#24

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LARGE TIME BEHAVIOUR OF SOLUTIONS TO THE NAVIER-STOKES EQUATIONS IN H^m SPACES

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ABSTRACT: In this paper we establish the decay of the homogeneous H^m norms for solutions to the Navier Stokes equations in two dimensions. The rates of decay are obtained by means of the Fourier splitting method. The rate obtained is optimal in the sense that it coincides with the rates for solutions to the heat system.

§1 INTRODUCTION

In this paper we study the large time behaviour in high Sobolev norms of the solutions to the incompressible Navier-Stokes equations in two spatial dimensions

$$
\begin{aligned}
u_t + u \cdot \nabla u + \nabla p &= \Delta u, \\
\operatorname{div} u &= 0, \\
u(x, 0) &= u_0(x),
\end{aligned}
\tag{1.1}
$$

where $u_0(x)$ will belong to an appropriate Sobolev space. The goal of the paper is to establish uniform decay rates in H^m spaces and for the L^∞ in space for the derivatives of all orders.

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The decay will follow using ideas of Kato [1] (see also Wiegner [8]) for the H^1 norm and a method first developed in [6, 7], which consists of Fourier analysis of the equation for higher Sobolev norms. This method can be applied to solutions of systems of equations for which there is an energy inequality combining L^2-norm of the solutions (or derivatives of the solution) with the L^2 norm of the gradient of the solution (or gradient of the appropriate derivative). In section two the decay of the gradient is established. This section is included for completeness since it is an immediate consequence of Kato's [1] decay results for small data and Wiegner's [8] decay results for the L^2 norm of the solutions in two space variables with arbitrary data.

Induction arguments on the number of derivatives and ideas developed in [6, 7] yield the decay for the homogeneous H^m norms for $m \geq 2$. More precisely, it is shown that the L^2 norm of the derivatives of order $|\alpha| = k$ decay at the same speed as the derivatives to solutions to the heat equation, that is, with a speed of $(t + 1)^{-(k+1)/2}$. Now the L^2 decay rates combined with the standard Gagliardo-Nirenberg [2] inequality will yield decay for the L^∞ norm of the derivatives of all orders.

Finally we would like to mention that this paper has also as goal to show another application to the Fourier splitting method which was developed to study decay of solutions to parabolic conservation laws [6] and solutions to the Navier-Stokes equations [7]. Results of similar type for L^p decay were obtained by several authors. See, for example, [3], [4], [5].

The following notation will be used.

$$u = (u_1, u_2), \alpha = (\alpha_1, \alpha_2)$$

$$\|u_i\| = \left(\int_{\mathbb{R}^2} |u_i|^p\right)^{1/p}$$

$$\|u\|_p^p = \sum_{i=1}^{2} \|u_i\|_p^p$$

$$D^\alpha = \frac{\partial^{\alpha_1+\alpha_2}}{\partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2}}$$

Let $e_1 = (1, 0)$, $e_2 = (0, 1)$, $D^{\alpha+1} = D^{\alpha+e_j}$ for either $j = 1$, or 2. If no confusion arises, we will use $D^{\alpha+1}u$ to indicate $D^{\alpha+e_j}u$.

§2. DECAY FOR THE L^q NORMS OF THE GRADIENT

The decay for the L^q norm of the gradient follows combining Kato's [1] decay results for small data and Wiegner's [8] results on decay rates for the solutions in two space dimensions with arbitrary data. This section is only included for completeness since it is a straightforward consequence of the results of Kato and Wiegner [1, 8].

Recall that in [1] Kato shows for $n = 2$

$$\|v(\cdot, t)\|_q \leq C\, t^{-1/2+1/q} \|v(\cdot, 0)\|_2, \qquad 2 \leq q \leq \infty \tag{2.1}$$

and

$$\|\nabla v(\cdot, t)\|_q \leq C\, t^{-1+1/q} \|v(\cdot, 0)\|_2, \qquad 2 \leq q < \infty \tag{2.2}$$

where v is a solution to the Navier-Stokes equations. By Wiegner's results on decay [8], we have for $u_0 \in L^2 \cap L^1$

$$\|u(\cdot, t)\|_{L^2}^{j} \leq C t^{-1/2} .$$

Hence starting with $v(\cdot, 0) = u(\cdot, t/2)$ in Kato's result it follows (by uniqueness of solutions in two dimensions)

$$\|u(\cdot, t)\|_q \leq C\, t^{-1+1/q}$$

and

$$\|\nabla u(\cdot, t)\|_q \leq C\, t^{-3/2+1/q} .$$

More precisely for the L^q norm of the gradient it follows that

Theorem 2.1: Let $u_0 \in L^1 \cap L^2(\mathbb{R}^2)$. Let $u\cdot(\cdot, t)$ be a solution to the Navier-Stokes equation with data u_0. Then

i. $\|u(\cdot, t)\|_q \leq C\, t^{-1+1/q}$ $\qquad 2 \leq q \leq \infty,$

ii. $\|\nabla u(\cdot, t)\|_q \leq C t^{-3/2+1/q}$ $\qquad 2 \leq q < \infty.$

If in addition we suppose $u_0 \in H^3(\mathbb{R}^2)$ then

iii. $\|\nabla u(\cdot, t)\|_\infty \leq C\, t^{-1/2}.$

Proof: Inequalities i and ii are as pointed out above an immediate consequence of Kato's [1] and Wiegner's [6] results. Inequality iii follows by Wiegner's decay result for the L^2 norm of the solution and the standard Gagliardo Nirenberg [2] inequality

$$\| \nabla u \|_\infty^2 \leq C \|u\|_\infty \| D^3 u \|_2 . \tag{2.3}$$

Note that if $u_0 \in H^3(\mathbb{R}^2)$ then the solution $u(\cdot, t)$ admits the uniform bound

$$\|D^3 u(\cdot, t)\|_2 \leq C_0,$$

where c_0 depends only on $\|D^3 u_0\|_2$.

§3. DECAY FOR HIGHER DERIVATIVES IN L^2 AND L^∞ NORMS

In this section we study the upper bounds of decay rates for the L^2 and L^∞ norms of the derivatives. We show that the derivatives of order α, $|\alpha| = k$, decay in L^2 at a rate of $(t+1)^{-(k+1)/2}$ and in L^∞ at a rate of $(t+1)^{-(k/2+1)}$. The L^2 and L^∞ rates of the derivatives are optimal since they coincide with the L^2 and L^∞ rates of the derivatives of the solution to the heat equation.

The decay will be obtained in several steps. The idea is to establish a differential inequality of the form

$$\frac{d}{dt}\int_{\mathbb{R}^2} |D^\alpha u|^2 dx \leq C1(t+1)^{-\mu} - C2 \int_{\mathbb{R}^2} |D^{\alpha+e_j} u|^2 dx \tag{3.1}$$

where by $e_j = j = 1, 2$ we mean the canonical basis of $\mathbb{R}^2$. (For simplicity in notation we will use $D^{\alpha+1}u$ to indicate $D^{\alpha+e_j}u$ when no confusion arises.) In (3.1) we will need to have $\mu = |\alpha| + 1$, then a slight modification of the arguments introduced in [6, 7] will yield the decay rate we are seeking. The first step will yield (3.1) with $\mu < |\alpha| + 1$ and then a bootstrapping argument involving the Gagliardo-Nirenberg [2] inequality will yield (3.1) with the right μ.

We start with an auxiliary lemma which will be needed for the bounds of the derivatives of order α, $|\alpha| \geq 3$.

Lemma 3.1: Let α, β be multi-indices $\alpha = (\alpha_1, \alpha_2)$, $\beta = (\beta_1, \beta_2)$. Let $u = (u_1, u_2)$ then the derivatives of u of order α and $\alpha + 1$ satisfy

$$\left| \int_{\mathbb{R}^2} [D^{\alpha}(u_i u_j)\, D^{\alpha+1} u_i] dx \right| \leq \int_{\mathbb{R}^2} \left| D^{\alpha+1} u \right|^2 dx \; +$$

$$C \sum_{\substack{\beta \leq [|\alpha|/2] \\ \beta \neq 0}} C_{\alpha,\beta} \| D^{\alpha-\beta} u \|_2^2 \, \| D^{\beta} u \|_\infty^2 + \tag{3.2}$$

$$+ C \| D^{\alpha-1} u \|_2^2 \left[\| u \|_\infty^4 + \| Du \|_\infty^2 \right]$$

Proof: By the generalized Leibnitz rule we have

$$I = \left| \int D^{\alpha}(u_i u_j) D^{\alpha+1} u_i \, dx \right| \leq \int \left| D^{\alpha+1} u_i \right| \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} \left| D^{\alpha-\beta} u_i D^{\beta} u \right| dx$$

where $\binom{\alpha}{\beta} = \binom{\alpha_1}{\alpha_2}\binom{\beta_1}{\beta_2}$ and $\alpha \leq \beta$ if and only if $\alpha_i \leq \beta_i$. Hence

$$I \leq \frac{1}{2} \int_{\mathbb{R}^2} |D^{\alpha+1} u_i|^2 dx + \frac{1}{2} \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} \int_{\mathbb{R}^2} |D^{\alpha-\beta} u_i D^{\beta} u_j|^2 dx$$

$$= (II + III)\frac{1}{2}.$$

We only need to bound III.

$$III = \sum_{\substack{\beta \leq \alpha/2 \\ \beta \neq 0}} \binom{\alpha}{\beta} \int_{\mathbb{R}^2} \left| D^{\alpha-\beta} u_i D^{\beta} u_j \right|^2 dx + \sum_{\substack{\alpha-\beta < \beta \\ \beta \neq 0}} \binom{\alpha}{\beta} \int_{\mathbb{R}^2} \left| D^{\alpha-\beta} u_i D^{\beta} u_j \right|^2 dx +$$

$$\int_{\mathbb{R}^2}|u_jD^\alpha u_i|dx+\int_{\mathbb{R}^2}|u_iD^\alpha u_j|dx$$

$$=J_1+J_2+J_3+J_4.$$

Note that the bounds for J_1 and J_2 will be the same and J_3 and J_4 also have the same bounds. Hence we only obtain the bounds for J_1 and J_3. For convenience in notation we drop the subindices.

(3.3) $$J_1\le\sum_{\substack{\beta\le\alpha/2\\ \beta\ne 0}} C\alpha\beta\int_{\mathbb{R}^2}\left|D^{\alpha-\beta}u\right|^2dx\,\|D^\beta u\|_\infty^2 \qquad i=1,2$$

Bounds for J_3 and J_4

As mentioned above it is sufficient to find a bound for J_3 since the bound for J_4 is analogous. For notation sake we omit the subindices. Integrating by parts yields

$$J_3\le\int_{\mathbb{R}^2}|uD^\alpha u|^2dx=-2\int_{\mathbb{R}^2}uDuD^{\alpha-1}uD^\alpha u\,dx-\int_{\mathbb{R}^2}u^2D^{\alpha-1}uD^{\alpha+1}u\,dx\le$$

$$\int_{\mathbb{R}^2}|u|^2|D^\alpha u|^2dx+\int_{\mathbb{R}^2}|Du|^2|D^{\alpha-1}u|^2dx-\int_{\mathbb{R}^2}|u|^2D^{\alpha-1}uD^{\alpha+1}udx$$

$$=A_1+A_2+A_3.$$

$$A_1\le\int_{\mathbb{R}^2}uDuD^{\alpha-1}uD^\alpha u\,dx+\int_{\mathbb{R}^2}|u|^2D^{\alpha-1}uD^{\alpha+1}u\,dx\le$$

$$A_1\le\int_{\mathbb{R}^2}uDuD^{\alpha-1}uD^\alpha u\,dx+\int_{\mathbb{R}^2}|u|^2D^{\alpha-1}uD^{\alpha+1}u\,dx\le$$

$$5\int_{\mathbb{R}^2}|u|^4|D^{\alpha-1}u|^2dx+\frac{1}{10}\int_{\mathbb{R}^2}|D^{\alpha+1}u|^2dx+$$

$$\le\frac{1}{2}\int_{\mathbb{R}^2}|uD^\alpha u|^2dx+\frac{1}{2}|Du|_\infty^2\int_{\mathbb{R}^2}|D^{\alpha-1}u|^2dx+$$

$$5|u|_\infty^4\int_{\mathbb{R}^2}|D^{\alpha-1}u|^2dx+\frac{1}{10}\int_{\mathbb{R}^2}|D^{\alpha+1}u|^2dx$$

$$A_2 \le |Du|_\infty^2 \int_{\mathbb{R}^2} |D^{\alpha-1}u|^2 dx$$

$$A_3 \le 5|u|_\infty^4 \int_{\mathbb{R}^2} |D^{\alpha-1}u|^2 dx + \frac{1}{10}\int_{\mathbb{R}^2} |D^{\alpha+1}u|^2 dx$$

Thus

$$\int_{\mathbb{R}^2} |uD^\alpha u|^2 dx \le 2\left(\frac{3}{2}|Du|_\infty^2 + 10|u|_\infty^4\right)\int_{\mathbb{R}^2} |D^{\alpha-1}u|^2 dx + \frac{1}{5}\int_{\mathbb{R}^2} |D^{\alpha+1}u|^2 dx .$$

Thus

$$(3.4)\quad J_3 + J_4 \le C\left[\left(|Du|_\infty^2 + |u|_\infty^4\right)\int_{\mathbb{R}^2} |D^{\alpha-1}u|^2 dx + \frac{2}{5}\int_{\mathbb{R}^2} |D^{\alpha+1}u|^2 dx\right]$$

Combining inequalities (3.3) and (3.4) gives inequality (3.2). □

The next theorem is the central one in the paper. It establishes the decay of the homogeneous H^m norms and of the L^∞ norms of the derivatives. The proof is done in two parts. The first one establishes the decay for the L^2 and L^∞ norms of derivatives up to order α, $|\alpha| \le 2$. Then Lemma (3.1) is used to establish the decay for the norms of derivatives of higher order by an inductive step. The decay rate of the homogeneous H^m norms obtained in this theorem is optimal. Finally at the end of the section we obtain a corollary which gives the decay of the derivatives for all L^p spaces $2 \le p \le \infty$.

Theorem 3.2: Let $u_0 \in H^m \cap L^1(\mathbb{R}^2)$, $m \ge 3$. Let $u(x, t)$ is a solution to the Navier-Stokes equations with data u_0. Then for $t \ge 1$

i. $|D^\alpha u|_2^2 \le C\alpha(t+1)^{-(|\alpha|+1)}$

ii. $|D^\alpha u|_\infty \le C\alpha(t+1)^{-(|\alpha|+1/2)}$

where $|\alpha| \le m$.

Proof. We first establish the result for $|\alpha| \le 1$ and then proceed by induction. We note first that (i) for $|\alpha| = 0, 1$, and (ii) for $|\alpha| = 0$ re consequences of Theorem (2.1) (Kato [1], Wiegner [2]). Once we have i, ii for $|\alpha| \le 3$ we proceed by induction using (3.2). To obtain (ii) for $|\alpha| = 1$ we need to obtain first some auxiliary estimates. Repeating the procedure three times will yield the right decay. From (iii) Theorem (2.1) (Kato [1], Wiegner [2]) it follows that

$$(3.6) \qquad |\nabla u|_\infty \le C(t+1)^{-1/2}, \qquad t \ge 1.$$

Now we obtain some auxiliary estimates for $|D^2u|_2$ and $|D^3u|_2$ which in conjunction with theorem (2.1) will give a new estimate for $|\nabla u|_\infty$ which is better than (3.6). With this estimate in hand we can improve the estimates for $|D^2u|_2$, $|D^3u|_2$. This procedure will end when we get the estimate (ii) for $|\nabla u|_\infty$.

Multiplying the Navier-Stokes equations by D^2u_i yields

$$(3.7) \qquad \frac{d}{dt}\int_{\mathbb{R}^2} |D^2u_i|^2 dx = 2\sum_{j=1}^{2} \int_{\mathbb{R}^2} D^{2+e_i}D^2(u_iu_j)dx - 2\int_{\mathbb{R}^2} |D^3u_i|^2$$

$$= I + II.$$

Since dir $u = 0$ the pressure term vanishes when we sum over i, hence we omit it.

By Lemma (3.1) and theorem 2.1 we have

$$I \le C|Du|_2^2(|Du|_\infty^2 + |u|_\infty^4) + |D^3u|_2^2$$

$$\le C[(t+1)^{-(2+1)} + (t+1)^{-(2+4)}] + |D^3u|_2^2 \le \frac{C_o}{(t+1)^3} + |D^3u|_2^2 \qquad t \ge 1$$

Hence

$$\frac{d}{dt}\int_{\mathbb{R}^2} |D^2u|^2 dx \le C(t+1)^{-3} - \int |D^3u|^2 dx$$

Let $v = D^2u$, then by Plancherel it follows that

$$\frac{d}{dt}\int |\hat{v}|^2 \le C(t+1)^{-3} - \int |\xi|^2 |\hat{v}|^2$$

Now we use the Fourier splitting argument introduced in [4], [5]. Let

$$S = \left\{ \xi : |\xi| \leq \left(\frac{4}{t+1} \right)^{1/2} \right\}.$$

Hence

$$\frac{d}{dt}\int |\hat{v}|^2 d\xi \leq +c(t+1)^{-3} - \frac{4}{t+1}\int_{S^c} |\hat{v}|^2 d\xi$$

and thus

$$\frac{d}{dt}\int_{\mathbf{R}^2} |\hat{v}|^2 d\xi + \frac{4}{t+1}\int_{\mathbf{R}^2} |v|^2 d\xi \leq C(t+1)^{-3} + \frac{8}{t+1}\int_{\mathbf{R}^2} |v|^2 d\xi.$$

Multiplying by $(t+1)^4$ yields

$$\frac{d}{dt}\left[(t+1)^4\int_{\mathbf{R}^2} |\hat{v}|^2 d\xi \leq C(t+1) + 4(t+1)^3\int_{S(t)} |\hat{v}|^2 d\xi\right]$$

$$\leq C(t+1) + 4(t+1)^2\int_{\mathbf{R}^2} |Du|^2 \leq C(t+1).$$

where the decay of $|Du|_2^2$ (obtained in theorm 2.1) ws used. Dividing by $(t+1)^4$ and integrating in space over $[1, t]$ yields

(3.8) $$\int_{\mathbf{R}^2} |D^2u|^2 dx \leq C_o(t+1)^{-2}$$

We use this last estimate to show that

(3.9) $$|D^3u|_{L^2}^2 \leq C_1(t+1)^{-2}.$$

Estimate (3.9) follows the same way as estimate (3.8). That is, multiplying the Navier-Stokes equation by D^3u integrating in space. Bounding the consecutive term by lemma (3.1) estimates (3.1) and the Fourier splitting method. Since the steps are an exact repetition of the steps which lead to (3.8) we omit them.

From (2.3), Theorems 2.1(i) and (3.9) it follows that

$$|\nabla u|_\infty \leq C(t+1)^{-1}$$

With this estimate in hand and following the same steps as before we obtain

$$\frac{d}{dt}\int_{\mathbb{R}^2} |D^2u|^2 dx \le \frac{C}{(t+1)^4} - 2\int_{\mathbb{R}^2} |D^3u|^2 dx\,.$$

Hence the Fourier splitting method yields

$$\int_{\mathbb{R}^2} |D^2u|^2 dx \le C(t+1)^{-3} \tag{3.10}$$

This is the best we can expect since the term coming from integrating on $S(t)$ will yield no better. In the same way we get

$$\int_{\mathbb{R}^2} |D^3u|^2 dx \le C(t+1)^{-3} \tag{3.11}$$

Now from (3.12), (2.1), Theorem (2.1)(i) it follows that

$$|\nabla u|_\infty^2 \le C(t+1)^{-3/2},$$

which is the desired estimate (ii) for $|\alpha| = 1$.

To obtain estimates i, ii for $|\alpha| = 2$ and $|\alpha| = 3$, one proceeds in the same fashion. That is part i for $|\alpha| = 2$ is (3.10). By Gagliardo Niremberg [2] we have the estimate

$$|D^j u|_\infty^2 \le C\left|D^{j-1}u\right|_{L^2}\left|D^{j+2}u\right|_{L^2}$$

Hence we have

$$|D^2u|_\infty \le C(t+1)^{-3/2}$$

Now applying the same procedure as before to obtain decay of $|D^4u|_{L^2}$. This decay bound will give a better decay rate for $|D^2u|_\infty$. As before iterating the process will yield

$$|D^2u|_\infty \le C(t+1)^{-2}$$

As before we can get from here

$$|D^3u|_\infty \le C(t+1)^{-5/2}$$

For $|\alpha| \geq 3$ we use induction to show if i, ii holds for $|\alpha| = k$ then

i′. $|D^\alpha u|_{L^2}^2 \leq C(t+1)^{-(|\alpha|+1)}, |\alpha| = k+i, i = 1, 2, 3$

ii. $|D^\alpha u|_\infty \leq C(t+1)^{-(|\alpha|/2+1)}, |\alpha| = k+1.$

Note that if i holds then ii follows by the Gagliardo and Nirenberg's inequality since

$$(3.12)\quad |D^\alpha u|_\infty^2 \leq C|D^{\alpha-1}u|_\infty |D^{\alpha+2}u|_2 \leq C(t+1)^{-(|\alpha|-1)/2-(|\alpha|+3)/2} = \leq C(t+1)^{-|\alpha|-2}$$

Now we proceed by induction: Let $|\alpha| = 3$ then theorem (3.1) i, ii, follows by the above arguments. To show that i′ holds for $|\alpha| = 4, 5, 6$, we proceed as in the following fashion. The idea is to show that

$$(3.13)\qquad \frac{d}{dt}\int_{\mathbb{R}^2} |D^\alpha u|^2 dx \leq \frac{C}{(t+1)^{|\alpha|+3}} - \int_{\mathbb{R}^2} |D^{\alpha+1}u|^2 dx$$

and then apply the Fourier splitting method to get estimate i. Multiply the NS systems by $D^\alpha u$ and integrate in space to yield

$$(3.14)\quad \frac{d}{dt}\int_{\mathbb{R}^2} |D^\alpha u|^2 dx = 2\sum_{j=1,i}^{2}\int_{\mathbb{R}^2} D^{\alpha+e_j} u D^\alpha u_i u_j dx - 2\int |D^{\alpha+1}u|^2 dx$$

The pressure term vanishes since div $u = 0$. By lemma 3.1, equation (3.1) we have

$$(3.15)\qquad \frac{d}{dt}\int_{\mathbb{R}^2} |D^\alpha u|^2 dx \leq M_\alpha(t) - \int_{\mathbb{R}^2} |D^{\alpha-1}u|^2 dx$$

where

$$M_\alpha(t) = C_1 \sum_{\substack{\beta \leq [\alpha/2] \\ \beta \neq 0}} \left|D^{\alpha-\beta}u\right|_2^2 \left|D^\beta u\right|_\infty^2 + C_2\left|D^{\alpha-1}u\right|_2^2\left[\left|u\right|_\infty^4 + \left|Du\right|_\infty^2\right]$$

Since $|\alpha| \geq 3$, $\varpi \leq [\alpha/2] = |\beta| \leq |\alpha| - 2$, hence by inductive hypothesis we have

$$M_\alpha(t) \leq C_1(t+1)^{-\mu} + C_2(t+1)^{-\gamma}$$

where

$$\mu = |\alpha| - |\beta| + |\beta| + 2 = |\alpha| + 3$$

$$\gamma = \min(|\alpha| + 4, |\alpha| + 3).$$

Hence

(3.16) $$M_\alpha(t) \le C(t+1)^{-(|\alpha|+3)}.$$

Hence combining (3.1) and this last estimate yields (3.13). Now the Fourier splitting technique yields the desired estimate i′, for $|\alpha| = 4, 5, 6$. To obtain i′, ii for $|\alpha| \ge 3$, we proceed by induction. Step $|\alpha| = 3$ is done. For $|\alpha| > 3$ we only need to show i′ since by 3.12, step ii will follow. Again the idea is to show 3.13 from where we can to apply the Fourier splitting technique to obtain the desired decay rates. Since that 3.14, 3.15 and 3.16 are valid for $|\alpha| > 3$ hence 3.13 follows. Now we work out the details of the Fourier splitting technique.

$$\frac{d}{dt}\int_{\mathbf{R}^2} |\hat{w}|^2 dx \le \frac{C}{(t+1)^{|\alpha|+3}} - \int_{\mathbf{R}^2} |\widehat{Dw}|^2 dx$$

Let

$$S(t) = \left\{ \xi : |\xi| \le \left(\frac{|\alpha| + 4}{(t+1)}\right)^{1/2} \right\}.$$

Then

$$\frac{d}{dt}\int_{\mathbf{R}^2} |\hat{w}|^2 dx \le C(t+1)^{-(\alpha+3)} - \frac{(|\alpha|+4)}{t+1}\int_{\mathbf{R}^2} |\hat{w}|^2 dx$$

Hence

$$\frac{d}{dt}\left[(t+1)^{|\alpha|+4}\int_{\mathbf{R}^2} |\hat{w}|^2 dx\right] \le C(t+1) + (|\alpha|+4)(t+1)^{|\alpha|+3}\int_{S(t)} |\widehat{D^2 u}|^2 d\xi \le$$

$$C(t+1) + (|\alpha|+4)(t+1)^{|\alpha|+2}\int_{\mathbf{R}^2} |D^{|\alpha|-1}u|^2 dx \le C[(t+1) + (|\alpha|+4)(t+1)^2]$$

Integrating in time once $[1, t]$ yields

$$\int_{\mathbf{R}^2} |D^{|\alpha|}u|^2 dx \le C_o(t+1)^{-(|\alpha|+4)} + C[(t+1)^{-(|\alpha|+2)} + (t+1)^{-(|\alpha|+1)}]$$

Hence

$$\int_{\mathbb{R}^2} |D^{|\alpha|}u|^2 dx \leq C(t+1)^{-(|\alpha|+1)}$$

To obtain the result for $|\alpha| = k+2$, $k+3$ one only has to note that the bound of $M_\alpha(t)$ only depends on $|D^\beta u|_{L^\infty}$ and $|D^{\alpha-\beta}|_{L^2}$ with $\beta \leq [\alpha/2]$ and since $|\alpha| \geq 3$ this norms decay of the appropriate rate is known by inductive hypothesis.

Now estimate (ii) follow by Gagliardo Nirenberg.

$$|D^\alpha u|_2^2 \leq C|D^{\alpha-1}u|_\infty |D^{\alpha+2}u|_2$$

Hence and last bounds

$$|D^\alpha u|_\infty \leq C(t+1)^{|\alpha|/2+1}$$

which concludes the proof.

The next corollary gives L^q decay of the derivatives, by interpolation. Since the L^∞ norm is not optimal we expect that the L^q decay can be improved.

Corollary 3.3: Let $u_0 \in H^\infty \cap L^1(\mathbb{R}^2)$. Let $u(x,t)$ be a solution to the Navier-Stokes equation with data u_0. Then

$$\left|D^j u\right|_{L^q} \leq C_q(t+1)^{-r(j)}$$

where $r(j) = \dfrac{j+2}{2} - \dfrac{1}{q}$.

Proof: Follows by Gagliardo-Niremberg's inequality and L^1 estimates obtained in theorem 3.1

$$\left|D^j u\right|_q \leq \left|D^{j+1}u\right|_2^a \left|u\right|_2^{1-a}$$

where

$$\frac{1}{q}=\frac{j}{2}+a\left(\frac{1}{2}-\frac{j+1}{2}\right)+(1-a)\frac{1}{2}$$

hence

$$a=1-\frac{1}{(j+1)q}, \quad 1-a=\frac{2}{(j+1)q}.$$

Thus by theorem 3.1

$$r(j)=\frac{j+2a}{2}+\frac{1-a}{2}=\frac{j+2}{2}-\frac{j+2}{(j+1)q}+\frac{1}{(j+1)q}.$$

Simplifying yields

$$r(j)=\frac{j+2}{2}-\frac{1}{q}.$$

□

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