Large-time Behaviour of Solutions to the Magneto-Hydrodynamics equations

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Abstract

In this paper we establish lower and upper bounds for the rate of decay of the total energy and of the magnetic energy of solutions to the magneto-hydrodynamics equations on $\mathbf{R}^n$, $2 \leq n \leq 4$. It is shown that weak solutions subject to large initial data outside a class of functions with *total radially equidistributed energy* decay algebraically (rather than exponentially). It is also proved that initial data with radially equidistributed energy gives rise to weak solutions which decay exponentially (rather than algebraically).

Key words: Magneto-hydrodynamics equations, decay of solutions
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1 Introduction

The purpose of this paper is to derive upper and lower bounds on the decay of the total energy and the magnetic energy of a viscous incompressible

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electrically conducting resistive fluid. The interactions between the fluid motions and the magnetic field are modelled by the magneto-hydrodynamics equations which, in non-dimensional form, can be written as (cf. [4], [11]):

$$\begin{aligned}\frac{\partial}{\partial t}u + (u\cdot\nabla)u - S(B\cdot\nabla)B + \nabla(P + \frac{1}{2}S|B|^2) &= \frac{1}{Re}\Delta u + f,\\ \frac{\partial}{\partial t}B + (u\cdot\nabla)B - (B\cdot\nabla)u &= -\frac{1}{Rm}\nabla\times(\nabla\times B),\\ \nabla\cdot u = 0, \quad \nabla\cdot B &= 0.\end{aligned}$$

Here u, P and B are non-dimensional quantities corresponding to the velocity of the fluid, its pressure, and the magnetic field; $f(x,t)$ represents a non-dimensional density of volume force, and $|B|^2/2$ is the magnetic pressure. The non-dimensional numbers appearing in these equations are the Reynolds number, Re, the magnetic Reynolds number Rm, and $S = M^2/ReRm$, where M is the Hartman number. For the sake of notational simplicity, and without restricting generality, we set all of these numbers to unity. Thus, upon such normalisation and letting $p = P + \frac{1}{2}S|B|^2$ denote the total pressure, we consider the magneto-hydrodynamics (MHD) equations in $\mathbf{R}^n$, $2 \leq n \leq 4$:

$$\text{(MHD)} \qquad \begin{aligned}\frac{\partial}{\partial t}u + (u\cdot\nabla)u - (B\cdot\nabla)B + \nabla p &= \Delta u + f,\\ \frac{\partial}{\partial t}B + (u\cdot\nabla)B - (B\cdot\nabla)u &= \Delta B,\\ \nabla\cdot u = 0, \quad \nabla\cdot B &= 0,\end{aligned}$$

supplemented with the initial conditions

$$u(x,0) = u_0(x), \quad B(x,0) = B_0(x).$$

There is no loss in generality in assuming that the forcing function is divergence free; i.e., that $\nabla\cdot f(t) = 0$ for all $t \geq 0$. We shall always make this assumption.

We show that weak solutions to the MHD equations, subject to large initial data outside a class of functions with total radially equidistributed energy, decay algebraically (rather than exponentially). In particular we prove that, for such solutions, the total energy (kinetic plus magnetic) and the magnetic energy have slowly decaying algebraic lower bounds. Moreover, we show in which cases the lower bounds are valid for the kinetic energy alone or the magnetic energy alone. Thus, our results reinforce mathematically the observation made by Chandrasekhar [3] that "the magnetic field in systems of large linear dimensions can endure for relatively long periods of time".

The following notation will be used throughout the paper: $H^m(\mathbf{R}^n)$ will denote the Hilbertian Sobolev space on $\mathbf{R}^n$ of index m, $m \geq 0$; $L^p(\mathbf{R}^n)$ will stand for the Lebesgue space equipped with its standard norm $\|\cdot\|_p$, $1 \leq p \leq \infty$;

$$\mathcal{V} = \{v \in [C_0^\infty(\mathbf{R}^n)]^n \,:\, \nabla \cdot u = 0\}, \quad H = \text{closure of } \mathcal{V} \text{ in } [L^2(\mathbf{R}^n)]^n.$$

In addition we introduce the following weighted Lebesgue spaces (and associated norms):

$$W_1 = \{v \,:\, \int_{\mathbf{R}^n} |x|^2 |v(x)| dx < \infty\}, \quad W_2 = \{v \,:\, \int_{\mathbf{R}^n} |x||v(x)|^2 dx < \infty\},$$

$$|v|_{W_1} = \int_{\mathbf{R}^n} |x|^2 |v(x)| dx, \quad |v|_{W_2} = \left[\int_{\mathbf{R}^n} |x||v(x)|^2 dx\right]^{1/2}.$$

Suppose that $(u_0, B_0) \in H \times H$, and let $f \in L^1(0,\infty; L^2(\mathbf{R}^n)^n)$. By a weak solution of the MHD equations we mean a function $(u, B) \in \mathcal{B} \times \mathcal{B}$, where

$$\mathcal{B} = C_w([0,\infty), H) \cap L^2_{loc}((0,\infty), [H^1(\mathbf{R}^n)]^n),$$

satisfying the integral relations

$$\begin{aligned} &\langle u(t), \phi(t)\rangle + \int_0^t -\langle u(s), \frac{\partial \phi}{\partial s}(s)\rangle + \langle \nabla u(s), \nabla \phi(s)\rangle \\ + \quad &\langle (u(s)\cdot\nabla)u(s) - (B(s)\cdot\nabla)B(s), \phi(s)\rangle ds = \langle u_0, \phi(0)\rangle + \int_0^t \langle f(s), \phi(s)\rangle ds, \\ &\langle B(t), \phi(t)\rangle + \int_0^t -\langle B(s), \frac{\partial \phi}{\partial s}(s)\rangle + \langle \nabla B(s), \nabla \phi(s)\rangle \\ + \quad &\langle (u(s)\cdot\nabla)B(s) - (B(s)\cdot\nabla)u(s), \phi(s)\rangle ds = \langle B_0, \phi(0)\rangle, \end{aligned} \tag{1}$$

for all smooth divergence-free vector fields $\phi(x,t)$ with compact support in $\mathbf{R}^n \times [0,\infty)$. Here $\langle \cdot, \cdot \rangle$ denotes the scalar product of the space $[L^2(\mathbf{R}^n)]^n$, and $C_w([0,\infty), H)$ is the space of weakly continuous functions from $[0,\infty)$ into H.

When $n = 2$, the existence of a unique strong solution is well known, whereas for $n \geq 3$ no uniqueness result without restricting the class of solutions is available (see, [5] and [21]). Nevertheless, all solutions to the MHD equations constructed so far do obey the energy inequality, stated in (2) below, or at least they may be approximated by solutions of this type. This is well known for solutions to the Navier-Stokes equations (see [13], [24]). The methods used to construct solutions to the Navier-Stokes equations ([2], [13], [22]) can be easily extended to a construction for the MHD equations, and the solutions will in the same fashion satisfy the energy inequality (2) for $2 \leq n \leq 4$. Thus, following the approach of Wiegner [24] for the incompressible Navier-Stokes equations, we shall assume in the rest of the paper that the solution $(u(t), B(t))$ to the MHD equations obeys:

$$\begin{aligned} \|u(t)\|_2^2 &+ \|B(t)\|_2^2 + 2\int_s^t (\|\nabla u(r)\|_2^2 + \|\nabla B(r)\|_2^2) dr \\ &\leq \|u(s)\|_2^2 + \|B(s)\|_2^2 + 2\int_s^t |\langle f(r), u(r)\rangle| dr, \end{aligned} \tag{2}$$

for $s = 0$, and almost all $s > 0$ and all $t \geq s$.

By our assumption that $f \in L^1(0, \infty; L^2(\mathbf{R}^n)^n)$, this implies (in the same fashion as in [24]) the existence of a constant $C = C(u_0, B_0, f)$ such that for $s = 0$, and almost all $s > 0$ and all $t \geq s$,

$$\begin{aligned} \|u(t)\|_2^2 &+ \|B(t)\|_2^2 + 2\int_s^t (\|\nabla u(r)\|_2^2 + \|\nabla B(r)\|_2^2) dr \\ &\leq \|u(s)\|_2^2 + \|B(s)\|_2^2 + C\int_s^t \|f(r)\|_2 dr. \end{aligned} \tag{3}$$

Indeed, if f is smooth and $\|f(t)\|_2 > 0$, then $h(t) = |\langle f(t), u(t)\rangle| \cdot \|f(t)\|_2^{-1}$ is a continuous function, and, by the Cauchy-Schwarz inequality,

$$\begin{aligned} h^2(t) \leq \|u(t)\|_2^2 &\leq \|u_0\|_2^2 + \|B_0\|_2^2 + 2\int_0^t h(r)\|f(r)\|_2 dr \\ &\leq \|u_0\|_2^2 + \|B_0\|_2^2 + \int_0^\infty \|f(r)\|_2 dr + \int_0^t h^2(r)\|f(r)\|_2 dr \\ &\equiv C_*(u_0, B_0, f) + \int_0^t h^2(r)\|f(r)\|_2 dr. \end{aligned}$$

Thence, by virtue of Gronwall's inequality, we have that $h(t) \leq C(u_0, B_0, f)$, and the required inequality follows from (2). The general case of $f \in L^1(0, \infty; L^2(\mathbf{R}^n)^n)$ is dealt with by approximation.

In Section 2 we analyze the lower and upper bounds of L^2 decay rates for solutions to the heat equation subject to a forcing term. The upper bounds are included for completeness. In this paper, we improve known upper bounds given in [16] and [9]. The derivation of the lower bounds for weak solutions to the MHD equations presented in this paper hinges on decay properties of solutions $(u_0(x,t), B_0(x,t))$ to the non-homogeneous heat system

$$\frac{\partial}{\partial t} u_0 = \Delta u_0 + f, \qquad u_0(x,0) = u_0(x),$$

(HS,f)

$$\frac{\partial}{\partial t} B_0 = \Delta B_0, \qquad B_0(x,0) = B_0(x),$$

with the same initial data, (u_0, B_0), and volume force, f, as in the MHD equations. We will impose certain conditions on the functions f and prove that these conditions guarantee that the solutions of the heat equation with forcing function f decay at the same rate as the corresponding solutions of the "free" heat equation. We need these conditions on comparing solutions of the MHD equations with solutions of the heat equation in our proof that, at least generically, the square L^2-norms of the solutions to the MHD equations have a rate of decay bounded below by $C(t+1)^{-n/2-1}$ (with $C > 0$).

Definition. Let $f : \mathbf{R}^n \times [0, \infty) \to \mathbf{R}^n$ be measurable, and assume $\nabla \cdot f(t) = 0$ for all $t \geq 0$. Let $\mu, \nu, \sigma \in \mathbf{R}$. We say

a $f \in A_\mu$ if there exists $C \geq 0$ such that

$$\|f(t)\|_2 \leq C(t+1)^{-\mu} \quad \text{for} \quad t \geq 0.$$

b $f \in B_\sigma$ if there exists $C \geq 0$ such that

$$|\hat{f}(\xi, t)| \leq C|\xi|^\sigma \quad \text{for} \quad t \geq 0, \xi \in \mathbf{R}^n.$$

c $f \in C_\nu$ if there exists $C \geq 0$ such that

$$\|f(t)\|_\infty \leq C(t+1)^{-\nu} \quad \text{for} \quad t \geq 0.$$

We can make these spaces into Banach spaces defining norms in the obvious way; for example,

$$\|f\|_{A_\mu} = \sup_{t\geq 0}(t+1)^\mu \|f(t)\|_2,$$

and similarly for $\|f\|_{B_\sigma}$, $\|f\|_{C_\nu}$.

We also have to assume frequently that $f \in L^1(0,\infty; L^2(\mathbf{R}^n)^n)$; we define the norm in this space by

$$\|f\|_{L^1(L^2)} = \int_0^\infty \|f(t)\|_2 \, dt.$$

For our lower bounds, we need to be able to estimate first order moments of the solution (u, B) of the MHD equations (cf. Lemma 6.1). For this purpose we assume $f \in L^1(0,\infty; W_1)$.

In order to derive the lower bounds, we use the ideas presented in [20] for solutions to the Navier-Stokes equations. We compare the decay of solutions of the MHD equations to those of the heat system with the same initial datum. The decay in L^2-norm of solutions of the heat equation (or system) depends mainly on the presence (or absence) of long waves in the initial datum; i.e., on the size of the Fourier transform of the datum near the origin.

We show that for initial datum $(u_0, B_0) \in H \times H$, the L^2-norm of the difference between the solutions of the MHD equations and the heat system (with the same datum) satisfies an upper bound which decays faster than the L^2-norm of the solution of the heat system. Specifically, we prove the following theorem

Theorem. *Let the initial datum* $(u_0, B_0) \in H \times H$ *and assume*

$$f \in L^1(0,\infty; L^2(\mathbf{R}^n)^n) \cap A_{n/4+1} \cap B_4 \cap C_{(n+3)/2}.$$

a) *If* $n = 2$ *and not all components of* $\hat{u}_0(0)$ *or* $\hat{B}_0(0)$ *are zero (in the sense defined at the beginning of Section 2), then*

$$\|u(t) - u_0(t)\|_2^2 + \|B(t) - B_0(t)\|_2^2 \leq C_{D_0}(t+1)^{-n/2-1}(1 + \log^2(t+1));$$

b) *if* $n \geq 3$, *or if* $n = 2$ *and* $(u_0, B_0) \in [H \cap L^1(\mathbf{R}^n)^n]^2$, *then*

$$\|u(t) - u_0(t)\|_2^2 + \|B(t) - B_0(t)\|_2^2 \leq C_{D_1}(t+1)^{-n/2-1}.$$

Cases a) and b) are mutually exclusive due to the following lemma established by Borchers [1].
LEMMA [B]. *Let $u \in L^1(\mathbf{R}^n)^n \cap H$. Then*

$$\int_{\mathbf{R}^n} u\,dx = 0.$$

Proof. Suppose first $u \in [C_0^\infty(\mathbf{R}^n)]^n$. Then

$$\int_{\mathbf{R}^n} u_1\,dx = \int_{\mathbf{R}^n} u \cdot \nabla x_1\,dx = \int_{\mathbf{R}^n} (\nabla \cdot u) x_1\,dx = 0.$$

The general case of $u \in L^1(\mathbf{R}^n)^n \cap H$ follows using approximations and passing to the limit.□

For the sake of completeness, we notice that case a) of the theorem can obtain; in fact there do exist functions $u \in H$ such that $\hat{u}(0) \neq 0$ in the sense of Section 2. For example, let $g \in L^2 \cap L^\infty_{\text{loc}}(\mathbf{R}^n)$ and define u by

$$\hat{u}(\xi) = (\frac{\xi_2}{|\xi|} g(\xi), -\frac{\xi_1}{|\xi|} g(\xi), 0, \ldots, 0).$$

Then $u \in H$, but $\hat{u}(0) \neq 0$ if g is bounded away from zero near the origin.

By showing that in case $(\hat{u}_0(0), \hat{B}_0(0)) \neq (0,0)$, the solution of the heat system satisfies the lower bounds

$$\|u_0(t)\|_2^2 \geq C_2(t+1)^{-n/2}, \quad \|B_0(t)\|_2^2 \geq C_2(t+1)^{-n/2},$$

the lower bound for the MHD equations easily follows from the estimates in cases a) and b) above.

For the difference between a weak solution $u(t)$ of the Navier-Stokes equations and a semigroup solution $u_0(t)$ of the heat system, Wiegner [24] showed that one has the estimate

$$\|u(t) - u_0(t)\|_2^2 \leq C(t+1)^{-n/2-1}.$$

It is worth mentioning, however, that whereas for the Navier-Stokes equations this inequality holds regardless whether or not the initial data has zero average, this is no longer true for the MHD equations in two dimensions (cf. Corollary 3.2). This distinctive feature stems from dimensional arguments and the fact that the nonlinear terms in the MHD equations involve both the velocity and the magnetic field; thus when we estimate the difference between solutions of the MHD equations and the heat system, the bound involves $\|\nabla u_0(t)\|_\infty$ which decays at a rate depending on whether or not the initial data has zero average. A more subtle situation occurs when the average of the initial data is equal to zero: in this instance there are no long waves present in the initial data and the corresponding solution of the heat equation in Fourier space may contain only short waves, so that the L^2-norm of the solution to the heat system decays very fast. However, for the MHD equations the nonlinear terms can produce a mixing of the Fourier modes which slows down the decay. As was shown in [20] for the Navier-Stokes equations, solutions with data outside of a class of radially equidistributed functions will immediately produce low Fourier modes. More precisely, it can be shown that in frequency space solutions to the MHD equations take the form

$$\begin{aligned} \hat{u}(\xi, t_0) &= \xi \cdot \alpha(\xi, t_0) + h(\xi, t_0), \\ \hat{B}(\xi, t_0) &= \xi \cdot \beta(\xi, t_0) + k(\xi, t_0), \end{aligned} \tag{4}$$

where α and β are homogeneous functions of degree zero, and h and k are $O(|\xi|^2)$. This is proved as in [20]. The appearance of long waves (i.e., low Fourier modes) does not seem to be sufficient to insure that the decay rate will slow down. We need to add a hypothesis on the solutions guaranteeing that these slow Fourier modes keep showing up in a sequence of times tending to infinity. We conjecture that this hypothesis is removable; whenever a slow Fourier mode has appeared it will persist as time goes to infinity. In other words, once long waves are present they will stay indefinitely. For the time being, we show that if the slow modes do not persist then the solution decays at a faster rate. We also show that the persistence of the slow modes is generic (Section 4). Next we compare the solution of the MHD equations with solutions of the heat system with initial data $(u(\cdot, t_m), B(\cdot, t_m))$, where $\{t_m\}$ is a sequence of times satisfying $\lim_{m\to\infty} t_m = \infty$. The form of the initial data given by (4), with t_0 replaced by t_m, guarantees that the corresponding solution, $(u_0(t_m), B_0(t_m))$, of the heat system satisfies

$$\|u_0(t_m)\|_2^2 + \|B_0(t_m)\|_2^2 \geq C_3(t_m+1)^{-n/2-1},$$

with C_3 independent of m. The derivation of the lower bound for the total energy of the solution to the MHD equations is now reduced to showing that the upper decay constant, C_{D_1}, for the difference, $(u(t)-u_0(t), B(t)-B_0(t))$, satisfies

$$C_{D_1} < C_3,$$

where C_3 is the constant corresponding to the lower bound for the solution of the heat system. A similar analysis yields the lower bound on the decay for the magnetic energy alone.

This part of the analysis is performed for initial data giving rise to solutions in $\mathcal{M}_0^c$, the complement of the set $\mathcal{M}_0$ of functions with total radially equidistributed energy. The set $\mathcal{M}_0$ is a generalisation of the class of radially equidistributed functions, M, introduced in [18] to obtain algebraic lower bounds for the Navier-Stokes equations. Given $u = (u_1, ..., u_n)$ and $B = (B_1, ..., B_n)$ in $[L^1(0, \infty; L^2(\mathbf{R}^n))]^n$, let

$$\tilde{\mathcal{A}}_{ij} = \int_0^\infty \int_{\mathbf{R}^n} (u_i u_j - B_i B_j) dx,$$

$$\tilde{\mathcal{C}}_{ij} = \int_0^\infty \int_{\mathbf{R}^n} (u_i B_j - B_i u_j) dx.$$

We also set

$$\langle x, B_0 \rangle_{ij} = \int_{\mathbf{R}^n} x_j B_i(x, 0)\, dx.$$

Then, introducing the matrices $\tilde{\mathcal{A}} = [\tilde{\mathcal{A}}_{ij}]$, $\tilde{\mathcal{C}} = [\tilde{\mathcal{C}}_{ij}]$, and $\langle x, B_0 \rangle = [\langle x, B_0 \rangle_{ij}]$, we define

$$\mathcal{M}_0 = \{(u, B) \in [L^1(0, \infty; L^2(\mathbf{R}^n)]^{2n} \,:\, \tilde{\mathcal{A}} \text{is scalar and } \tilde{\mathcal{C}} = \langle x, B_0 \rangle\}.$$

We note that $\mathcal{M}_0$ is very small. More precisely, if $(u, B) \in [L^1(0, \infty; L^2(\mathbf{R}^n))]^2$, then generically $(u, B) \in \mathcal{M}_0^c$. This will be established in Section 4 (cf. Lemma 4.2 and Corollary 4.3). In other words, generically solutions are not in $\mathcal{M}_0$, the set where slow Fourier modes do not persist. We show that the lower bound holds if and only if the solution is not in $\mathcal{M}_0$ (cf. Section 4, Theorem 4. 3). We strengthen this result in the final section where we give examples of solutions to the MHD equations not in $\mathcal{M}_0$ which decay exponentially (rather than algebraically). These examples, of solutions which decay exponentially rather than algebraically, are of two types. Solutions where the magnetic field B is zero and the velocity is simultaneously a solution to the Navier-Stokes and the heat equations; secondly, solutions where both the velocity u and the magnetic field B satisfy heat systems. The example we have is valid in all even space dimensions and shows that the classical radial solution of the Navier-Stokes equations is also a solution of the homogenenous heat equation. It extends in a quite technical way the 2 dimensional example given in [20]. The analysis done here gives a general method for constructing simultaneous solutions to the heat equation and the Navier-Stokes equations. It raises the question whether only solutions which also satisfy the heat equation can decay exponentially.

Our results are also a first step to showing that, in the far field, the behaviour of solutions to the MHD equations and the heat system are quite different. Whereas solutions to the heat system may have very fast algebraic or exponential decay when the initial data is rapidly oscillating, the nonlinear terms in the MHD equations generically alter the decay of solutions to a slow algebraic rate when started from the same initial data. We note that, with minor modifications, the analysis presented here can be adapted to the case when the second equation in (MHD) also contains a forcing term. We have the following result.

Theorem. *Let* $(u_0, B_0) \in [W_2 \cap H]^2$,

$$f \in L^1(0, \infty; L^2(\mathbf{R}^n)^n \cap W_1) \cap A_{n/4+1} \cap B_4 \cap C_{(n+3)/2},$$

and let $(u(x,t), B(x,t))$ *be a weak solution of the MHD equations with initial datum* $(u(x,0), B(x,0)) = (u_0(x), B_0(x))$.

a) If $\hat{u}_0(0) \neq 0$ or $\hat{B}_0(0) \neq 0$, *then*

$$C_0(t+1)^{-n/2} \leq \|u(\cdot,t)\|_2^2 + \|B(t)\|_2^2 \leq C_1(t+1)^{-n/2}.$$

b) If $(u_0, B_0) \in [W_2 \cap H \cap [L^1(\mathbf{R}^n)]^n]^2$ *(so that*

$$\hat{u}_0(0) = (2\pi)^{-n/2} \int_{\mathbf{R}^n} u_0(x)\, dx = 0, \hat{B}_0(0) = (2\pi)^{-n/2} \int_{\mathbf{R}^n} B_0(x)\, dx = 0,$$

by Borchers' Lemma) and $(u, B) \notin \mathcal{M}_0$,*then*

$$C_2(t+1)^{-n/2-1} \leq \|u(\cdot,t)\|_2^2 + \|B(\cdot,t)\|_2^2 \leq C_3(t+1)^{-n/2-1}.$$

We note that if $\Re(\nabla \hat{u}_0(0)) \neq 0$ the solutions stay in $\mathcal{M}_0^c$.

In this paper we also obtain a very precise expression for the first term of the Taylor expansion in frequency space of solutions to the MHD equations. The expansion is valid, as well, for solutions to the Navier-Stokes equations setting $B \equiv 0$. In fact, the MHD equations reduce to the Navier-Stokes equations when $B \equiv 0$ so that the methods presented here are also suitable for the analysis of the decay of solutions to the Navier-Stokes equations with a forcing term f. The lower bound on the decay rate of the total energy for the MHD equations is the same as that of the kinetic energy for the incompressible Navier-Stokes equations, with the set $\mathcal{M}_0$ replaced by its counterpart

$$M = \{u \in L^1\left((0,\infty), L^2(\mathbf{R}^n)^n\right) \,:\, [r_{ij}] \text{ is scalar, where } r_{ij} = \int_0^\infty \int_{\mathbf{R}^n} u_i u_j dx\}.$$

We note that in this paper we corrected a gap that was found in the corresponding result for the Navier-Stokes equations [20].

Finally, we mention that upper bounds for the MHD equations have also been derived in [16] and [9].

2 Bounds for solutions of the heat equation

We begin with the homogeneous heat equation. If $\psi \in L^1_{\text{loc}}$ (or, more generally if ψ is a distribution), we say ψ has a zero of order ρ at the origin if in some neighborhood V of the origin

$$\psi(\xi) = \mu(\xi) + h(\xi) \qquad \text{for}\, \xi \in V$$

where μ is a non-zero function homogeneous of degree ρ, continuous on $\mathbf{R}^n\backslash\{0\}$, and $h(\xi) = O(|\xi|^{\rho+\epsilon})$ for $\xi \to 0$, some $\epsilon > 0$. If ψ has a zero of order 0, we will say (with some abuse of language) that ψ does not vanish at 0, write $\psi(0) \neq 0$. This generalisation of the concept of a zero at the origin will be used exclusively in case $\psi(\xi) = \hat{u}(\xi)$, $u \in L^2(\mathbf{R}^n)$. Moreover, u will be the solution of a heat equation or part of the system of solutions of the MHD equations for some (fixed) time value. If ρ is a non-negative integer and $|x|^\rho u \in L^1$, then $\hat{u}$ is ρ-times continuously differentiable (with bounded derivatives) and $\hat{u}$ has a zero of order ρ at 0 in the generalised sense if and only if it has a zero of order ρ in the conventional sense. In particular, if $u \in L^1$, then $\hat{u}(0) \neq 0$ if and ond only if $\hat{u}$ has a zero of order 0 at the origin. Since we also have for $u \in L^1$ that

$$\hat{u}(0) = (2\pi)^{-n/2} \int_{\mathbf{R}^n} u(x)\, dx,$$

so that $\hat{u}(0) = 0$ if and only if u has a zero average. With some abuse of language, we shall say that a function $u \in L^2(\mathbf{R}^n)$ has zero average if its Fourier transform $\hat{u}$ has a zero of order at least 1 at 0. Also, with some abuse of language, we shall say that a function $u \in L^2$ satisfies $\hat{u}(0) \neq 0$ if $\hat{u}$ has a zero of order 0 at the origin; i.e., if $\hat{u}$ differs from a homogeneous function of degree 0 by a quantity of order $O(|\xi|^\epsilon)$ for $\xi \to 0$, where $\epsilon > 0$. If $u \in W_1$, these concepts reduce to their usual interpretation.

Lemma 2.1 *Let $u_0 \in L^2(\mathbf{R}^n)$ and let $v = v(x,t)$ be the solution of the homogeneous heat equation $v_t = \Delta v$ with initial datum $v(0) = u_0$. If $\hat{u}_0$ has a zero of order $\rho \geq 0$ at the origin, then there exist constants $C_0 > 0$, $C_1 \geq 0$ such that*

$$C_0(t+1)^{-\frac{n}{2}-\rho} \leq \|v(t)\|_2^2 \leq C_1(t+1)^{-\frac{n}{2}-\rho}$$

for all $t \geq 0$.

Proof. Let $\delta > 0$. By Plancherel's Theorem,

$$\|v(t)\|_2^2 = \int_{\mathbf{R}^n} |\hat{u}_0(\xi)|^2 e^{-2t|\xi|^2}\, d\xi = A_\delta(t) + B_\delta(t),$$

with

$$A_\delta(t) = \int_{|\xi|\leq\delta} |\hat{u}_0(\xi)|^2 e^{-2t|\xi|^2}, B_\delta(t) = \int_{|\xi|>\delta} |\hat{u}_0(\xi)|^2 e^{-2t|\xi|^2}.$$

Since

$$B_\delta(t) \leq e^{-2\delta^2 t}\|u_0\|_2^2,$$

$B_\delta(t)$ decays exponentially and it suffices to see that $A_\delta(t)$ decays at the desired rate. We select $\delta > 0$ so that

$$\hat{u}_0(\xi) = \mu(\xi) + h_1(\xi)$$

for $|\xi| < \delta$, where μ is homogeneous of order ρ and $h_1(\xi)$ is bounded by const$\cdot|\xi|^{\rho+\epsilon}$, where $\epsilon > 0$. The result follows, since

$$\int_{|\xi|<\delta} |\mu(\xi)|^2 e^{-2t|\xi|^2}\, d\xi \sim t^{-\rho-n/2}$$

while

$$\int_{|\xi|<\delta} |2\mu(\xi)h_1(\xi) + h_1(\xi)^2)|e^{-2t|\xi|^2}\, d\xi \leq \text{ const}\, t^{-\rho-n/2+\epsilon}.$$

We notice that we can replace t by $t+1$ in the estimates, since $\|v(t)\|_2^2$ is bounded at 0. □

In particular, if $\hat{u}_0(0) \neq 0$, we get

Corollary 2.2 *Let $u_0 \in L^2(\mathbf{R}^n)$ and let $v = v(x,t)$ be the solution of the homogeneous heat equation $v_t = \Delta v$ with initial datum $v(0) = u_0$. Assume $\hat{u}_0(0) \neq 0$. Then there exist positive constants C_0, C_1, depending only on the norms of the datum, such that*

$$C_0(t+1)^{-\frac{n}{2}} \leq \|v(t)\|_2^2 \leq C_1(t+1)^{-\frac{n}{2}}$$

for $t \geq 0$.

We want to explore a case in which the initial datum has zero average in a bit more detail.

Lemma 2.3 *Let $u_0 \in [L^2(\mathbf{R}^n)]^n$ and let $v(x,t)$ be the solution of the homogeneous heat system with initial datum u_0. Suppose in addition that there is $\delta > 0$ such that the Fourier transform of u_0 admits the representation*

$$\hat{u}_0(\xi) = P(\xi)\xi + h(\xi), \qquad \text{for } |\xi| \leq \delta,$$

where P and h satisfy the following conditions:
i)P is a homogeneous, $n \times n$ matrix-valued function of degree zero with

$$\|P\| = \sup_{|\xi|=1} |P(\xi)| < \infty$$

(where $|P(\xi)|$ denotes a matrix norm of $P(\xi)$).
ii)$|h(\xi)| \leq M_0|\xi|^2$, for some $M_0 \geq 0$ and all $\xi \in \mathbf{R}^n$, $|\xi| \leq \delta$.
Then

$$\|v(t)\|_2^2 = c_n(\delta,t)\left(\int_{S^{n-1}} |P(\omega)\omega|^2\,d\omega\right) t^{-\frac{n}{2}-1} + O(t^{-\frac{n}{2}-\frac{3}{2}}) \tag{5}$$

where

$$\left(\frac{\delta^2}{2}\right)^{\frac{n}{2}+1} e^{-2\delta^2} \leq c_n(\delta,t) \leq \left(\frac{1}{2}\right)^{\frac{n}{2}+2} \Gamma(\frac{n+2}{2}) \tag{6}$$

for $t \geq 1$, and

$$|O(t^{-\frac{n}{2}-\frac{3}{2}})| \leq Ct^{-\frac{n}{2}-\frac{3}{2}}$$

with C depending only on M_0, $\|P\|$ and δ. If, in addition,

$$\sigma_1 := \int_{S^{n-1}} |P(\omega)\omega|^2\,d\omega \neq 0 \tag{7}$$

then there exist positive constants C_0, C_1 with C_1 depending only on M_0, $\|u_0\|_2$, $\|P\|$, and σ_1, C_0 depending on all these quantities and on δ such that

$$C_0(t+1)^{-\frac{n}{2}-1} \le \|v(t)\|_2^2 \le C_1(t+1)^{-\frac{n}{2}-1}.$$

Proof. We have

$$\begin{aligned}\left| \|v(t)\|_2^2 - \int_{|\xi|\le\delta} e^{-2t|\xi|^2} |\hat{u}_0(\xi)|^2 \, d\xi \right| &= \int_{|\xi|\ge\delta} e^{-2t|\xi|^2} |\hat{u}_0(\xi)|^2 \, d\xi \\ &\le e^{-2t\delta^2} \|u_0\|_2^2.\end{aligned}$$

It suffices to show that $\int_{|\xi|\le\delta} e^{-2t|\xi|^2} |\hat{u}_0(\xi)|^2 \, d\xi$ is of the form $c_n(\delta, t)\sigma_1 t^{-n/2-1} + O(t^{-n/2-3/2})$. For $|\xi| \le \delta$,

$$|\hat{u}_0(\xi)|^2 = |P(\xi)\xi|^2 + \psi(\xi)$$

where

$$|\psi(\xi)| \le C|\xi|^3$$

and C depends only on M_0 and $\|P\|$. Equation (5) follows multiplying $|\hat{u}_0(\xi)|^2$ by $e^{-2t|\xi|^2}$ and integrating over $|\xi| \le \delta$. In fact,

$$\int_{|\xi|\le\delta} |\xi|^3 e^{-2t|\xi|^2} \, d\xi \le \int_{\mathbf{R}^n} |\xi|^3 e^{-2t|\xi|^2} \, d\xi = C \cdot t^{-\frac{n}{2}-\frac{3}{2}},$$

where C is a constant, while

$$\begin{aligned}\int_{|\xi|\le\delta} e^{-2t|\xi|^2} |P(\xi)\xi|^2 \, d\xi &= \left(\int_0^\delta r^{n+1} e^{-2tr^2} dr\right) \left(\int_{S^{n-1}} |P(\omega)\omega|^2 \, d\omega\right) \\ &= \frac{1}{2}\sigma_1 \left(\int_0^{2t\delta^2} s^{\frac{n}{2}} e^{-s} \, ds\right) (2t)^{-\frac{n}{2}-1}.\end{aligned}$$

Notice that

$$(\delta^2)^{\frac{n}{2}+1} e^{-2\delta^2} \le \int_0^{2t\delta^2} s^{\frac{n}{2}} e^{-s} \, ds \le \Gamma(\frac{n+2}{2})$$

for $t \ge 1$. The final statement of the lemma is an easy consequence of (5) and the boundedness near 0 of $\|v(t)\|_2$. $\square$

Having derived upper and lower bounds on the decay of solutions of the homogeneous heat equation, we now consider the heat equation with a forcing function f. Let v satisfy

$$\begin{aligned} \frac{\partial}{\partial t} v &= \Delta v + f \\ v(x,0) &= u_0(x). \end{aligned}$$

We split $v = u + w$, where u solves the free heat equation $u_t = \Delta u$, with $u(0) = u_0$, and w solves $w_t = \Delta w + f$ with initial datum $w(0) = 0$. Then

$$\frac{1}{2}\|u(t)\|_2^2 - \|w(t)\|_2^2 \leq \|v(t)\|_2^2 \leq 2\|u(t)\|_2^2 + 2\|w(t)\|_2^2$$

and to see that $\|u(t)\|_2$ and $\|v(t)\|_2$ have the same rate of decay, it suffices to show that $\|w(t)\|_2$ decays at a faster rate than $\|u(t)\|_2$. We have

Lemma 2.4 *Let v be the solution of $v_t = \Delta v + f$, assume the Fourier transform $\hat{u}_0$ of the initial datum of v has a zero of order $\beta \geq 0$ at 0 and that $f \in A_\mu \cap B_\sigma$, where $\mu > \frac{1}{2}(\frac{n}{2} + \beta + 2)$ and $\sigma > \beta + 2$. There exist positive constants C_1, C_2, such that*

$$C_1(1+t)^{-n/2-\beta} \leq \|v(t)\|_2^2 \leq C_2(1+t)^{-n/2-\beta}$$

for all $t \geq 0$.

Proof. In view of Lemma 2.1 and by the remarks preceding this lemma, it suffices to prove $\|w(t)\|_2^2$ decays at a faster rate than $t^{-n/2-\beta}$, where $v = u + w$ as above. Writing

$$\begin{aligned} w(t) &= \int_0^t e^{-(t-s)\Delta} f(s)\, ds \\ &= \int_0^{t/2} e^{-(t-s)\Delta} f(s)\, ds + \int_{t/2}^t e^{-(t-s)\Delta} f(s)\, ds = I_1 + I_2. \end{aligned}$$

By Plancherel,

$$\begin{aligned} \|e^{-(t-s)\Delta} f(s)\|_2^2 &= \int_{\mathbf{R}^n} e^{-2(t-s)|\xi|^2} |\hat{f}(\xi,s)|^2\, d\xi \\ &\leq \|f\|_{B_\sigma}^2 \int_{\mathbf{R}^n} |\xi|^{2\sigma} e^{-2(t-s)|\xi|^2}\, d\xi = C\|f\|_{B_\sigma}^2 (t-s)^{-n/2-\sigma}, \end{aligned}$$

so that

$$\|I_1\|_2 \leq C\|f\|_{B_\sigma} \int_0^{t/2} (t-s)^{-n/4-\sigma/2}\, ds = C\|f\|_{B_\sigma} t^{-n/4-\sigma/2+1}.$$

By Plancherel (or otherwise), $\|e^{-(t-s)\Delta} f(s)\|_2 \leq \|f(s)\|_2$, hence

$$\|I_2\|_2 \leq \int_{t/2}^t \|f(s)\|_2\, ds \leq \|f\|_{A_\mu} \int_{t/2}^t (s+1)^{-\mu}\, ds \leq C\|f\|_{A_\mu}(t+1)^{-\mu+1}.$$

¿From these bounds we conclude (after squaring and with a constant C depending on $\|f\|_{A_\mu}$, $\|f\|_{B_\sigma}$) that for $t \geq 1$

$$\|w(t)\|_2^2 \leq C[(t+1)^{-n/2-\sigma+2} + (t+1)^{-2\mu+2}] \leq C(t+1)^{-n/2-\beta-\epsilon}$$

where $\epsilon = \min(\sigma - 2 - \beta, 2\mu - \beta - n/2 - 2) > 0$. □

As for the case of the free heat equation, the main cases in which Lemma 2.4 is applied are contained in the following Lemma.

Lemma 2.5 *Let v be the solution of $v_t = \Delta v + f$ with initial datum $v(0) = u_0$. Then*

1. *If $\hat{u}_0(0) \neq 0$ and $f \in A_\mu \cap B_\sigma$ with $\mu > n/4 + 1$, $\sigma > 2$, then there exist constants C_0 and C_1, depending on norms of u_0 and f, such that*
$$C_0(t+1)^{-n/2} \leq \|u(\cdot,t)\|_2^2 \leq C_1(t+1)^{-n/2}.$$
2. *If $\hat{u}_0(\xi) = P(\xi)\xi + h(\xi)$ for $|\xi| \leq \delta$, where $\delta > 0$ and P, h are as in Lemma 2.3, and $f \in A_\mu \cap B_\sigma$ where $\mu > n/4 + 3/2$, $\sigma > 3$, then there exist constants C_2 and C_3, depending on norms of u_0 and f, such that*
$$C_2(t+1)^{-n/2-1} \leq \|u(\cdot,t)\|_2^2 \leq C_3(t+1)^{-n/2-1}.$$

To show that the solution (u, B) to the MHD equations with initial data (u_0, B_0) is asymptotically equivalent to the solution of the heat system with the same initial data, we need the following auxiliary result concerning the decay of the L^∞-norm of the solution of the heat system.

Lemma 2.6 *Suppose $\alpha \geq 0$, and let $v(x,t)$ denote the solution of the heat system* (HS,f) *with $f \in C_\nu$, where $\nu = \dfrac{n}{4} + \dfrac{\alpha}{2} + 1$. If*

$$\|v(\cdot,t)\|_2^2 \leq C(t+1)^{-\alpha}, \quad t \geq 0.$$

Then
(a) $\|\nabla v(\cdot,t)\|_\infty^2 \leq C(t+1)^{-n/2-\alpha-1}, \quad t \geq 1$;
(b) $\|v(\cdot,t)\|_\infty^2 \leq C(t+1)^{-n/2-\alpha}, \quad t \geq 1$.

Proof. Let D^i denote the identity operator if $i=0$, the gradient operator ∇ if $i=1$. Let

$$K(t)(x) = K(x,t) = (4\pi t)^{-n/2} e^{-\frac{|x|^2}{4t}}$$

be the heat kernel; we can then write

$$D^i v(t) = D^i K(t/2) * v(t/2) + \int_{t/2}^{t} D^i K(t-s) * f(s)\, ds$$

so that

$$\|D^i v(t)\|_\infty \leq \|D^i K(t/2)\|_2 \|v(t/2)\|_2 + \int_{t/2}^{t} \|D^i K(t-s)\|_1 \|f(s)\|_\infty\, ds.$$

Standard estimates yield

$$\begin{aligned} \|D^i K(t)\|_2 &= C t^{-\frac{1}{2}(n/2+i)} \\ \|D^i K(t)\|_1 &= C t^{-\frac{i}{2}}, \end{aligned}$$

hence

$$\|D^i v(t)\|_\infty \leq C t^{-\frac{1}{2}(\frac{n}{2}+i)} \|v(t/2)\|_2 + C \int_{t/2}^{t} (t-s)^{-i/2} \|f(s)\|_\infty\, ds.$$

Using $f \in C_\nu$ and the hypothesis on the decay of $\|v(t)\|_2$, we get

$$\begin{aligned} \|D^i v(t)\|_\infty &\leq C t^{-\frac{1}{2}(\frac{n}{2}+i)}\, t^{-\frac{1}{2}\alpha} + C t^{-\frac{1}{2}(\alpha+\frac{n}{2}+2)} \int_{t/2}^{t} (t-s)^{-i/2}\, ds \\ &\leq C t^{-\frac{1}{2}(\alpha+\frac{n}{2}+i)}, \end{aligned}$$

since $0 \leq i/2 < 1$ (and $(t-s)^{-i/2}$ integrated from $t/2$ to t is of order $t^{1-i/2}$). The proof is complete. □

3 Fourier representation and upper bounds for solutions of the MHD equations

In this section, and in the sequel, (u, B) will denote a solution of the MHD equations with initial datum (u_0, B_0) and divergence free forcing function f. In deriving upper bounds for the decay of (u, B) we mimic the methods developed to establish decay of solutions of the Navier Stokes equations in [18] and in [24], most notably the Fourier splitting method (introduced by one of the authors in [17]). The Fourier splitting method has already been used by Mohgooner and Sarayker for the MHD equations (cf. [16]), but their rate is not optimal. Our approach is somewhat formal (if $n \geq 3$) since we are applying ordinary time derivatives to functions which may only be weakly differentiable in time. We remark, however, that the rigorous proof follows applying the method to approximating sequences to solutions, similar to those constructed for the Navier-Stokes equations by Leray [12], and by Caffarelli, Kohn and Nirenberg [2] in case $n = 3$; by Sohr, Wiegner and von Wahl [22], and by Kajikiya and Miyakawa [7] if $n = 2, 3, 4$. The decay results established for the approximating sequence will be valid for the limiting weak solution.

We begin introducing some notation. Let

$$H = (H_1, \ldots, H_n) = (u \cdot \nabla)u - (B \cdot \nabla)B + \nabla p,$$

$$M = (M_1, \ldots, M_n) = (u \cdot \nabla)B - (B \cdot \nabla)u,$$

so that the MHD equations become

$$u_t - \Delta u = -H + f, B_t - \Delta B = -M.$$

Fourier transforming with respect to the space variables and solving the resulting first order equations in time, we get

$$\begin{aligned} \hat{u}(\xi, t) &= e^{-t|\xi|^2}\hat{u}_0(\xi) - \int_0^t e^{-(t-s)|\xi|^2}\left(\hat{H}(\xi, s) - \hat{f}(\xi, s)\right) ds, \\ \hat{B}(\xi, t) &= e^{-t|\xi|^2}\hat{B}_0(\xi) - \int_0^t e^{-(t-s)|\xi|^2}\hat{M}(\xi, s)\, ds. \end{aligned} \tag{8}$$

Since $\nabla \cdot u = \nabla \cdot B = \nabla \cdot f = 0$, applying the divergence operator to the first set of MHD equations gives

$$\Delta p = -\sum_{k,j=1}^{n} \frac{\partial^2}{\partial x_k \partial x_j}(u_k u_j - B_k B_j),$$

where we used

$$\nabla \cdot ((v \cdot \nabla) w) = \sum_{k,j} \frac{\partial^2}{\partial x_k \partial x_j}(v_k w_j) \qquad \text{if } \nabla \cdot v = 0.$$

Hence

$$\hat{p}(\xi, t) = -\frac{1}{|\xi|^2} \sum_{k,j} \xi_k \xi_j \left(\widehat{u_k u_j} - \widehat{B_k B_j} \right). \tag{9}$$

Notice also that

$$(v \cdot \nabla) w = \sum_j \frac{\partial}{\partial x_j}(v_j w) \qquad \text{if } \nabla \cdot v = 0.$$

It follows that

$$\begin{aligned}
\hat{H}(\xi, t) &= i \sum_j \xi_j \left(\widehat{u_j u} - \widehat{B_j B} \right) - i \sum_{k,j} \frac{\xi_k \xi_j}{|\xi|^2} \left(\widehat{u_k u_j} - \widehat{B_k B_j} \right) \xi, \\
\hat{M}(\xi, t) &= i \sum_j \xi_j \left(\widehat{u_j B} - \widehat{B_j u} \right).
\end{aligned}$$

Setting

$$a_{kj} = \widehat{u_k u_j}, b_{kj} = \widehat{B_k B_j}, c_{kj} = \widehat{u_j B_k},$$

we write all this in a somewhat more compact way introducing the $n \times n$ matrices $A = [A_{kj}]$, $C = [C_{kj}]$, $\mu = [\mu_{kj}]$, where

$$A_{kj}(\xi, t) = a_{kj}(\xi, t) - b_{kj}(\xi, t), C_{kj}(\xi, t) = c_{kj}(\xi, t) - c_{jk}(\xi, t),$$

and

$$\mu_{kj}(\xi) = \frac{\xi_k \xi_j}{|\xi|^2}.$$

Then

$$\begin{aligned}
\hat{H}(\xi, t) &= i(I - \mu(\xi)) A(\xi, t) \xi, \\
\hat{M}(\xi, t) &= i C(\xi, t) \xi.
\end{aligned}$$

Since

$$\begin{aligned}|a_{k,j}(\xi,t)|+|b_{k,j}(\xi,t)| &\leq \|\widehat{u_k u_j}\|_\infty + \|\widehat{B_k B_j}\|_\infty \leq (2\pi)^{-n/2}\left(\|u_k u_j\|_1 + \|B_k B_j\|_1\right)\\ &\leq (2\pi)^{-n/2}\left(\|u_k\|_2\|u_j\|_2 + \|B_k\|_2\|B_j\|_2\right)\end{aligned}$$

we see that

$$|A(\xi,t)|^2 \leq 2\sum_{k,j}\left(|a_{kj}(\xi,t)|^2 + |b_{k,j}(\xi,t)|^2\right) \leq 2(2\pi)^{-n}\left(\|u(t)\|_2^2 + \|B(t)\|_2^2\right)^2 \tag{10}$$

for all $\xi \in \mathbf{R}^n$, $t \geq 0$; since $\mu(\xi)$ (hence also $I-\mu(\xi)$) is an orthogonal projection matrix for each $\xi \in \mathbf{R}^n\backslash\{0\}$, we get

$$\left|\hat{H}(\xi,t)\right| \leq 2(2\pi)^{-n/2}\left(\|u(t)\|_2^2 + \|B(t)\|_2^2\right)|\xi| \tag{11}$$

for all (ξ,t). Similarly,

$$|C(\xi,t)|^2 \leq 4\sum_{k,j}|c_{kj}(\xi,t)|^2 \leq (2\pi)^{-n}\left(\|u(t)\|_2^2 + \|B(t)\|_2^2\right)^2 \tag{12}$$

so that

$$\left|\hat{M}(\xi,t)\right| \leq (2\pi)^{-n/2}\left(\|u(t)\|_2^2 + \|B(t)\|_2^2\right)|\xi|. \tag{13}$$

for all ξ, t.

We denote by (v,w) the solution of the (HS,f) system with the same initial datum (u_0,B_0). We set $D=(D_1,D_2)=(u-v,B-w)$. We now obtain several auxiliary estimates which will be needed later. First we have

$$\begin{aligned}\frac{d}{dt}\|D(t)\|_2^2 &= \frac{d}{dt}(\|D_1(t)\|_2^2 + \|D_2(t)\|_2^2)\\ &= 2\langle D_1, \Delta D_1 - H\rangle + 2\langle D_2, \Delta D_2 - M\rangle.\end{aligned}$$

Recall that the form $(u,v,w)\mapsto\langle u,(v\cdot\nabla w)\rangle$ is skew symmetric in the first and third entries (if u,v,w are divergence free), moreover divergence free vector fields are orthogonal to gradients. We thus get, after some integration by parts,

$$\begin{aligned}\frac{d}{dt}\|D(t)\|_2^2 &= -2\|\nabla D(t)\|_2^2 - 2\langle D_1, u\cdot\nabla v\rangle + 2\langle B, B\cdot\nabla v\rangle - 2\langle B, u\cdot\nabla w\rangle\\ &\quad +2\langle u, B\cdot\nabla w\rangle\\ &= -2\|\nabla D(t)\|_2^2 - 2\langle D_1, u\cdot\nabla v\rangle + 2\langle D_1, B\cdot\nabla w\rangle - 2\langle D_2, u\cdot\nabla w\rangle\\ &\quad +2\langle D_2, B\cdot\nabla v\rangle.\end{aligned} \tag{14}$$

We proceed to apply the Fourier splitting method. Let $g(t) \geq 0$ for $t \geq 0$ (to be determined later) and let $G(t) = \exp(2\int^t g(s)^2\,ds)$, so that $G' = 2g^2G$. By (14),

$$\begin{aligned} \frac{d}{dt}\left(G(t)\|D(t)\|_2^2\right) &= 2[g(t)^2\|D(t)\|_2^2 - \|\nabla D(t)\|_2^2 - \langle D_1, u\cdot\nabla v\rangle \\ &\qquad + \langle B_1, B\cdot\nabla w\rangle - \langle D_2, u\cdot\nabla w\rangle + \langle D_2, B\cdot\nabla v\rangle]G(t). \end{aligned}$$

Estimating each of the last four terms on the right hand side by

$$|\langle q_1, (q_2\cdot\nabla)q_3\rangle| \leq \|q_1\|_2\|q_2\|_2\|\nabla q_3\|_\infty,$$

we get

$$\begin{aligned} \frac{d}{dt}\left(G(t)\|D(t)\|_2^2\right) &\leq 2G(t)\left(g(t)^2\|D(t)\|_2^2 - \|\nabla D(t)\|_2^2\right) \\ &\quad +2(t)\left(\|u(t)\|_2^2 + \|B(t)\|_2^2\right)^{1/2}\|D(t)\|_2(\|\nabla v\|_\infty + \|\nabla w\|_\infty) \end{aligned} \tag{15}$$

For some of our results it suffices to use a simpler inequality. Noticing that $\langle D_1, u\cdot\nabla v\rangle = \langle u, u\cdot\nabla v\rangle$ and using the first inequality in (14) we get (estimating as before)

$$\begin{aligned} \frac{d}{dt}\left(G(t)\|D(t)\|_2^2\right) &\leq 2G(t)\left(g(t)^2\|D(t)\|_2^2 - \|\nabla D(t)\|_2^2\right) \\ &\quad +2G(t)\|D(t)\|_2\left(\|u(t)\|_2^2 + \|B(t)\|_2^2\right)^{1/2}(\|\nabla v\|_\infty + \|\nabla w\|_\infty) \end{aligned} \tag{16}$$

Now, by Plancherel's theorem,

$$\begin{aligned} g(t)^2\|D(t)\|_2^2 - \|\nabla D(t)\|_2^2 &= \int_{\mathbf{R}^n}(g(t)^2 - |\xi|^2)|\hat{D}(\xi,t)|^2\,d\xi \\ &\leq \int_{|\xi|\leq g(t)}(g(t)^2 - |\xi|^2)|\hat{D}(\xi,t)|^2\,d\xi \\ &\leq g(t)^2\int_{|\xi|\leq g(t)}|\hat{D}(\xi,t)|^2\,d\xi. \end{aligned}$$

This inequality is used to derive both the upper and the lower bounds for the decay of the L^2-norm of (u, B). Using it in (16), we get

$$\begin{aligned}\frac{d}{dt}\left(G(t)\|D(t)\|_2^2\right) &\leq 2g(t)^2 G(t)\int_{|\xi|\leq g(t)}|\hat{D}(\xi,t)|^2\,d\xi \\ &\quad + 2(\|u(t)\|_2^2+\|B(t)\|_2^2)(\|\nabla v\|_\infty+\|\nabla w\|_\infty)G(t).\end{aligned} \tag{17}$$

For the purpose of finding upper bounds, it will suffice to bound

$$\int_{|\xi|\leq g(t)}|\hat{D}(\xi,t)|^2\,d\xi$$

as follows. Since

$$D_{1t}=\Delta D_1 - H, D_{2t}=\Delta D_2 - M$$

and $D(0)=(0,0)$, we have (Fourier transforming and solving the first-order equation in t)

$$\begin{aligned}\hat{D}_1(\xi,t) &= -\int_0^t e^{-(t-s)|\xi|^2}\hat{H}(\xi,s)\,ds,\\ \hat{D}_2(\xi,t) &= -\int_0^t e^{-(t-s)|\xi|^2}\hat{M}(\xi,s)\,ds.\end{aligned}$$

By (11), (13) ,

$$|\hat{D}(\xi,t)|=\left(|\hat{D}_1(\xi,t)|^2+|\hat{D}_2(\xi,t)|^2\right)^{1/2}\leq C_n|\xi|\int_0^t(\|u(s)\|_2^2+\|B(s)\|_2^2)\,ds, \tag{18}$$

where from now on C_n denotes a constant depending only on the dimension n, not always the same in all formulas.

We are ready to prove our result on upper bounds. It can be summarised by saying that the solutions of the MHD equation decay at the same rate as the corresponding solutions of the heat system; the difference $D(t)$ decays at a faster rate.

Theorem 3.1 *Let $(u_0, B_0)\in H\times H$, and let $f\in L^1(0,\infty;[L^2(\mathbf{R}^n)]^n)\cap C_\nu$, $\nu = n/4+\alpha/2+1$, and $0<\alpha\leq n/2+1$. Assume that the solution (v,w) of the (HS,f) system satisfies*

$$\|v(t)\|_2^2+\|w(t)\|_2^2\leq K(t+1)^{-\alpha} \tag{19}$$

for all $t\geq 0$, some constant $K\geq 0$.

1. *There exists a constant C, depending only on the L^2-norm of the initial datum (u_0, B_0), on the $L^1(0,\infty; L^2(\mathbf{R}^n)^n) \cap C_\nu$-norm of f, and on K such that*
$$\|u(t)\|_2^2 + \|B(t)\|_2^2 \le C(t+1)^{-\alpha}$$
for $t \ge 0$.

2. *If $1 \le \alpha \le n/2+1$, then there is a constant C, depending only on the L^2-norm of the initial datum (u_0, B_0), on the $L^1(0,\infty; L^2(\mathbf{R}^n)^n) \cap C_\nu$-norm of f, and on K such that*
$$\|D(t)\|_2^2 \le \begin{cases} C(t+1)^{-n/2-1} & \text{if } \ 1 < \alpha \le n/2+1 \quad , \\ C(t+1)^{-n/2-1}(1+\log^2(t+1)) & \text{if } \ \alpha = 1 \quad . \end{cases}$$
for $t \ge 0$.

Remark. Note that if $f \in A_\mu \cap B_\sigma$ with $\mu > (1/2)(\alpha+2)$ and $\sigma > \alpha - n/2 + 2$, then (19) holds (see Lemma 2.4).

Proof. All constants C, C_1, C_2, appearing in this proof depend only on the dimension n, the L^2-norm of the initial datum (u_0, B_0), the $L^1(0,\infty; L^2(\mathbf{R}^n)^n) \cap C_\nu$-norm of f, and on K. We write $\phi(t) = \|u(t)\|_2^2 + \|B(t)\|_2^2$, $\Phi(t) = \int_0^t \phi(s)\,ds$. The energy inequality implies

$$\phi(t) \le \phi(0) + \int_0^\infty \|f(t)\|_2\,dt = C$$

for all $t \ge 0$; hence $\Phi(t) \le Ct$ for all $t \ge 0$. By (18),

$$|\hat{D}(\xi,t)| \le C_n \Phi(t)|\xi|;$$

hence

$$\int_{|\xi| \le g(t)} |\hat{D}(\xi,t)|^2\,d\xi \le C_n \Phi(t)^2 \int_{|\xi|\le g(t)} |\xi|^2\,d\xi \le C_n \Phi(t)^2 g(t)^{n+2}. \tag{20}$$

Using this in (17), we get

$$\frac{d}{dt}\left(G(t)\|D(t)\|_2^2\right) \le C_n \left[g(t)^{n+4}\Phi(t)^2 + 2\phi(t)(\|\nabla v(t)\|_\infty + \|\nabla w(t)\|_\infty)\right] G(t).$$

We estimate the last term on the right by Lemma 2.6 obtaining

$$\frac{d}{dt}\left(G(t)\|D(t)\|_2^2\right) \le C\left[g(t)^{n+4}\Phi(t)^2 + (t+1)^{-\frac{n}{4}-\frac{\alpha}{2}-\frac{1}{2}}\phi(t)\right]G(t). \qquad (21)$$

We now select $g(t) = (\gamma/2(t+1))^{1/2}$,

$$G(t) = e^{2\int_0^t g(s)^2\,ds} = (t+1)^\gamma,$$

where $\gamma > \dfrac{n}{2} + 2$. The last displayed inequality becomes

$$\frac{d}{dt}\left((t+1)^\gamma\|D(t)\|_2^2\right) \le C\Phi(t)^2(t+1)^{-\frac{n}{2}-2+\gamma} + C\phi(t)(t+1)^{-\frac{n}{4}-\frac{\alpha}{2}-\frac{1}{2}+\gamma}. \quad (22)$$

The choice of γ insures that all powers of $t+1$ appearing in the last inequality are positive. Integrating from 1 to t, we get (Φ is increasing and $\Phi(t) \ge \int_1^t \phi(s)\,ds$)

$$\begin{aligned}
(t+1)^\gamma\|D(t)\|_2^2 &\le 2^\gamma\|D(1)\|_2^2 + C\int_1^t (s+1)^{-\frac{n}{2}-2+\gamma}\,ds\,\Phi(t)^2 \\
&\quad + C(1+t)^{-\frac{n}{4}-\frac{\alpha}{2}-\frac{1}{2}+\gamma}\int_1^t \phi(s)\,ds \\
&\le 2^\gamma\|D(1)\|_2^2 + C(t+1)^{-\frac{n}{2}-1+\gamma}\,\Phi(t)^2 + C(1+t)^{-\frac{n}{4}-\frac{\alpha}{2}-\frac{1}{2}+\gamma}\Phi(t).
\end{aligned}$$

By the energy inequality satisfied by solutions of the MHD equations, and by the corresponding one for solutions of the heat equation, we can estimate the L^2-norm of $D(1) = (u(1)-v(1), B(1)-w(1))$ in terms of the L^2-norm of (u_0, B_0) and the $L^1(0,\infty; L^2(\mathbf{R}^n)^n)$ norm of f. We thus get, dividing by $(t+1)^\gamma$,

$$\|D(t)\|_2^2 \le C(t+1)^{-\gamma} + C(t+1)^{-\frac{n}{2}-1}\,\Phi(t)^2 + C(1+t)^{-\frac{n}{4}-\frac{\alpha}{2}-\frac{1}{2}}\Phi(t) \qquad (23)$$

for $t \ge 1$, hence also for $t \ge 0$ since $\|D(t)\|_2^2$ is bounded for all t (by the energy inequality). Using $(u, B) = (v, w) + D$, we get

$$\begin{aligned}
\phi(t) &= \|u(t)\|_2^2 + \|B(t)\|_2^2 \le 2\|v(t)\|_2^2 + 2\|w(t)\|_2^2 + 2\|D(t)\|_2^2 \\
&\le 2K(t+1)^{-\alpha} + 2\|D(t)\|_2^2 \\
&\le 2K(t+1)^{-\alpha} + C(t+1)^{-\gamma} + C(t+1)^{-\frac{n}{2}-1}\,\Phi(t)^2 + C(1+t)^{-\frac{n}{4}-\frac{\alpha}{2}-\frac{1}{2}}\Phi(t).
\end{aligned}$$

Since $\gamma > \frac{n}{2} + 2 > \alpha$ we can replace $(t+1)^{-\gamma}$ by $(t+1)^{-\alpha}$ and get (notice that $\frac{n}{4} + \frac{\alpha}{2} + \frac{1}{2} = \frac{1}{2}(\frac{n}{2} + 1 + \alpha))$

$$\phi(t) \le C_1(t+1)^{-\alpha} + C_2(t+1)^{-\frac{n}{2}-1}\Phi(t)^2. \tag{24}$$

We *claim* that $\lim_{t\to\infty} \phi(t) = 0$. Since ϕ is bounded, hence $\Phi(t)$ is bounded by Ct, this is clear from (24) if $n \ge 3$. For $n = 2$, we proceed as in [19] for the Navier-Stokes equations: We return to (21), taking now

$$g(t)^2 = \frac{3}{2(t+1)\log(t+1)}, \quad G(t) = \exp(2\int_{e-1}^t g(s)^2\,ds) = \log^3(t+1).$$

Instead of (22) we get

$$\begin{aligned} \frac{d}{dt}\left(\log^3(t+1)\|D(t)\|_2^2\right) &\le C(t+1)^{-3}\Phi(t)^2 + C(t+1)^{-1-\frac{\alpha}{2}}\phi(t)\log^3(t+1) \\ &\le \frac{C}{t+1} + \frac{C}{(t+1)^{1+\frac{\alpha}{2}}}\log^3(t+1), \end{aligned}$$

where we have used again that $\Phi(t)$ is bounded by Ct. Integrating from 0 to t, we get

$$\log^3(t+1)\|D(t)\|_2^2 \le C\log(t+1) + C\int_0^t \frac{\log^3(s+1)}{(s+1)^{1+\frac{\alpha}{2}}}\,ds \le C\log(t+1) + C.$$

Dividing by $\log^3(t+1)$, we see that $\lim_{t\to\infty}\|D(t)\|_2 = 0$. The *claim follows* since

$$\phi(t) \le 2K(t+1)^{-\alpha} + 2\|D(t)\|_2^2.$$

Moreover, it is clear from all these bounds that the rate of decay of ϕ at ∞ (as defined in the Appendix, after Lemma 6.2), depends only on K, the L^2-norm of (u_0, B_0) and the $L^1(0,\infty; L^2(\mathbf{R}^n)^n)$-norm of f. This holds for $n = 2$ as well as for $n > 2$. We can now apply Lemma 6.2 of the Appendix (with $\lambda = 1 + n/2$, so $\lambda \ge 2$ and $\min\{\alpha, \lambda\} = \alpha$) to conclude the proof of part 1 of the theorem. To prove part 2, we assume $\alpha \ge 1$. We use (15) instead of (16), which allows us to replace (17) by the improved version

$$\begin{aligned} \frac{d}{dt}\left(G(t)\|D(t)\|_2^2\right) &\le 2g(t)^2G(t)\int_{|\xi|\le g(t)} |\hat{D}(\xi,t)|^2\,d\xi \\ &\quad + 2\phi(t)^{1/2}\|D(t)\|_2(\|\nabla v\|_\infty + \|\nabla w\|_\infty)G(t) \end{aligned} \tag{25}$$

Working as before, using (20) to estimate the first term on the right hand side of (25) and invoking Lemma 2.6), with $g(t) = (\gamma/2(t+1))61/2$, $G(t) = (t+1)^\gamma$, we get instead of (22),

$$\frac{d}{dt}\left((t+1)^\gamma \|D(t)\|_2^2\right) \leq C\Phi(t)^2(t+1)^{-\frac{n}{2}-2+\gamma} + C\phi(t)^{1/2}\|D(t)\|_2(t+1)^{-\frac{n}{4}-\frac{\alpha}{2}-\frac{1}{2}+\gamma}.$$

We integrate from 0 to t, estimating (by part 1), $\phi(t)$ by $C(t+1)^{-\alpha}$ and also (as before)

$$\begin{aligned}\int_0^t \Phi(s)^2(s+1)^{-\frac{n}{2}-2+\gamma}\,ds &\leq \Phi(t)^2\int_0^t (s+1)^{-\frac{n}{2}-2+\gamma}\,ds \\ &= (\gamma-\frac{n}{2}-1)^{-1}\Phi(t)^2\left((t+1)^{-\frac{n}{2}-1+\gamma}-1\right) \leq C\Phi(t)^2(t+1)^{-\frac{n}{2}-1+\gamma}.\end{aligned}$$

We get, after dividing by $(t+1)^\gamma$

$$\|D(t)\|_2^2 \leq C\Phi(t)^2(t+1)^{-\frac{n}{2}-1} + C(t+1)^{-\gamma}\int_0^t (s+1)^{-\frac{n}{4}-\alpha-\frac{1}{2}+\gamma}\|D(s)\|_2\,ds.$$

Setting $Y(t) = \sup_{0\leq s\leq t}(s+1)^{n/4+1/2}$ the last inequality implies

$$Y(t)^2 \leq C\Phi(t)^2 + C(t+1)^{1-\alpha}Y(t),$$

thus

$$Y(t) \leq C\Phi(t) + C(t+1)^{1-\alpha}.$$

Since

$$\Phi(t) = \int_0^t \phi(s)\,ds \leq \begin{cases} C & \text{if } \alpha > 1 \\ C\log(t+1) & \text{if } \alpha = 1\end{cases}$$

for all $t \geq 0$, part 2 follows at once. $\Box$

The following corollary summarises the consequences of Theorem 3.1 which will be needed in the sequel. The hypothesis $f \in C_{(n+3)/2}$ insures $f \in C_\nu$ for $\nu \leq n/4 + \alpha/2 + 1$, $\alpha = n/2$ or $\alpha = n/2+1$.

Corollary 3.2 *Let $(u_0, B_0) \in H \times H$, and let $f \in L^1(0,\infty; L^2(\mathbf{R}^n)^n) \cap C_{(n+3)/2}$. Assume that the solution (v, w) of the (HS,f) system satisfies*

$$\|v(t)\|_2^2 + \|w(t)\|_2^2 \leq K(t+1)^{-\alpha}$$

for all $t \geq 0$, some constant $K \geq 0$, where $\alpha = n/2$ or $\alpha = n/2+1$.

1. *There exists a constant C such that*

$$\|u(t)\|_2^2 + \|B(t)\|_2^2 \leq C(t+1)^{-\alpha}$$

   for $t \geq 0$.

2. *If $n = 2$ and $\alpha = n/2 = 1$, there is a constant C, such that*

$$\|D(t)\|_2^2 \leq C(t+1)^{-2}(1 + \log^2(t+1))C(t+1)^{-3/2}$$

   for $t \geq 0$.

3. *If $n > 2$ or if $n = 2$ but $\alpha = n/2 + 1 = 2$, there is a constant C, such that*

$$\|D(t)\|_2^2 \leq C(t+1)^{-\frac{n}{2}-1}$$

   for $t \geq 0$.

We would like to mention that the result $\|D(t)\|_2^2 \leq C(t+1)^{-n/2-1}$, in the case in which $D(t)$ is the difference between a solution of the Navier-Stokes equation and the solution of the heat equation with the same initial datum was first established by Wiegner [24] for $n \geq 2$, using a different approach. It is also worth mentioning that Zhang Linghai has recently made some improvements to the the Fourier splitting method for $n = 2$ (cf. [25]), which we have used in our proof above.

4 The lower bounds

In this section we derive lower bounds for the decay rate of the L^2-norms of solutions to the MHD equations. Our lower bounds take the form

$$\phi(t) = \|u(t)\|_2^2 + \|B(t)\|_2^2 \geq C(t+1)^{-\alpha}$$

for $t \geq 0$, where $C > 0$ and $\alpha = n/2$ or $\alpha = n/2 + 1$ (see below). We assume from now on that our forcing function is in the set

$$\mathcal{F} = L^1(0, \infty; L^2(\mathbf{R}^n)^n \cap W_1) \cap A_{n/4+4} \cap B_4 \cap C_{(n+3)/2}.$$

The choice of this set is motivated by the immediately verifiable fact that if $f \in \mathcal{F}$, then f satisfies the hypotheses of Lemma 2.5, Lemma 2.6 and

Theorem 3.1 with $\alpha \leq n/2+1$, and Corollary 3.2. We need something like $f \in L^1(0,\infty; L^2 \cap W_1)$ to be able to apply Lemma 6.1 of the Appendix.

We may assume that our solutions are smooth. In fact, a general solution can be approximated by smooth solutions, and it is an easy matter to verify that such an approximation can be done without affecting the constants appearing in our estimates. It follows that general solutions satisfy the lower bounds for a.e. t. However, we notice that by the results of the last section, since $\phi(t)$ decays in time, we can find $t_0 > 0$ such that $\phi(t)$ is arbitrarily small for $t \geq t_0$. By the energy inequality (2), $\psi(t) = \|\nabla u(t)\|_2^2 + \|\nabla B(t)\|_2^2$ is integrable over $0 \leq t < \infty$, hence $\liminf_{t\to\infty} \psi(t) = 0$. It follows we can find $t_0 \geq 0$ for which $\phi(t_0)\psi(t_0)$ is arbitrarily small. It is a well known classical result that, for $2 \leq n \leq 3$, the solution (u, B) is smooth for $t \geq t_0$ if $\phi(t_0)\psi(t_0)$ is sufficiently small (see [12], [6] for the case of the Navier-Stokes equations, the MHD case is similar). For $n = 4$ it is also known that, for t sufficiently large, the solution is strong. The lower bounds are then valid everywhere for $t \geq t_0$. Since we are assuming that our solutions satisfy the strong energy inequality (hence cannot vanish for some t without also vanishing for all $t' \geq t$),the lower bound also has to hold everywhere for $t \leq t_0$, possibly with a smaller positive constant.

The case $\alpha = n/2$ is easily dealt with. Assuming that $\hat{u}_0(0) \neq 0$ or $\hat{B}_0(0) \neq 0$, we proved in the last section that

$$\begin{aligned} \|D(t)\|_2^2 &\leq C(t+1)^{-n/2-1} \qquad \text{if} \quad n \geq 3 \\ \|D(t)\|_2^2 &\leq C(t+1)^{-n/2-1}(1+\log^2(t+1)) \qquad \text{if} \quad n = 2, \end{aligned}$$

while, by Lemma 2.5, $\|v(t)\|_2^2 \geq C(t+1)^{-n/2}$. It follows that

$$\|u(t)\|_2^2 + \|B(t)\|_2^2 \geq C(t+1)^{-n/2},$$

and, in fact,

$$\|u(t)\|_2^2 \geq C(t+1)^{-n/2}$$

if $\hat{u}_0(0) \neq 0$,

$$\|B(t)\|_2^2 \geq C(t+1)^{-n/2}$$

if $\hat{B}_0(0) \neq 0$. Thus we will assume for the rest of the section that $\hat{u}_0(0) = \hat{B}_0(0) = 0$.

The proof of the lower bounds is based on Fourier analysis of the equations satisfied by the difference between the solution of the MHD equations and the solution of the heat system with the same initial data. We begin with two auxiliary Lemmas. The first one calculates the value of the constant σ_1 of Lemma 2.3 for the solution of a heat equation whose initial value is u(T).

Lemma 4.1 *Let $P(\xi) = I - \mu(\xi)$, where I is the $n \times n$ identity matrix and*

$$\mu(\xi) = \frac{1}{|\xi|^2}(\xi_k \xi_j)_{1 \le k,j \le n} \tag{26}$$

for $\xi = (\xi_1, \dots, \xi_n) \in \mathbf{R}^n \setminus \{0\}$ and assume $S = (s_{kj})$ is a symmetric matrix. Then

$$\sigma_1 = \int_{S^{n-1}} P(\omega) S\omega \cdot S\omega \, d\omega = \frac{\pi^{n/2}}{n(n+2)\Gamma(\frac{n}{2})} \left(\sum_{k \neq j} (s_{kk} - s_{jj})^2 + 2n \sum_{k \neq j} s_{kj}^2 \right).$$

Proof. We shall be using the following easily established formulas:

$$\int_{S^{n-1}} \omega_k^2 \, d\omega = \frac{2\pi^{n/2}}{n\Gamma(\frac{n}{2})}$$

for $k = 1, \dots, n$;

$$\int_{S^{n-1}} \omega_k^2 \omega_j^2 \, d\omega = \begin{cases} \frac{2\pi^{n/2}}{n(n+2)\Gamma(\frac{n}{2})} & \text{if } k \neq j \\ \\ \frac{6\pi^{n/2}}{n(n+2)\Gamma(\frac{n}{2})} & \text{if } k = j \end{cases}$$

for $1 \le k, j \le n$. Since S is symmetric, we get for $\omega \in S^{n-1}$,

$$(I - \mu(\omega)) S\omega \cdot S\omega = \sum_{k,j,p} s_{kp} s_{jp} \omega_k \omega_j - \sum_{k,j,p,q} s_{kp} s_{jq} \omega_p \omega_q \omega_k \omega_j. \tag{27}$$

We integrate over S^{n-1} noticing that only terms in which all powers of the ω_k's are even have non-vanishing integrals. In the first sum appearing in (27), these are the terms with $k = j$. In the second sum, we have to consider the terms for which all indices are equal and the terms in which the indices appear in pairs. We get

$$\int_{S^{n-1}} (I - \mu(\omega)) S\omega \cdot S\omega \, d\omega$$

$$= \sum_{k,j} s_{k,j}^2 \int_{S^{n-1}} \omega_k^2 \, d\omega - \sum_k s_{k,k}^2 \int_{S^{n-1}} \omega_k^4 \, d\omega - \sum_{1 \le k < j \le n} (2 s_{kk} s_{jj} + 4 s_{kj}^2) \int_{S^{n-1}} \omega_k^2 \omega_j^2 \, d\omega$$

$$= \sum_{k,j} s_{k,j}^2 \frac{2\pi^{n/2}}{n\Gamma(\frac{n}{2})} - \sum_k s_{k,k}^2 \frac{6\pi^{n/2}}{n(n+2)\Gamma(\frac{n}{2})} - \sum_{k \neq j} s_{kk} s_{jj} \frac{2\pi^{n/2}}{n(n+2)\Gamma(\frac{n}{2})} - 2 \sum_{k \neq j} s_{kj}^2 \frac{2\pi^{n/2}}{n(n+2)\Gamma(\frac{n}{2})}.$$

Decomposing now

$$\sum_{k.j} s_{kj}^2 = \sum_k s_{kk}^2 + \sum_{k \neq j} s_{kj}^2,$$

grouping terms together and using

$$(n-1) \sum_k s_{kk}^2 - \sum_{k \neq j} s_{kk} s_{jj} = \frac{1}{2} \sum_{k \neq j} (s_{kk} - s_{jj})^2,$$

the expression for σ_1 follows. □

The Lemma we have just proved will be used to calculate the constant σ_1 of Lemma 2.3 for solutions v of the heat equation with forcing function f and initial datum $u(T)$, for some $T > 0$. In this case, we will have $S = \mathcal{A}(T)$, where, for $t \geq 0$, we define the $n \times n$ matrix

$$\mathcal{A}(t) = [\mathcal{A}_{kj}(t)] = \int_0^t A(0,s) \, ds.$$

We will also need the $n \times n$ matrix

$$\mathcal{C}(t) = [\mathcal{C}_{kj}(t)] = \int_0^t C(0,s) \, ds.$$

In these formulas, A, C are the matrices defined in Section 3. By (10), (12) and Theorem 3.1, we see that $A(\xi, \cdot), C(\xi, \cdot) \in L^1(0, \infty)$ for every $\xi \in \mathbf{R}^n$ if either $n > 2$ or if $n = 2$ and u_0, B_0 have 0 averages. In estimating the solution of the heat system with initial value $(u(t), B(t))$, for some $t = T > 0$, the following quantities play a role. We define for $t \geq 0$,

$$\kappa_1(t) = \frac{\pi^{n/2}}{n(n+2)\Gamma(\frac{n}{2})} \left(\sum_{k \neq j} (\mathcal{A}_{kk}(t) - \mathcal{A}_{jj}(t))^2 + 2n \sum_{k \neq j} \mathcal{A}_{kj}(t)^2 \right)$$

and

$$\kappa_2(t) = -\frac{2\pi^{n/2}}{n\Gamma(\frac{n}{2})} \sum_{k \neq j} \left(\frac{\partial \hat{B}_{0k}}{\partial \xi_j}(0) - i\mathcal{C}_{kj}(t) \right)^2 = \frac{2\pi^{n/2}}{n\Gamma(\frac{n}{2})} \sum_{k \neq j} \left| \frac{\partial \hat{B}_{0k}}{\partial \xi_j}(0) - i\mathcal{C}_{kj}(t) \right|^2.$$

The first quantity, κ_1, comes directly from applying Lemma 4.1. Both quantities, κ_1 and κ_2, are related to the first order terms of the Taylor expansion of the Fourier transforms of u, B as is made explicit in the proof of Theorem 4.5. Notice that if u, B are real valued (as we assume), then the matrices $\mathcal{A}(t)$, $\mathcal{C}(t)$ are real. In fact, all entries are Fourier transforms evaluated at $\xi = 0$ of real valued functions. Notice moreover that $(\partial/\partial\xi_j)\hat{B}_{0k}(0) - i\mathcal{C}_{kj}(t)$ is purely imaginary. We also define

$$\kappa(t) = \kappa_1(t) + \kappa_2(t)$$

and

$$\tilde{\kappa}_i = \kappa_i(\infty) = \lim_{t\to\infty} \kappa_i(t) \qquad \text{for } i = 1, 2,$$
$$\tilde{\kappa} = \tilde{\kappa}_1 + \tilde{\kappa}_2.$$

By the remarks following the introduction of $\mathcal{A}, \mathcal{C}$, the functions $\mathcal{A}_{kj}(t)$, $\mathcal{C}_{kj}(t)$, are integrable over $[0, \infty)$, hence $\tilde{\kappa}$ is finite (recall that we are assuming that $\hat{u}_0(0) = \hat{B}_0(0) = 0$, which is needed for integrability in case $n = 2$).

Definition. Assume $(u_0, B_0) \in [H \cap W_1(\mathbf{R}^n)^n]^2$, $f \in \mathcal{F}$ and let (u, B) be the corresponding solution of the MHD equations. We say $(u, B) \in \mathcal{M}_1$ iff

1. $D_\xi \hat{u}_0(0) = 0$, and
2. $\tilde{\kappa}_1 = 0$.

We say $(u, B) \in \mathcal{M}_2$ iff

1. $\dfrac{\partial \hat{B}_{0k}}{\partial \xi_k}(0) = 0$ for $k = 1, \ldots, n$, and
2. $\tilde{\kappa}_2 = 0$.

We set $\mathcal{M} = \mathcal{M}_1 \cap \mathcal{M}_2$. Note that by the definition of $\tilde{\kappa}_1$ and $\tilde{\kappa}_2$, a pair (u, B) has $\tilde{\kappa}_1 = \tilde{\kappa}_2 = 0$ if and only if the matrices $\tilde{\mathcal{A}}, \tilde{\mathcal{C}}$ of the Introduction satisfy $\tilde{\mathcal{A}}$ is scalar and $\tilde{\mathcal{C}}$ coincides with $\langle x, B_0\rangle$. It follows that the set $\mathcal{M}$ is a subset of the set $\mathcal{M}_0$ of the Introduction.

We note that $\mathcal{M}_0$ is very small. More precisely, if $(u, B) \in \left[L^1\left(0, \infty; L^2(\mathbf{R}^n)^n\right)\right]^2$, then generically $(u, B) \in \mathcal{M}_0^c$. To see this, we set

$$\Gamma^1_{ij} = \{(u, B) \in \left[L^1\left(0, \infty; L^2(\mathbf{R}^n)^n\right)\right]^2 : \mathcal{A}_{ii} - \mathcal{A}_{jj} = 0\},$$

$$\Gamma^2_{i,j} = \{(u,B) \in \left[L^1\left(0,\infty; L^2(\mathbf{R}^n)^n\right)\right]^2 : \mathcal{A}_{ij} = 0\},$$

for $1 \leq i \neq j \leq n$ and set

$$\Gamma^1 = \cap_{i,j}\Gamma^1_{i,j},\ \Gamma^2 = \cap_{i,j}\Gamma^2_{i,j}.$$

We then have

Lemma 4.2 *The manifolds Γ_1 and Γ_2 are transversal.*

Proof. To show that Γ^1 and Γ are transversal it suffices to show that any two submanifolds $\Gamma^1_{i,j}$, $\Gamma^2_{i,j}$ are transversal. This will be established by showing that $\Gamma^2_{i,j}$ can be obtained from $\Gamma^2_{i,j}$ by a double rotation by an angle of 45^0 (and vice-versa). Let

$$Q_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, Q = \begin{pmatrix} Q_1 & 0 \\ 0 & Q_1 \end{pmatrix}.$$

Set $(v_i, v_j, w_i, w_j)^t = Q(u_i, u_j, B_i, B_j)^t$. A simple computation yields

$$v_i\, v_j - w_i\, w_j = \frac{1}{2}[(u_i^2 - B_i^2) - (u_j^2 - B_j^2)].$$

The pair(u, w) with $v_k = u_k, w_k = B_k$ for $k \neq i, j$, v_i, v_j, w_i, w_j defined as above, is thus in Γ^2_{ij}. Since the eigenvalues of the orthogonal matrix Q_1 are equal to $1/\sqrt{2}(1 \pm i)$, we see that Q_1 is a rotation by an angle of 45^0 and Q is the double rotation we were referring to above. □

Corollary 4.3 *Let $(u,B) \in [L^1(0,\infty; L^2(\mathbf{R}^n)^n)]^2$. Then generically $(u,B) \in \mathcal{M}^c$.*

Proof. Immediate; it is just a restatement of Lemma 4.2 □

We will also need the following auxiliary lemma.

Lemma 4.4 *Let V be an $n \times n$ matrix of complex entries, let B be a symmetric $n \times n$ matrix of real entries. If for every $\omega \in S^{n-1}$ we have*

$$V\omega - i\,(I - \mu(\omega))\,B\omega = 0,$$

then $V = 0$ and B is a scalar matrix.

Proof. Let ω be an eigenvector of B. Noticing that $\mu(\omega)\omega = \omega$, we get $V\omega = 0$. Since B is real symmetric, it has a complete set of eigenvectors and it follows that $V = 0$. It is not too hard to see that for every $\omega \in S^{n-1}$, 1 is an eigenvalue of multiplicity one of $\mu(\omega)$ with eigenvector ω. Since now $B\omega = \mu(\omega)B\omega$ we conclude that $B\omega = \lambda\omega$ for some $\lambda \in \mathbf{R}$. This proves that every vector is an eigenvector of B, which is only possible if B is a scalar matrix. □

We now prove our main theorem.

Theorem 4.5 *Assume $2 \leq n \leq 4$ and let $f \in \mathcal{F}$, $\alpha > n/2+1$. Let $(u_0, B_0) \in [H \cap H^1(\mathbf{R}^n) \cap L^1(\mathbf{R}^n)^n \cap W_1 \cap W_2]^2$ (which, by Borchers' Lemma B implies*

$$\int_{\mathbf{R}^n} u_0(x)\, dx = \int_{\mathbf{R}^n} B_0(x)\, dx = 0).$$

Let (u, B) be the corresponding solution of the MHD equations. Then

1. *If $(u, B) \notin \mathcal{M}$, then there exist positive constants M_0, M_1 such that*

$$M_0(t+1)^{-n/2-1} \leq \|u(t)\|_2^2 + \|B(t)\|_2^2 \leq M_1(t+1)^{-n/2-1}.$$

2. *If $(u, B) \in \mathcal{M}$, then for every $\epsilon > 0$ there exists T_ϵ such that*

$$\|u(t)\|_2^2 + \|B(t)\|_2^2 \leq \epsilon(t+1)^{-n/2-1}$$

for $t \geq T_\epsilon$.

Proof. We will denote by $O_t(\xi)$ a quantity depending on ξ, t, bounded in $|\xi| \leq \delta$ for each $t \geq 0$, where $\delta > 0$ depends only on f. (If $f \equiv 0$, then one can assume $O_t(\xi)$ is bounded for all $\xi \in \mathbf{R}^n$.) Since A, C are continuously differentiable in ξ with bounded partials (cf. Appendix, Lemma 6.1; here is the only place where we need to assume $n \leq 4$), $A(\xi, t) = A(0, t) + O_t(\xi)|\xi|$, $C(\xi, t) = C(0, t) + O_t(\xi)|\xi|$. It follows that

$$\begin{aligned} \hat{H}(\xi, t) &= i\,(I - \mu(\xi))\,A(0, t)\xi + O_t(\xi)|\xi|^2, && (28)\\ \hat{M}(\xi, t) &= iC(0, t)\xi + O_t(\xi)|\xi|^2. && (29) \end{aligned}$$

We use this expansion in (8), as well as the fact that since $f \in \mathcal{F}$ we have $\hat{f}(\xi, t) = O_t(\xi)|\xi|^4$. In addition, we expand $\hat{u}_0$ and $\hat{B}_0$ in Taylor series around the origin up to terms of order 2 (the hypothesis $(u_0, B_0) \in W_1$ implies that $\hat{u}_0, \hat{B}_0$ are twice continuously differentiable with bounded partials) and use that $e^{-t|\xi|^2} = 1 + O_t(\xi)|\xi|^2$. We get

$$\hat{u}(\xi,t) = \hat{u}_0(0) + D_\xi \hat{u}_0(0)\xi - i(I - \mu(\xi))\mathcal{A}(t)\xi + O_t(\xi)|\xi|^2, \tag{30}$$
$$\hat{B}(\xi,t) = \hat{B}_0(0) + D_\xi \hat{B}_0(0)\xi - i\mathcal{C}(t)\xi + O_t(\xi)|\xi|^2, \tag{31}$$

where, for an $\mathbf{R}^n$-valued function g of the variable $\xi \in \mathbf{R}^n$, $D_\xi g$ denotes the Jacobian matrix with (k,j)-th entry $\partial g_k/\partial \xi_j$. For part 1, set

$$\begin{aligned} P_1(\xi,t) &= D_\xi \hat{u}_0(0) - i(I - \mu(\xi))\mathcal{A}(t), \\ P_2(\xi,t) &= P_2(t) = D_\xi \hat{B}_0(0) - i\mathcal{C}(t), \end{aligned}$$

so that (since $\hat{u}_0(0) = \hat{B}_0(0) = 0$)

$$\begin{aligned} \hat{u}(\xi,t) &= P_1(\xi,t)\xi + O_t(\xi)|\xi|^2, \\ \hat{B}(\xi,t) &= P_2(\xi,t)\xi + O_t(\xi)|\xi|^2. \end{aligned} \tag{32}$$

Observe that if $D_\xi \hat{u}_0(0) = 0$, then $P_1(\xi,t) = i(I - \mu(\xi))\mathcal{A}(t)$ so that by Lemma 4.1

$$\int_{S^{n-1}} |P_1(\omega,t)\omega|^2 \, d\omega = \kappa_1(t),$$

while a simple computation shows that

$$\kappa_2(t) = \int_{S^{n-1}} |P_2(t)\omega|^2 \, d\omega$$

for all $t \geq 0$. Assume first $\tilde{\kappa}_1 > 0$ or $D_\xi \hat{u}_0(0) \neq 0$. We *claim* that there exist $T_0 > 0$, $\rho > 0$ such that

$$\int_{S^{n-1}} |P_1(\omega,t)\omega|^2 \, d\omega \geq \rho \tag{33}$$

for all $t \geq T_0$. In fact, otherwise

$$\liminf_{t\to\infty} \int_{S^{n-1}} |P_1(\omega,t)\omega|^2 \, d\omega = 0.$$

Since $A(0,\cdot) \in L^1(0,\infty)$, we can define

$$\tilde{\mathcal{A}} = \int_0^\infty A(0,s)\, ds = \lim_{t\to\infty} \mathcal{A}(t)$$

and get

$$\int_{S^{n-1}} |\tilde{P}_1(\omega)\omega|^2 \, d\omega = 0 \tag{34}$$

where

$$\tilde{P}_1(\xi) = D_\xi \hat{u}_0(0) - i(I - \mu(\xi))\tilde{\mathcal{A}}.$$

Since $\tilde{P}_1$ is homogeneous, (34) is possible only if $\tilde{P}_1(\xi)\xi = 0$ for all $\xi \in \mathbf{R}^n \backslash 0$. By Lemma 4.4 we conclude that $D_\xi \hat{u}_0(0) = 0$ and $\tilde{\mathcal{A}}$ is a scalar matrix (i.e., $\tilde{\kappa}_1 = 0$), contradicting our assumption. *The claim is established.* Assume now $\tilde{\kappa}_2 > 0$; then (since P_2 does not depend on ξ) it is clear that there exist $T_0 \geq 0$, $\rho > 0$ such that

$$\int_{S^{n-1}} |P_2(t)\omega|^2 \, d\omega \geq \rho \tag{35}$$

for $t \geq T_0$. It follows that with the hypothesis of part 1, one of (33), (35) holds. Let $T \geq T_0$ (to be determined later) and let $(v(t), w(t))$ be the solution of the (HS,f) system with f replaced by $f_T = f(\cdot + T)$ and initial datum $(v(0), w(0)) = (u(T), B(T))$. In view of the representation (32) (with $t = T$) of the initial datum of (v, w) we get from Lemma 2.3 that there exists a constant $c_n > 0$, depending only on n, δ, such that

$$\|v(t)\|_2^2 + \|w(t)\|_2^2 \geq c_n \rho t^{-n/2-1} + O(t^{-n/2-3/2}).$$

We set $D(t) = (D_1(t), D_2(t)) = (u(t+T), B(t+T)) - (v(t), w(t))$ so that D satisfies

$$D_{1t}(t) = \Delta D_1(t) - H(t+T), D_{2t}(t) = \Delta D_1(t) - M(t+T),$$

and $D(0) = 0$. We will apply the Fourier splitting method, taking this time $g(t) = (\gamma/(2t))^{1/2}$, $G(t) = t^\gamma$, $t \geq \max\{1, \gamma/(2\delta^2)\}$. Inequality (17) is still valid, with $u(t)$, $B(t)$ replaced by $u(t+T)$, $B(t+T)$. The squares of the L^2-norms of $u(t+T)$, $B(t+T)$, $v(t)$, $w(t)$ decay at the same rate as $t^{-n/2-1}$ for $t \to \infty$;using Lemma 2.6 to estimate the L^∞-norm of ∇v and ∇w, inequality (17) implies

$$\frac{d}{dt}\left(t^\gamma \|D(t)\|_2^2\right) \leq t^{\gamma-1} \int_{2t|\xi|^2 \leq \gamma} |\hat{D}(\xi, t)|^2 \, d\xi + C_T t^{\gamma-n-2} \tag{36}$$

where C_T is a constant depending on T. Inequality (18) also holds in this case, once more with u, B translated by T; i.e.,

$$|\hat{D}(\xi,t)| \leq C|\xi| \int_0^t (\|u(s+T)\|_2^2 + \|B(s+T)\|_2^2)\, ds \leq C|\xi| \int_T^\infty (\|u(s)\|_2^2 + \|B(s)\|_2^2)\, ds.$$

Since $\|u(t)\|_2^2$,$\|B(t)\|_2^2$ behave like $t^{-n/2-1}$ for $t \to \infty$, we get

$$|\hat{D}(\xi,t)| \leq C|\xi| T^{-n/2}.$$

Thus

$$\int_{2t|\xi|^2 \leq \gamma} |\hat{D}(\xi,t)|^2\, d\xi \leq CT^{-n} \int_{2t|\xi|^2 \leq \gamma} |\xi|^2 = CT^{-n} t^{-n/2-1}.$$

Using this in (36) gives

$$\frac{d}{dt}\left(t^\gamma \|D(t)\|_2^2\right) \leq CT^{-n} t^{\gamma-n/2-2} + C_T t^{\gamma-n-2}.$$

Taking $\gamma > n+2$ large enough so that all powers of t in the last inequality are positive, integrating from 1 to t, and dividing by t^γ gives

$$\|D(t)\|_2^2 \leq CT^{-n} t^{-n/2-1} + C_T t^{-n-1}$$

where C, C_T depend now also on γ and we collected all terms with powers of $t^{-\gamma}$ into the last term. Taking now T large enough so that the coefficient CT^{-n} of $t^{-n/2-1}$ in the last inequality is less than $\frac{1}{4} c_n \rho$, we get

$$\begin{aligned} \|u(t+T)\|_2^2 + \|B(t+T)\|_2^2 &\geq \left(\|v(t)\|_2 + \|w(t)\|_2 - \|D(t)\|_2\right)^2 \\ &\geq \frac{1}{2}(\|v(t)\|_2^2 + \|w(t)\|_2^2) - \|D(t)\|_2^2 \geq \frac{1}{4} c_n \rho t^{-n/2-1} + O(t^{-n/2-3/2}) \end{aligned}$$

proving part 1. For the proof of part 2, it suffices to observe that if we set

$$\sigma_1(t) = \int_{S^{n-1}} |P_1(\omega,t)\omega|^2\, d\omega + \int_{S^{n-1}} |P_2(t)\omega|^2\, d\omega,$$

then, under the hypotheses of part 2, $\lim_{t\to\infty} \sigma_1(t) = 0$. In fact, since $P_2(t) = D_\xi \hat{B}_0(0) - i\mathcal{C}(t)$, it is clear that

$$\lim_{t\to\infty} \int_{S^{n-1}} |P_2(t)\omega|^2\, d\omega = 0.$$

Concerning the other surface integral, notice that, since we are now assuming $D_\xi \hat{u}_0(0) = 0$, we have (as remarked in the proof of part 1)

$$\int_{S^{n-1}} |P_1(\omega, t)\omega|^2 \, d\omega = \kappa_1(t).$$

The assertion about σ_1 follows, since we are assuming $\tilde{\kappa}_1 = \lim_{t\to\infty} \kappa_1(t) = 0$. By Lemma 2.3, taking T large enough so

$$d_n \sigma_1(T) \leq \frac{\epsilon}{4},$$

where $d_n = (\frac{1}{2})^{n/2+2}\Gamma(n/2+1)$, so that by Lemma 2.3 (see inequality (6)), letting again $(v(t), w(t))$ be the solution of the (HS,f) system with initial datum $(u(T), B(T))$, we get

$$\|v(t)\|_2^2 + \|w(t)\|_2^2 \leq \frac{\epsilon}{4} t^{-n/2-1} + O(t^{-n/2-3/2})$$

for $t \to \infty$. We define and estimate $D(t)$ as in part 1. Taking T large enough, we get

$$\|D(t)\|_2^2 \leq \frac{\epsilon}{4} t^{-n/2-1} + O(t^{-n-1}).$$

The result follows applying the elementary inequality $(a+b)^2 \leq 2a^2 + 2b^2$. □

Remark. It should be emphasized that while it may be difficult to always decide when a given solution (u, B) is not in $\mathcal{M}$, there is a trivial way of getting solutions not in $\mathcal{M}$ and which, therefore, satisfy the conclusions of part 1 of Theorem 4.5. In fact, it suffices to select the initial datum (u_0, B_0) so that

$$\int_{\mathbf{R}^n} u_0(x) \, dx = \int_{\mathbf{R}^n} B_0(x) \, dx = 0,$$

but there exist j, k such that

$$\int_{\mathbf{R}^n} x_j u_{0k}(x) \, dx \neq 0.$$

In fact, in this case $D_\xi \hat{u}_0(0) \neq 0$.

Noticing that in the proof of Theorem 4.5 the roles of u and B can be kept fairly independent, we actually proved

Corollary 4.6 *Let $f \in \mathcal{F}$. Let $(u_0, B_0) \in [H \cap W_1]^2$ and assume that*

$$\int_{\mathbf{R}^n} u_0(x)\, dx = \int_{\mathbf{R}^n} B_0(x)\, dx = 0.$$

Let (u, B) be the corresponding solution of the MHD equations. Then

1. *If $u \notin \mathcal{M}_1$, then there exist positive constants M_0, M_1 such that*

$$M_0(t+1)^{-n/2-1} \leq \|u(t)\|_2^2 \leq M_1(t+1)^{-n/2-1}.$$

2. *If $B \notin \mathcal{M}_2$, then there exist positive constants M_2, M_3 such that*

$$M_2(t+1)^{-n/2-1} \leq \|B(t)\|_2^2 \leq M_3(t+1)^{-n/2-1}.$$

Notice that if $\Re(\nabla \hat{u}_0(0)) \neq 0$, then $u \in \mathcal{M}_1^c$.

5 The counterexample.

Assuming an even number of space dimensions, we present an example of a solution $(u(t), B(t))$ to the MHD equations, subject to initial data in the class $\mathcal{M}$ of functions with radially equidistributed energy, which is exponentially decaying in both the velocity and the magnetic field component. This is achieved by first constructing functions u which are simultaneously solutions to the MHD equations and the heat system, and are such that $(u \cdot \nabla)u$ is a gradient, i.e.:

$$\begin{aligned} (u \cdot \nabla)u &= -\nabla p, \\ u_t - \Delta u &= -(u \cdot \nabla)u - \nabla p. \end{aligned}$$

By letting $(u(t), B(t)) = (u(t), u(t))$, we obtain a solution to the MHD equations, since then $(u \cdot \nabla)B - (B \cdot \nabla)u = 0$ and $B_t = \Delta B$. The problem is, therefore, reduced to constructing a radial solution $u(t)$ of the heat equation which decays exponentially.

Theorem 5.1 *Suppose that n is even and let*

1. $w : [0, \infty) \times \mathbf{R} \to \mathbf{R}$ *be such that the function* $v(x,t) = w(|x|, t)$ *is a solution of the heat equation* $v_t = \Delta v$*; so*

$$w_t = w_{rr} + \frac{n-1}{r} w_r;$$

2. $g(r,t) = r^{-n} \int_0^r s^{n-1} w(s,t) ds$*;*

3. $A = (a_{ij})$ *is an* $n \times n$ *matrix with real entries such that*

$$A^2 = \lambda I \quad \textit{for some } \lambda \in \mathbf{R},$$

$$x^t A x = 0 \quad \textit{for all } x \in \mathbf{R}^n.$$

Then the function $u(x,t) = g(|x|, t) A x$ *satisfies:*

a) $u_t = \Delta u$*;*

b) *there exists a function* p *such that* (u, p) *is a solution of the Navier-Stokes equations*

$$\begin{aligned} u_t - \Delta u + (u \cdot \nabla) u + \nabla p &= 0, \\ \nabla \cdot u &= 0. \end{aligned}$$

Remark. Let us note that, since we are assuming n is even, such a matrix A can be constructed by choosing, for example,

$$A = \begin{bmatrix} T & O \dots O \\ O & T \dots O \\ \dots & \dots \ \dots \\ O & O \dots T \end{bmatrix},$$

where T is the rotation matrix

$$T = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

For n odd, no such matrix can exist.

Proof. Denoting by A_j the j^{th} column of the matrix A, it is easily seen that

$$\begin{aligned} \frac{\partial u}{\partial x_j} &= (g_r \frac{x_j}{r})(Ax) + gA_j, &(37)\\ \frac{\partial^2 u}{\partial x_j^2} &= (g_{rr} \frac{x_j^2}{r^2})(Ax) + g_r \frac{r^2 - x_j^2}{r^3} Ax + 2g_r \frac{x_j}{r} A_j. &(38)\end{aligned}$$

We note that the i^{th} component of $\sum_{j=1}^n x_j A_j$ is

$$\left(\sum_{j=1}^n x_j A_j \right)_i = \sum_{j=1}^n x_j a_{ij} = (Ax)_i,$$

so that

$$\sum_{j=1}^n x_j A_j = Ax.$$

Using this in (38), we obtain

$$\Delta u = (g_{rr} + \frac{n+1}{r} g_r)(Ax). \tag{39}$$

Since $u_t = g_t(Ax)$, in order to prove **a)** it suffices to see that

$$g_t = g_{rr} + \frac{n+1}{r} g_r. \tag{40}$$

To establish (40), let us first notice that

$$w_t = w_{rr} + \frac{n-1}{r} w_r = r^{1-n}(r^{n-1} w_r)_r,$$

so that

$$g_{rr} + \frac{n+1}{r} g_r = r^{-n-1}(r^{n+1} g_r)_r$$

and

$$g_t = r^{-n} \int_0^r s^{n-1} w_t ds = r^{-n} \int_0^r (s^{n-1} w_s)_s = \frac{1}{r} w_r.$$

Thence

$$\begin{aligned} g_{rr} + \frac{n+1}{r} g_r &= r^{-n-1} \left[r^{n+1} (\frac{1}{r} w - n r^{-n-1} \int_0^r s^{n-1} w ds) \right]_r \\ &= r^{-n-1} \left[r^n w - n \int_0^r s^{n-1} w ds \right]_r \\ &= r^{-n-1} \left[r^n w_r + n r^{n-1} w - n r^{n-1} w \right] = \frac{1}{r} w_r, \end{aligned}$$

and hence (40). From (37) we have that

$$\begin{aligned} (u \cdot \nabla) u_i &= \sum_{j=1}^n u_j \frac{\partial u_i}{\partial x_j} \\ &= \sum_{j=1}^n g(Ax)_j \left[g_r \frac{x_j}{r} (Ax)_i + g a_{ij} \right] \\ &= \frac{1}{r} g g_r (x^t A x)(Ax)_i + g^2 (A^2 x)_i. \end{aligned}$$

Consequently,

$$(u \cdot \nabla) u = \frac{1}{r} g g_r (x^t A x) A x + g^2 (A^2 x).$$

By our hypotheses on the matrix A, we obtain

$$(u \cdot \nabla) u = \lambda g^2 x,$$

hence

$$\frac{\partial}{\partial x_j} ((u \cdot \nabla) u)_i = \lambda \frac{\partial}{\partial x_j} (g^2 x_i) = 2 \lambda g g_r \frac{x_i x_j}{r} + \lambda g^2 \delta_{ij} = \frac{\partial}{\partial x_i} ((u \cdot \nabla) u)_j,$$

so that $(u \cdot \nabla) u$ is curl-free. This implies the existence of a function p such that

$$\nabla p = -(u \cdot \nabla) u.$$

Thence

$$u_t - \Delta u + (u \cdot \nabla) u + \nabla p = 0.$$

Finally,

$$\nabla \cdot u = \nabla g(|x|, t) \cdot A x + g(|x|, t) Tr(A) = 0,$$

since $Tr(A) = 0$ (by $x^t Ax = 0$ for all $x \in \mathbf{R}^n$) and $\nabla g(|x|, t)$ is parallel to x. $\square$

We give two examples, based on this theorem, which illustrate that solutions to the MHD equations in the class $\mathcal{M}$ can decay exponentially.

Example 1. Choose $(u(x,t), B(x,t), p(x,t)) = (u(x,t), 0, p(x,t))$, where u and p are the functions constructed in Theorem 5.1. Then

$$\begin{aligned} u_t - \Delta u + (u \cdot \nabla)u - (B \cdot \nabla)B + \nabla p = u_t - \Delta u &= 0, \\ B_t - \Delta B + (u \cdot \nabla)B - (B \cdot \nabla)u &= 0, \\ \nabla \cdot u &= 0, \\ \nabla \cdot B &= 0. \end{aligned}$$

The essence of the example is that the function u is the solution to the heat system, and so, by choosing appropriate initial data, u can be guaranteed to decay exponentially. Let us suppose, using the notation introduced in the statement of Theorem 5.1, that v is a solution to the heat equation $v_t = \Delta v$ subject to the initial condition $v(x,0) = v_0(x)$, where the function v_0 is such that $\hat{v}_0(\xi) = 0$ for all ξ with $|\xi| \leq \delta$. Then it is easy to show that v decays exponentially, so that, by construction, $u(x,t) = g(|x|,t)Ax$ decays exponentially.

Note that the matrix $[m_{ij}]$ with entries

$$m_{ij}(t) = \int_{\mathbf{R}^n} (u_i u_j - B_i B_j) dx = \int_{\mathbf{R}^n} u_i u_j dx$$

is scalar for all $t \geq 0$, and

$$e_{ij}(t) = \int_{\mathbf{R}^n} (u_i B_j - B_i u_j) dx = 0 \quad t \geq 0,$$

so that not only is the solution in $\mathcal{M}$ but satisfies a much stronger condition. If we define

$$\begin{aligned} \mathcal{N} \;\; = \;\; & \{(u,B) \in [L^2(\mathbf{R}^n)]^2 : \\ & [\int_{\mathbf{R}^n} (u_i u_j - B_i B_j)\, dx] \quad \text{is a scalar matrix} \\ & \text{and} \quad [\int_{\mathbf{R}^n} (u_i B_j - u_j B_i)\, dx] \quad \text{is the 0 matrix}\}, \end{aligned}$$

we see that in this case $(u(t), B(t)) \in \mathcal{N}$ for all $t \geq 0$. We conjecture, in fact, that this example (as well as the next one) is in some way typical : $(u, B) \in \mathcal{M}$ iff $(u(0), B(0)) \in \mathcal{N}$ iff $(u(t), B(t)) \in \mathcal{N}$ for all $t \geq 0$.

Example 2. In this example we choose $(u(x,t), B(x,t), p(x,t)) = (u(x,t), u(x,t), 0)$, where u is a solution of the heat equation. Then

$$\begin{aligned} u_t - \Delta u + (u \cdot \nabla) u - (B \cdot \nabla) B + \nabla p &= u_t - \Delta u = 0, \\ B_t - \Delta B + (u \cdot \nabla) B - (B \cdot \nabla) u &= B_t - \Delta B = 0, \\ \nabla \cdot u &= 0, \\ \nabla \cdot B &= 0. \end{aligned}$$

Let us choose v_0 to be the same as in the first example. Then $u(t)$ and $B(t) = u(t)$ decay exponentially in time. Now

$$m_{ij}(t) = \int_{\mathbf{R}^n} (u_i u_j - B_i B_j) dx = 0, \quad t \geq 0,$$

and

$$e_{ij}(t) = \int_{\mathbf{R}^n} (u_i B_j - B_i u_j) dx = 0, \quad t \geq 0,$$

so that the initial data is again in the class $\mathcal{N}$.

6 Appendix

This section is devoted to the proof of some auxiliary results.

Lemma 6.1 *Let $2 \leq n \leq 4$ and let (u_0, B_0) belong to $[H^1(\mathbf{R}^n) \cap H \cap W_2]^2$, $f \in L^1(0, \infty; \cap W_1) \cap C_\nu$ for some $\nu \geq 0$. Suppose that $(u(t), B(t))$ are regular global solutions of the MHD equations with initial data (u_0, B_0). Then, for all $t \geq 0$,*

$$\begin{aligned} |\nabla_\xi a_{ij}(\xi, t)| &\leq C(t), \\ |\nabla_\xi b_{ij}(\xi, t)| &\leq C(t), \\ |\nabla_\xi c_{ij}(\xi, t)| &\leq C(t), \end{aligned}$$

where $a_{ij} = \widehat{u_i u_j}$, $b_{ij} = \widehat{B_i B_j}$, and $c_{ij} = \widehat{u_i B_j}$. Here $C(t)$ depends only on t, $\|u_0\|_2$, $\|B_0\|_2$ and norms of f.

Proof. Clearly,

$$|\nabla_\xi a_{ij}(\xi,t)| \le C\int_{\mathbf{R}^n} |x||u_i u_j|dx \le C\int_{\mathbf{R}^n} |x||u|^2\,dx.$$

Similarly,

$$\begin{aligned}
|\nabla_\xi b_{ij}(\xi,t)| &\le C\int_{\mathbf{R}^n} |x||B|^2\,dx \\
|\nabla_\xi c_{ij}(\xi,t)| &\le C\int_{\mathbf{R}^n} |x||u||B|\,dx \le C\int_{\mathbf{R}^n} (|x||u|^2\,dx + |x||B|^2)\,dx.
\end{aligned}$$

It thus suffices to prove

$$\int_{\mathbf{R}^n} |x|\left(|u|^2\,dx + |x||B|^2\right)\,dx \le C(t). \tag{41}$$

Dot-multiplying both sides of the first MHD equation with $|x|u$, of the second MHD equation with $|x|B$, adding and integrating over $\mathbf{R}^n$, we get after some integration by parts

$$\begin{aligned}
\frac{d}{dt}\int_{\mathbf{R}^n} |x|(|u|^2+|B|^2)\,dx &= -\int_{\mathbf{R}^n} |x|\left(|\nabla u|^2 + |\nabla B|^2\right)\,dx + \frac{n-1}{2}\int_{\mathbf{R}^n} \frac{|u|^2+|B|^2}{|x|}\,dx \\
&\quad -\frac{1}{2}\int_{\mathbf{R}^n} \frac{(x\cdot u)|u|^2}{|x|}\,dx - \int_{\mathbf{R}^n} \frac{(x\cdot B)(u\cdot B)}{|x|}\,dx - \frac{1}{2}\int_{\mathbf{R}^n} \frac{(x\cdot u)|B|^2}{|x|}\,dx \\
&\quad -\int_{\mathbf{R}^n} \frac{1}{|x|}(x\cdot u)p\,dx + \int_{\mathbf{R}^n} |x|u\cdot f\,dx.
\end{aligned}$$

In deriving this formula we used repeatedly that for divergence free vector fields w_1, w_2, w_3 in $H^1(\mathbf{R}^n)^n$ we have

$$\int_{\mathbf{R}^n} |x|w_1\cdot(w_2\cdot\nabla w_3)\,dx + \int_{\mathbf{R}^n} |x|w_3\cdot(w_2\cdot\nabla w_1)\,dx = -\int_{\mathbf{R}^n} \frac{(w_1\cdot w_3)(x\cdot w_2)}{|x|}\,dx.$$

An obvious estimate now gives

$$\begin{aligned}
\frac{d}{dt}\int_{\mathbf{R}^n} |x|(|u|^2+|B|^2)\,dx &\le \frac{n-1}{2}\int_{\mathbf{R}^n} \frac{1}{|x|}(|u|^2+|B|^2)\,dx + \frac{1}{2}\int_{\mathbf{R}^n} |u|^3\,dx \\
&\quad +2\int_{\mathbf{R}^n} |u||B|^2\,dx + \int_{\mathbf{R}^n} |u|\,|p|\,dx + \int_{\mathbf{R}^n} |x||u||f|\,dx. \\
&= I + II + III + IV + V
\end{aligned} \tag{42}$$

Integrals II, III and IV can be estimated by the L^2 norm of u times the square of the L^4 norm of (u, B); i.e.,

$$II + III + IV \leq C\|u\|_2(\|u\|_4^2 + \|B\|_4^2). \tag{43}$$

This is obvious (using Hölder's inequality) for II and III. For IV recall the expression (9) for p in terms of u, B. ¿From it we get at once

$$\|p\|_2 = \|\hat{p}\|_2 \leq \sum_{k,j} \|u_k u_j - B_k B_j\|_2 \leq C\|u\|_2(\|u\|_4^2 + \|B\|_4^2).$$

If $q > 2n/(n-1)$ we have

$$\begin{aligned} \int_{\mathbf{R}^n} \frac{1}{|x|}(|u|^2 + |B|^2)\,dx &= \int_{|x|\leq 1} \frac{1}{|x|}(|u|^2 + |B|^2)\,dx + \int_{|x|>1} \frac{1}{|x|}(|u|^2 + |B|^2)\,dx \\ &\leq C(\|u\|_q^2 + \|B\|_q^2) + (\|u\|_2^2 + \|B\|_2^2), \end{aligned}$$

where

$$C = \left(\int_{|x|\leq 1} |x|^{-q/(q-2)}\,dx\right)^{(q-2)/q} < \infty$$

by the choice of q. We thus have

$$I \leq C\left(\|u\|_2^2 + \|B\|_2^2 + \|u\|_q^2 + \|B\|_q^2\right). \tag{44}$$

To estimate V we recall that $f \in C_\nu$ allows us to estimate $|f(t)|$ by $C(t+1)^{-\nu} \leq C$ for all $t \geq 0$ so that

$$V = \int_{\mathbf{R}^n} |x||f|^{1/2}|f|^{1/2}|u||f|\,dx \leq C\int_{\mathbf{R}^n} |x||f|^{1/2}|u|\,dx \leq C\left(\|f\|_{W_1} + \|u\|_2^2\right). \tag{45}$$

By the Sobolev embedding theorem, H^1 is continuously embedded in L^4 if $2 \leq n \leq 4$. Selecting $q = 2n/(n-2)$ if $n \geq 3$, $q \in (4, \infty)$ if $n = 2$, we have $q > 2n/(n-1)$ and, by the Sobolev embedding theorem, H^1 is continuously embedded in L^q. We can thus estimate the L^4 norm in (43) and both the L^2 and L^q norms in (44) by the H^1 norm. We also estimate the L^2 norm of u in (45) by the H^1 norm of (u, B). We thus get from (42), (43), (44), (45)

$$\frac{d}{dt}\int_{\mathbf{R}^n} |x|(|u|^2 + |B|^2)\,dx \leq C\left(|u|_{H_1}^2 + |B|_{H^1}^2\right)(1 + \|u\|_2) + C\|f\|_{W_1} \tag{46}$$

We integrate both sides of (46) with respect to t from 0 to t. By the energy inequality

$$\begin{aligned}\int_0^t (\|u(s)\|_{H^1}^2 + \|B(s)\|_{H^1}^2)\,ds &= \int_0^t (\|u(s)\|_2^2 + \|B(s)\|_2^2)\,ds + \int_0^t (\|\nabla u(s)\|_2^2 + \|\nabla B(s)\|_2^2)\,ds \\ &\le t(\|u_0\|_2^2 + \|B_0\|_2^2) + (\|u_0\|_2^2 + \|B_0\|_2^2),\end{aligned}$$

while

$$\int_0^t \|f(s)\|_{W_1}\,ds \le C.$$

We obtained

$$\int_{\mathbf{R}^n} |x|(|u|^2 + |B|^2)\,dx \le C(t+1)$$

for all $t \ge 0$. Since this is an inequality of type (41), we are done. □

The following Gronwall style lemma was used in the proof of Theorem 3.1.

Lemma 6.2 *Let $\phi : \mathbf{R}^+ \to \mathbf{R}^+$ be measurable and bounded. Let $\lambda \ge 2$, $\alpha > 0$, and assume that there exist constants C_1, C_2 such that*

$$\phi(t) \le \frac{C_1}{(t+1)^\alpha} + \frac{C_2}{(t+1)^\lambda}\left(\int_0^t \phi(s)\,ds\right)^2 \tag{47}$$

for all $t \ge 0$. If $\lim_{t\to\infty}\phi(t) = 0$, then there exists a constant C, depending only on α, λ, C_1, C_2, the sup norm of ϕ, and the rate of decay of ϕ at ∞ (as explained in the remarks following the proof) such that

$$\phi(t) \le C(t+1)^{-\min\{\alpha,\lambda\}}$$

for almost all $t \ge 0$.

Proof. Assume first there exists $\delta > 0$, $C_3 \ge 0$, depending only on α, λ and the rate of decay of ϕ at ∞ such that $\phi(t) \le C_3(t+1)^{-\delta}$. If $\delta > 1$ then

$$\int_0^t \phi(s)\,ds \le C_3 \int_0^\infty (s+1)^{-\delta}\,ds = \frac{C_3}{\delta - 1}$$

and the conclusion follows. If $0 < \delta \le 1$ (decreasing δ if necessary) we can assume $0 < \delta < 1$ and bound one factor of $\int_0^t \phi(S)\,ds$ in (47) by $C_3'(t+1)^{1-\delta}$ to get

$$\phi(t) \le \frac{C_1}{(t+1)^\alpha} + \frac{C_2 C_3'}{(t+1)^\lambda - 1 + \delta}\int_0^t \phi(s)\,ds;$$

$C_3' = C_3/(1-\delta)$. With $\psi(t) = (t+1)^{\lambda-1+\delta}\phi(t)$ we get

$$\psi(t) \le C_1(t+1)^{\lambda-1+\delta-\alpha} + C_2C_3'\int_0^t (s+1)^{-\lambda+1-\delta}\psi(s)\,ds.$$

Since

$$\int_0^\infty (s+1)^{-\lambda+1-\delta}\,ds = \frac{1}{\lambda-2+\delta} < \infty$$

because $\lambda \ge 2$, $\delta > 0$, Gronwall's lemma implies

$$\psi(t) \le C_4 \sup_{0\le s\le t}(s+1)^{\lambda-1+\delta-\alpha} \tag{48}$$

with $C_4 = C_1\exp\{C_2C_3'/(\lambda-2+\delta)\}$.If $\lambda-1+\delta-\alpha \ge 0$, then (48) becomes $\psi(t) \le (t+1)^{\lambda-1+\delta-\alpha}$; i.e., $\phi(t) \le (t+1)^{-\alpha}$, proving the lemma. On the other hand, if $\lambda-1+\delta-\alpha < 0$, then (48) becomes $\psi(t) \le C_4$, hence $\phi(t) \le C_4(t+1)^{-(\lambda-1+\delta)}$. Since $\lambda-1+\delta > 1$ we are now in the first case in which $\delta > 1$. The lemma is proved, modulo the existence of δ, C_3. To see they exist, we use an argument suggested by the referee, which shows that this starting δ can be chosen arbitrarily in the interval $(0,1)\cap(0,\alpha]$. Given such a δ, we let ϵ be such that $0 < \epsilon \le (1-\delta)/(4C_2)$. Set $K = \sup_{t\ge 0}\phi(t)$ and let T_1 be such that $\phi(t) \le \epsilon$ for $t \ge T_1$. Then, if $t \ge T_1$,

$$\begin{aligned}
\phi(t) &\le \frac{C_1}{(t+1)^\alpha} + \frac{C_2}{(t+1)^\lambda}\left(\int_0^{T_1}\phi(s)\,ds + \int_{T_1}^t \phi(s)\,ds\right)\int_0^t \phi(s)\,ds \\
&\le \frac{C_1}{(t+1)^\alpha} + \frac{2C_2}{(t+1)^\lambda}\left(KT_1 + \epsilon(t-T_1)\right)\int_0^t \phi(s)\,ds \\
&\le \frac{C_1}{(t+1)^\alpha} + \frac{2C_2}{(t+1)}\left(\frac{KT_1}{(t+1)^{\lambda-1}} + \epsilon\right)\int_0^t \phi(s)\,ds
\end{aligned}$$

(where we bounded $(t-T_1)/(t+1)^{\lambda-1} \le 1$). Selecting now T_0 such that $t \ge T_0$ implies $C_2KT_1(t+1)^{1-\lambda} < (1-\delta)/4$, recalling $\delta \le \alpha$ and $C_2\epsilon \le (1-\delta)/4$, we get

$$\phi(t) \le \ C_1(t+1)^{-\delta} + \frac{1-\delta}{2(t+1)}\int_0^t \phi(s)\,ds.$$

Setting now $\psi(t) = \sup_{0\le s\le t}(s+1)^\delta\phi(s)$, we get

$$\psi(t) \le C_1 + \frac{1-\delta}{2(1+t)^{1-\delta}}\int_0^t (s+1)^{-\delta}\,ds\psi(t) \le C_1 + \frac{1}{2}\psi(t),$$

thus $\psi(t) \leq 2C_1$; i.e., $\phi(t) \leq 2C_1(1+t)^{-\delta}$ for $t \geq T_0$. This completes the proof of the lemma $\Box$

Remark: If we define for $\epsilon > 0$

$$T_\phi(\epsilon) = \sup\{t \,:\, \phi(t) \geq \epsilon\},$$

then we can call T_ϕ the "rate of decay of ϕ at ∞." Notice that in the proof $T_1 = T_\phi(\epsilon)$ with ϵ depending only on C_2, δ hence (since δ depends only on α) only on C_2, α). Since T_0 depends only on λ, T_1 and K (the sup of ϕ), we see that the constant C_3 depends only on α, λ, C_1, C_2, the sup norm of ϕ, and the rate of decay of ϕ at ∞, and so do all the other constants appearing in the proof of the lemma.

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