi \nabla_\xi a_{ij}(\bar{\xi}),$$

where $a_{ij} = a_{ij}(\xi, t) = \widehat{u_i u_j}(\xi, t)$, $a_{ij}^0 = a_{ij}(0, t)$.

Proof: Let t_0 be given by Lemma 2.4. It is only necessary to show that for $t \leq t_0$, $\nabla_{\xi_k} a_{ij}$ is well-defined or equivalently that

$$\int_{\mathbb{R}^n} |x|\, |u|^2\, dx < \infty.$$

Multiply the Navier-Stokes equations by $|x|u_i$ and integrating in space

$$\frac{d}{dt}\int_{\mathbb{R}^n} |x| \frac{|u_i|^2}{2}\, dx = -\sum_{j=1}^{n} \int_{\mathbb{R}^n} |x|\, u_i \partial_j u_j u_i\, dx - \int_{\mathbb{R}^n} |x| u_i\, \partial_i p \partial x + \int_{\mathbb{R}^n} |x| u_i \Delta u_i dx.$$

Note that the first term on the right can be bounded as follows.

$$-\sum_{j=1}^{n} \int |x|\, u_i u_j \partial_j u_i\, dx \leq C \int_{\mathbb{R}^n} |x|\, |u|^2 dx + C \int |x|\, |u|^2 |\nabla u|^2 dx \leq (C + |\nabla u|)\left(\int |x|\, |u|^2 dx\right) L^\infty(\mathbb{R}^n \times [0, T_0]$$
$$\leq C_1 \int |x|\, |u|^2 dx.$$

Further integration by parts in (2.1) yields after summation over the i index

$$\sum_{i=1}^{n} \frac{d}{dt}\int_{\mathbb{R}^n} |x| \frac{|u_i|^2}{2}\, dx = C_1 \int |x|\, |u|^2 dx.$$

$$+ \sum_{i=1}^{n} \int_{\mathbb{R}^n} u_i p\, dx - \sum_{i,j}^{n} \int_{\mathbb{R}^n} u_i \partial_j u_i dx - \int_{\mathbb{R}^n} |x|\, |\nabla u_i|^2 dx$$

Here we supposed that the integrated terms tend to zero.

$\Big] + 2$. Let $u = (u_1, u_2, \ldots, u_n)$
re exists $t_0 > 0$, such that for $t \leq$

w that for $t \leq t_0$, $\nabla_{\xi_k} a_{ij}$ is well-

e

$\partial x + \int_{\mathbb{R}^n} |x| u_i \Delta u_i dx.$

$|\nabla u|\left(\int |x|\,|u|^2 dx\right) L^\infty(\mathbb{R}^n \times [0, T_0]$

index

$\nabla u_i|^2 dx$

$$\leq C_1 \int_{\mathbb{R}^n} |x|\,|u|^2 dx + n\left(\int_{\mathbb{R}^n} |u|^2 dx \int_{\mathbb{R}^n} |p|^2 dx\right)^{1/2} + n^2\left(\int_{\mathbb{R}^n} |u|^2 dx \int_{\mathbb{R}^n} |\nabla u|^2 dx\right)^{1/2} -$$

$$- \int_{\mathbb{R}^n} |x|\,|\nabla u_i|^2 dx \leq C_1 \int_{\mathbb{R}^n} |x|\,|u|^2 dx + C_2(t_0, n, |u_0|_{L^2}, |\nabla u_0|_{L^2}) - \int_{\mathbb{R}^n} |x|\,|\nabla u_i|^2 dx.$$

Hence by Gronwall's inequality for $t \leq t_0$

$$\int |x|\,|u(x, t)|^2 dx \leq k \exp C_1 t_0,$$

where $k = k(t_0, n, |u_0|_{L^2}, |\nabla u_0|_{L^2})$. And the lemma is proven. □

Lecture Notes in Mathematics

1530

J. G. Heywood K. Masuda
R. Rautmann S. A. Solonnikov (Eds.)

The Navier-Stokes Equations II – Theory and Numerical Methods

Proceedings, Oberwolfach 1991

Springer-Verlag

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