25

Estimates for the Pressure and the Fourier Transform for Solutions and Derivatives to the Navier-Stokes Equations

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ABSTRACT. Several bounds on the solutions and the pressure are obtained for regular solutions to the Navier-Stokes equations in two and three dimensions on the whole space. More precisely, we obtain L^1 estimates for the Fourier transform of the solution and its derivatives of all orders, L^q estimates for the pressure and its derivatives of all orders, estimates which show that if the solutions in three dimensions should blow up, this must happen at a certain algebraic rate.

1. Introduction. It is well known that a good understanding on the pressure term would lead to a better knowledge of the behavior of solutions to the 3-dimensional Navier-Stokes equations. In this paper we work on a first step in this direction establishing bounds on the derivatives of all orders of the pressure in L^P spaces, to $1 \leq p \leq \infty$ for regular solutions to the Navier-Stokes equations in two and three spatial dimensions on the whole space

$$u_t + u\nabla u + \nabla p = \Delta u + f, \tag{1.1}$$

$$\text{div } u = 0.$$

Here the initial data $u(x,0) = u_0(x)$ will possibly be large and belong to appropriate spaces which will be specified below. The forcing term f belongs to L^2 or to an H^m space and will be divergence free. In what follows we refer to equations (1.1) by NS.

We note here that for regular solutions on bounded domains with smooth boundary and smooth boundary values, the pressure would also be regular and, as an immediate consequence, the pressure would be bounded in all L^p spaces.

In Section 2 we show that the Fourier transform of regular solutions and derivatives of all orders of the NS are bounded in L^1. The bounds follow by Fourier analysis of the equation and are essential to obtain the pressure bounds

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in Section 3. More precisely, in Section 3 we establish L^q bounds, $2 \leq q \leq \infty$, for the pressure and all its derivatives. The L^q estimates on the pressure itself are included for completeness since such estimates follow also by the classical Calderon-Zygmund theory [8].

In the last section we show that estimates can be used to show that, if solutions to the three-dimensional Navier-Stokes equation should blow up, they have to do it at a certain algebraic rate. A similar problem was treated by a different approach in the pioneering 1934 paper of Leray [4]. More precisely, we show that if there is a first point T_0 such that

$$\|\nabla u(\cdot,t)\|_2 = \infty,$$

then it is not possible that there exist $\varepsilon_0 > 0$ such that

$$\|\nabla u(\cdot,t)\|_2 \leq C(T_0 - t)^{-1/4+\varepsilon_0}$$

for $t \in [T_0 - \delta, T_0)$, some $\delta > 0$.

It is interesting to note that this blow up estimate is consistent with Prodi's results [4] where he derived a differential inequality

$$\frac{d}{dt}\varphi(t) \leq C\varphi^3, \quad \text{with } \varphi(t) = \|\nabla u(\cdot,t)\|_2^2,$$

which implies that for $T* = [2C\varphi(0)^2]^{-1}$,

$$\varphi(t)^{1/2} \leq \left\{\frac{1}{2c}\left[\frac{1}{T_* - t}\right]\right\}^{1/4}.$$

For a general reference on blow up results of the solutions to the Navier-Stokes equations we refer the reader to [9].

We would like to recall that for both bounded and unbounded domains it is shown in [5] that the pressure belongs to $L^{5/3}$ and its gradient is in $L^{5/4}$. Moreover, for the unbounded case a very detailed exposition of the case can be found in W. von Wahl [10]. The results in [7] and [10] hold for any weak solution, and there is not even any requirements of energy-inequalities. In this paper we rely on the solution being regular or, equivalently, on its having either a bound in H^1 or L^∞, which, as is well known, are estimates that yield regularity. We find that the results of Sohr and W. van Wahl [7, 10] are extremely interesting and use hard techniques very different from the ones in his paper. More recently, in [3], it was shown that the pressure is $L^{q,s} = L^q(0,\infty; L^s(\Omega))$ for Ω a bounded domain.

The following notation will be used.

$$\|g\|_p = \left(\int_{\mathbb{R}^n} |g|^p\, dx\right)^{1/p} \qquad \text{for } g : \mathbb{R}^n \to R.$$

If $u = (u_1, \dots, u_n)$, then

$$\|u\|_p^p = \sum_{i=0}^{n} \int_{\mathbb{R}^n} |u_i|^p \, dx$$

$$D^\alpha = \frac{\partial^\alpha}{\partial \alpha_1^{\alpha_1} \cdots, \partial x_n^{\alpha_n}}, \quad \alpha = (\alpha_1, \dots, \alpha_n),\ \alpha_i > 0,\ |\alpha| = \sum \alpha_i$$

$$\partial_i = \frac{\partial}{\partial x_i} = \nabla_i, \quad \nabla = (\partial_i, \dots, \partial_n) = \text{grad},$$

$$\hat{u}(\xi, t) = F(u(\xi, t)) = \int_{\mathbb{R}^n} u(x,t) e^{-2\pi i x \cdot \xi} \, dx.$$

2. L^1 bounds of the Fourier transform of the solution and its derivatives. We first obtain an L^1 bound on the Fourier transform of the solution and then use this bound to estimate the L^1 norm of the Fourier transform of all higher derivatives of the solution.

The idea is to take advantage of the expression of the solution in Fourier space and obtain a bound of the L^1 norm in terms of the H^1 norm of the solution. The next lemma is auxiliary and establishes in Fourier space a bound on the derivatives of all orders of the gradient of the pressure in terms of a multiple of the correspnding derivative of the convective term.

Lemma 2.1. *Let $u_0 \in H \cap H^1(\mathbb{R}^n)$, $f \in L^2$ and* div $f = 0$. *If $u(x,t)$ is a solution to the NS equations with data u_0, then*

(i) $$\left|\widehat{D^\alpha p}(\xi, t)\right| \le \sum_{r,k} \left|\widehat{D^\alpha u_r u_k}(\xi, t)\right|,$$

(ii) $$\left|\widehat{D^\beta \nabla_j p}(\xi, t)\right| \le 3 \left|D^\beta u \nabla u(\xi, t)\right|$$

for $|\beta| \ge 0$, $|\alpha| \ge 0$, and $t \in [0, T_0]$, T_0 chosen so that $D^\alpha u(\cdot, t)$ exists for $t \in [0, T_0]$.

Proof. Take the divergance of the (NS) equation. Since f and u are divergence free,

$$\nabla p = -\sum_{i,j} \partial_i \partial_j u_i u_j.$$

Since $u \in H$, the Fourier transform is well defined. Taking the Fourier transform of the last inequality yields

$$|\xi|^2 \hat{p} = -\sum_{i,j} \xi_i \xi_j \widehat{u_i u_j}.$$

Note that $u_0 \in H^1$; hence for $2D$ the derivatives D_u^α are well defined for all time and for 3D, $t \in [0, T_0]$, for some T_0. Hence the derivatives of the gradients of the pressure satisfy

$$\widehat{D^\alpha p}(\xi, t) = -i \sum_{r,k} \xi_r \xi_k \widehat{D^\alpha u_r u_k} |\xi|^{-2}. \tag{2.1}$$

Thus, by the Schwarz inequality, since $|\xi_r \xi_k| \ |\xi|^{-2} \leq 1$,

$$\left|\widehat{D^\alpha p}(\xi, t)\right| \leq \sum_{r,k} \left|\widehat{D^\alpha u_r u_k}(\xi, t)\right|.$$

Hence, letting $D^\alpha = D^\beta \nabla j$, it follows easily from (2.1)

$$\begin{aligned} |\widehat{D^\beta \nabla_j p}(\xi, t)| &\leq \left| \xi_j \sum_{k,r} \frac{\xi_r \xi_k}{|\xi|^2} \widehat{D^\beta u_r u_k} \right| \\ &\leq \sum_{k,r} |\xi_k \widehat{D^\beta u_r u_k}| \\ &\leq 3 |\widehat{D^\beta u \nabla u}(\xi, t)|. \end{aligned}$$

This last lemma will be essential for all the estimates of the L^1 norms of the Fourier transform of the solution and derivatives of all orders. □

The next lemma is also auxiliary and is a crucial step in the process to establish that the L^1 norm of the Fourier transform of the solution and of the derivatives of all orders of the solution are bounded. The argument we present in this lemma includes some suggestion made by M. Wiegner which shortened the orginal version. The lemma is an extension of the standard Gronwall inequality.

Lemma 2.2. *Let $\varphi(t)$ satisfy $\varphi(t) \geq 0$ and*

$$\varphi(t) \leq A_n + B_n \int_o^t \frac{1}{(t-s)^{n/4}} \varphi(s)\, ds, \qquad t \in [0, T_0], \tag{2.2}$$

where $n = 2$ or 3, and A_n, B_n given constants, then

$$\varphi(t) \leq 2A_n \exp \varepsilon^{-1} \left(1 - \frac{n}{4}\right) T_0,$$

where $\varepsilon^{1-n/4} B_n = \frac{1}{2}(1 - n/4)$.

Proof. We first show by contradiction that $\varphi(t)$ is bounded in $[0, T_0]$. Suppose $\varphi(t)$ is not bounded in $[0, T_0]$. Let $s_0 \in [0, T_0)$ be the first point such that $\varphi(s_0) = \infty$. We will show that this leads to a contradiction. For any $t_0 < s_0$ choose $t \in [0, t_0]$ such that

$$\varphi(t) = \max_{0 \le s \le t_0} \{\varphi(s)\}. \tag{2.3}$$

Rewrite (2.2) as follows:

$$\varphi(t) \le A_n + B_n \int_0^{t-\varepsilon} \frac{1}{(t-s)^{n/4}} \varphi(s)\, ds + B_n \int_{t-\varepsilon}^{t} \frac{1}{(t-s)^{n/4}} \varphi(s)\, ds$$

where ε is chosen such that

$$\left(1 - \frac{n}{4}\right)^{-1} \varepsilon^{1-n/4} B_n = \frac{1}{2}. \tag{2.4}$$

Then, by the choice of t in (2.3), the last inequality yields

$$\varphi(t) \le A_n + B_n \frac{1}{\varepsilon^{n/4}} \int_0^t \varphi(s)\, ds + B_n \left(1 - \frac{n}{4}\right)^{-1} \varepsilon^{1-n/4} \varphi(t).$$

Hence combining with (2.4) yields

$$\varphi(t) \le 2A_n + 2B_n \varepsilon^{-n/4} \int_0^t \varphi(s)\, ds.$$

Now, using (2.4), Gronwall's inequality yields,

$$\varphi(t) \le 2A_n \exp(\varepsilon^{-n/4} t) = \Gamma_n(t). \tag{2.5}$$

Since (2.5) is valid for all $t \in [0, s_0)$ and $\Gamma_n(t)$ is increasing, we have

$$\varphi(t) \le \Gamma_n(s_0).$$

Hence by (2.2) it follows that

$$\begin{aligned}\varphi(s_0) &\le A_n + B_n \int_0^{s_0} \frac{1}{(t-s)^{n/4}} \Gamma_n(s_0)\, ds \\ &= A_n + \left(1 - \frac{n}{4}\right)^{-1} B_n \Gamma_n(s_o) s_o^{1-n/4} < \infty.\end{aligned}$$

We have reached a contradiction and hence $\varphi(t) < \infty$ for all $t \in [0, T_0]$. Now repeating the argument that led to inequality (2.5) with $t < s_0$, we obtain

$$\varphi(t) \le \Gamma_n(T_0)$$

and the proof of the lemma is complete. □

Theorem 2.3. *Let $u_0 \in L^1 \cap H \cap H^1$ and $\hat{u}_0 \in L^1$, $f \in H$. Let $u(x,t)$ be a solution to the Navier-Stokes equations with data u_0. Suppose that $u(x,t) \in H^1(\mathbb{R}^n)$ for $t \in [0,T_0]$, $n = 2,3$ (note that T_0 could be equal to infinity). Then*

$$\|\hat{u}(\xi,t)\|_{L^1(\mathbb{R}^n)} \leq C(T_0), \quad \text{for } t \in [0,T_0]$$

where $C(T_0)$ depends on T_0, $\|\hat{u}\|_{L^1}$ and $\sup_{0\leq t\leq T_0} \|\nabla u(\cdot,t)\|_2$.

Proof. If $T_0 = \infty$, the result above will be established in $[0,T]$ for arbitrary T. Take the Fourier transform of the (NS) equations to obtain for the j^{th} component,

$$\hat{u}_{j,t} + |\xi|^2 \hat{u}_j = -\widehat{u\nabla u_j} - \widehat{\nabla_j p} + \hat{f}_j = -\hat{H}_j. \tag{2.6}$$

Integrating the last equation in time for $t \in [0,T_0]$ yields

$$\hat{u}_j(\xi,t) = \hat{u}_j^0(\xi)e^{-|\xi|^2 t} - \int_0^t \hat{H}_j(\xi,s)e^{-|\xi|^2(t-s)}ds.$$

Hence

$$\int_{\mathbb{R}^n} |\hat{u}(\xi,t)|\, d\xi \leq \int_{\mathbb{R}^n} |\hat{u}_j(\xi)|\, d\xi + \int_0^t \int_{R^n} |e^{-|xi|^2(t-s)}\hat{H}_j|\, d\xi\, ds,\ n = 1,2.$$

Thus, by the Schwarz inequality and using the bound of Lemma 2.1(ii) with $\beta = 0$ (to estimate the $\nabla_j p$), we obtain

$$\int_{\mathbb{R}^n} |\hat{u}(\xi,t)|\, d\xi \leq \int_{\mathbb{R}^n} |\hat{u}_j^0(\xi)|\, d\xi$$
$$+ 4\int_0^t \left(\int_{\mathbb{R}^n} e^{-2|\xi|^2(t-s)}\, d\xi\right)^{1/2} \left[\left(\int_{\mathbb{R}^n} |u\nabla u_j|^2\right)^{1/2} dx + (\|f\|_{l^2})\right].$$

Recall that

$$\left(\int_{\mathbb{R}^n} e^{-2|\xi|^2(t-s)}\, ds\right)^{1/2} = 2^{-n/2}\omega(n)\frac{1}{(t-s)^{n/4}}, \quad \text{for } \omega(n) = \left(\int_{\mathbb{R}^n} e^{-\eta^2}\, d\eta\right)^{1/2}.$$

Hence

$$\|\hat{u}_j(\xi,t)\|_{L^1} \leq \|\hat{u}_j^0\|_1 + 2^{-n/2}\omega(n)4\int_0^t \frac{1}{(t-s)^{-n/4}}\left(\|\widehat{u\nabla u_j}\|_2 + \|f\|_2|\right) ds.$$

By Young's inequality it follows that

$$\|\hat{u}_j(\cdot,t)\|_1 \le \alpha_n + \int_{\mathbb{R}^n} \beta_n \int_0^t \frac{1}{(t-s)^{n/4}} \|\hat{u}_j(\cdot,s)\|_1 \, \|\widehat{\nabla u}(\cdot,s)\|_2 \, ds, \tag{2.7}$$

where

$$\alpha_n = \|\hat{u}_0\|_1 + 2^{-n/2}\omega(n)4T_0 \operatorname*{ess\,sup}_{s\in(0,T_0)} \|f(s)\|_{L^2}$$

$$\beta_n = 2^{-n/2}\omega(n)12.$$

Note that if $T_0 = \infty$, then we obtain estimates in arbitrary intervals $[0,T]$ and hence $K_n = K_n(T)$ where T_0 is replaced by T. Hence we can always work in a finite time interval. Since $\|\nabla u\|_2 \le M_0$, (2.7) can be rewritten as

$$\|\hat{u}_j(\cdot,t)\|_1 \le \alpha_n + \beta_n M_0 \int_0^t \frac{1}{(t-s)^{n/4}} \|\hat{u}_j(\cdot,s)\|_1 \, ds.$$

Now by Lemma 2.2 with $\varphi(t) = \|\hat{u}_j(\cdot,t)|_1$, $A_n = \alpha_n$, $B_n = \beta_0 M_0$ yields

$$\|\hat{u}_j(\cdot,t)\|_1 \le 2\alpha_n \exp \varepsilon^{-1}\left(1 - \frac{n}{4}\right) T_0,$$

where $\varepsilon^{1-n/4} B_n = \frac{1}{2}(1 - n/4)$. This completes the proof of Theorem 2.3. □

Remark 2.4. We note that the theorems above and the lemma do not extend in an immediate fashion to dimensions $n \ge 4$ since then $\int_0^t 1/(t-s)^{n/4}$ is no longer integrable.

The next theorem establishes an L^1 bound for the Fourier transform of the gradient and all higher order derivatives of the solution to the Navier-Stokes equations. The proof is inductive and uses the results on short time regularity by Leray [4] for solutions to the 3-D Navier-Stokes equations. More precisely, recall the following results due to Leray [4].

Theorem 2.5 (Leray [3]).

(i) *If $u_0 \in H^1(\mathbb{R}^2) \cap H$, then the corresponding solution to the 2-D Navier-Stokes equations belongs to $H^m(\mathbb{R}^2)$ for all m, i.e., there exist constants K_m such that*

$$\|u(\cdot,t)\|_{H^m} \le K_m(T), \qquad 0 \le t \le T.$$

(ii) *If $u_0 \in H^1(\mathbb{R}^3) \cap H$ and for $[0,\infty)$ the corresponding solutions to the 3-D Navier-Stokes equations $u(\cdot,t) \in H^1(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$, $u(\cdot,t) \in H^m(\mathbb{R}^3)$ for all m, i.e., there exist constants K_m such that*

$$\|u(\cdot,t)\|_{H^m} \le K_m(T) \qquad \text{for } t \in [0,T].$$

Proof. See Leray [3]. □

Theorem 2.6. *Let $u_0 \in L^1 \cap H \cap H^1(\mathbb{R}^n)$ and $\|\widehat{D^\alpha u_0}\|_1 \leq \Gamma_0$, $n = 2, 3$. Let $f \in H^m(\mathbb{R}^n)$ and $\operatorname{div} f = 0$. Let $u(x,t)$ be a solution to the Navier-Stokes equations with data u_0 and forcing term f. If $n = 3$, also suppose that $u(\cdot,t) \in H^1(\mathbb{R}^3)$ for $t \in [0,T_0]$. Then, for $|\alpha| \leq m$,*

$$\|\widehat{D^\alpha u}(\cdot,t)\|_1 \leq C_1 \qquad \text{for } t \in [0,T_0],$$

where $C_1 = C_1(\|D^\beta u\|_2,\ \|\widehat{D^\beta u_0}\|_1\ ,\ T_0)$, $\beta \leq \alpha$. If $n = 2$, any $T_0 > 0$ is satisfactory.

Proof. If $T_0 = \infty$, then the proof is done on an arbitrary interval $[0,T]$ and C_1 depends on T instead of T_0. We proceed by induction in $|\alpha|$.

1. If $|\alpha| = 0$, the conclusion of the theorem is given by Theorem 2.3.
2. Suppose the theorem is true for $|\alpha| = k$, that is $\|\widehat{D^\gamma u_r}\|_1 \leq M_\ell$ where $|\gamma| = \ell \leq k$. Let $|\alpha| = k+1$, $k+1 \leq m$. Take the D^α derivative of the j^{th} component of the solution to the Navier-Stokes equations. Then take the Fourier transform to obtain

$$\widehat{D^\alpha u_{j,t}} + |\xi|^2 \widehat{D^\alpha u_j} = -\widehat{D^\alpha u \nabla u_j} - \widehat{D^\alpha \nabla_j p} - \widehat{D^\alpha f} = -\widehat{D^\alpha H_j}. \tag{2.8}$$

Hence integrating in time yields

$$\widehat{D^\alpha u_j} = \widehat{D^\alpha u_j^0} e^{|\xi|^2 t} - \int_0^t e^{-|\xi|^2(t-s)} \widehat{D^\alpha H_j} ds. \tag{2.9}$$

Recall that by Lemma 2.1 (ii),

$$|\widehat{D^\alpha \nabla_j p}(\xi,t)| \leq 3|\widehat{D^\alpha u \nabla u}(\xi,t)|.$$

Thus $\widehat{D^\alpha H_j}$ (see definition of $\widehat{D^\alpha H_j}$ in (2.8)) can be bounded by

$$|\widehat{D^\alpha H_j}(\xi,t)| \leq 4|\widehat{D^\alpha u \nabla u}(\xi,t)| + |\widehat{D^\alpha f}(\xi,t)|.$$

Combining the last inequality with (2.9) and integrating in space yields

$$\begin{aligned} &\int_{\mathbb{R}^n} |\widehat{D^\alpha u_j}(\xi,t)|\, d\xi \\ &\leq \int_{\mathbb{R}^n} |\widehat{D^\alpha u_0}(\xi)|\, d\xi + 4\int_0^t \int_{\mathbb{R}^n} e^{-|\xi|^2(t-s)} |\widehat{D^\alpha u \nabla u}|\, d\xi \\ &\qquad + \int_0^t \int_{\mathbb{R}^n} e^{-|\xi|^2(t-s)} |\widehat{D^\alpha f}|\, ds \end{aligned} \tag{2.10}$$

$$= \mathrm{I} + \mathrm{II} + \mathrm{III}.$$

The hardest term to bound is II. We start with this term. For this, we use the generalized Leibnitz rule. Let e_i be the i^{th} element of the canonical basis in $\mathbb{R}^n$. Then

$$\widehat{D^\alpha(u\nabla u_j)} \le \sum_i \widehat{D^\alpha(u_i\partial_i u_j)}$$
$$= \sum_i \widehat{D^{\alpha+e_i}(u_i u_j)} = \sum_i \sum_{\beta\le\alpha+e_i} \binom{\alpha+e_i}{\beta} \widehat{Du_i^\beta\, D^{\alpha+e_i-\beta}u_j}$$

since u is divergence free. Here

$$\binom{\mu}{\nu} = \prod_{i=1}^n \binom{\mu_i}{\nu_i} \quad \begin{cases} \mu = (\mu_1,\dots,\mu_n) \\ \nu = (\nu_1,\dots,\nu_n) \end{cases}$$

and $\nu \le \mu$ (i.e., $\nu_i \le \mu_i$, $i = 1,\dots,n$). Thus,

$$\widehat{D^\alpha(u\nabla u_j)} \le \sum_i \sum_{\substack{\beta\le\alpha+e_i\\ \beta\neq\alpha\\ \beta\neq e_r,\ r=1,\dots,n}} \binom{\alpha+e_i}{\beta} \widehat{D^\beta u_i\, D^{\alpha+e_i-\beta}u_j} + \sum_{i=1}^3 a_i\, \widehat{D^\alpha u_i\, D^{e_i}u_j} + \sum_{\substack{i=1\\ r=1,\dots,n}} b_{i,r} \widehat{D^{e_r}u_i\, D^{\alpha+e_i-e_r}u_j} \tag{2.11}$$

where

$$a_i = \binom{\alpha+e_i}{\alpha}, \quad b_{i,r} = \binom{\alpha+e_i-e_r}{e_r}.$$

Hence to bound II it suffices to obtain bounds for

(i) $$\int_0^t \int_{\mathbb{R}^n} e^{-|\xi|^2(t-s)} |\widehat{D^\beta u_i D^{\alpha+e_i-\beta}u_j}|\, d\xi\, ds = P_1,$$

where $\beta \le \alpha + e_i$ and $\beta \neq \alpha$ and $\beta = e_r$, $r = 1,\dots,n$.

(ii) $$\int_0^t \int_{\mathbb{R}^n} e^{-|\xi|^2(t-s)} |\widehat{D^\alpha u_i\, D^{e_i}u_j}|\, d\xi\, ds = P_2,$$

(iii) $$\int_0^t \int_{\mathbb{R}^n} e^{-|\xi|^2(t-s)} |\widehat{D^{e_r}u_i\, D^{\alpha-e_i-e_r}u_j}|\, d\xi\, ds = P_3.$$

Bound of P_1. By Schwarz's and Young's inequalities,

$$P_1 \le C \int_0^t \left(\int_{\mathbb{R}^n} e^{-2|\xi|^2(t+s)}\, d\xi \right)^{1/2} \|\widehat{D^\gamma u}\|_1 \, \|D^\mu u\|_2 \, ds,$$

where $\gamma = \beta$ if $|\beta| \le |\alpha + e_i - \beta|$ and hence $\mu = \alpha + e_i - \beta$; otherwise, $\gamma = \alpha + e_i - \beta$ and $\mu = \beta$. With this choice of γ we insure $|\gamma| \le k$. Recall that, by inductive hypothesis,

$$\|\widehat{D^\gamma u_r}\| \le M_e \qquad \text{where } |\gamma| = \ell \le k,$$

and by Theorem 2.5,

$$\|D^\mu u\|_2 \le K_s \qquad \text{for } s = |\mu|.$$

Hence, letting $M = \max(M_\ell)$, $\ell \le k$, $K = \max K_s$, $s \le m$, then

$$P_1 \le MKC\omega(n) \int_0^t \frac{1}{(t-s)^{n/4}} \|D^\gamma u\|_1 \, ds, \tag{2.12}$$

$$\text{where } \omega(n) = \left(\int_{\mathbb{R}^n} e^{-\eta^2}\, d\eta \right)^{1/2}.$$

The estimates of P_2 and P_3 are similar:

$$P_2 \le KC\omega(n) \int_0^t \frac{1}{(t-s)^{n/4}} \|\widehat{D^\alpha u}\|_1 \, ds, \tag{2.13}$$

$$P_3 \le KC\omega(n) \int_0^t \frac{1}{(t-s)^{n/4}} \|\widehat{D^\nu u}\|_1 \, ds, \quad \nu = \alpha + e_i - e_r. \tag{2.14}$$

Note that $|\nu| = |\alpha|$. Combining the estimates for P_1, P_2, P_3 given in (2.12), (2.13), and (2.14), respectively, yields the following bound for II in (2.10):

$$\mathrm{II} \le \alpha_n + \beta_n \int_0^t \frac{1}{(t-s)^{n/4}} \left(\|\widehat{D^\alpha u}\|_1 + \|\widehat{D^\nu u}\|_1 \right) ds \tag{2.15}$$

where $\alpha_n = MK\tau T_0^{1-n/4}(1-n/4)^{-1}\omega(n)$, $\beta_n = KC\omega(n)$.

The bounds for I and III in (2.10) follow easily by the hypothesis in the data u_0 and the forcing function f. That is,

$$\mathrm{I} \le \|\widehat{D^\alpha u_0}\|_{L^1} \le \Gamma_0 \tag{2.16}$$

$$\mathrm{III} \le \omega(n) \int_0^t \frac{1}{(t-s)^{n/4}} \|D^\alpha f\|_2 \, ds \le \omega(n) \left(1 - \frac{n}{4}\right)^{-1} T_0^{-1-n/r} B = \Gamma_1 . \tag{2.17}$$

Note that for III we used Schwarz's inequality and B is such that $\|f\|_{H^m} \leq B$ with $|\alpha| \leq m$. Such a B exists by hypothesis. Define

$$\|\widehat{D^{k+1}u}(\xi,t)\|_1 = \sum_{\substack{\alpha=k+1\\ j}} \int_{\mathbb{R}^n} |\widehat{D^\alpha u_j}(\xi,t)|\, d\xi\,.$$

Now sum (2.9) over j and all α such that $|\alpha| \leq k+1$. Combine the bounds obtained in (2.15), (2.16), and (2.17) with (2.10) to yield

$$\|D^{k+1}u\|_1 \leq \Gamma_0 + \Gamma_1 + \alpha_n + 2\beta_n \int_0^t \frac{1}{(t-s)^{n/4}} \|\widehat{D^{k+1}u}\|_1\, ds\,.$$

Now apply Lemma 2.2 with $\varphi(t) = \|D^{k+1}u(t)\|_1$, $A_n = \Gamma_0 + \Gamma_1 + \alpha_n$, $B_n = 2\beta_n$. Hence $\|\widehat{D^{k+1}u}(\xi,t)\| \leq 2A_n \, \exp \varepsilon^{-1}(1-n/4)T_0$, where $\varepsilon^{1-n/4}B_n = \frac{1}{2}(1-n/4)$ and $t \in [0,T_0]$. This completes the proof of Theorem 2.6. □

3. L^q estimates for the pressure and its derivative of all orders. The results of the last section are used to bound the pressure and its derivatives of all orders in L^q for $q=2$ and $q=\infty$. Standard interpolation arguments will give a bound for the L^q norms with $q \in (2,\infty)$. The bounds for the pressure are included for completeness since they are an immediate consequence of the results of the last section and will not lengthen the arguments. The pressure bounds can also be obtained using the classical Calderón–Zigmund theory [8].

Theorem 3.1. *Let $u_0 \in L^1 \cap H \cap H^1(\mathbb{R}^n)$, $\widehat{D^\alpha u_0} \in L^1$ and $f \in H^m(\mathbb{R}^n)$, $n = 2,3$, div $f = 0$. Let $u(x,t)$ be a solution to the* (NS) *equations. If $n=3$, suppose that $\|\nabla u(\cdot,t)\|_2 \leq M_0$ for $t \in [0,T_0]$. Then*

$$\left\|D^\alpha p(\cdot,t)\right\|_q \leq C_q, \qquad t \in [0,T_0],\ |\alpha| \leq m,$$

where C_q depends only on the norms of the data, M_0, T_0, and q.

Proof. Recall that by Lemma 2.1, it follows that

$$|\widehat{D^\alpha p}(\xi,t)| \leq C|\widehat{D^\alpha u^2}(\xi,t)|. \tag{3.1}$$

Hence, by Plancherel, Lemma 2.1 and the Leibnitz generalized rule,

$$\left\|D^\alpha p\right\|_2^2 = \left\|\widehat{D^\alpha p}\right\|_2^2 \leq C \sum_{\substack{\beta\leq\alpha\\ i,j}} K_{\alpha,\beta} \left\|D^\beta \widehat{u_i\, D^{\alpha-\beta}} \mu_j\right\|_2^2,$$