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On the decay of higher-order norms of the solutions of Navier–Stokes equations

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We show that an energy decay $\|u(t)\|_2 = O(t^{-\mu})$ for solutions of the Navier–Stokes equations on $\mathbb{R}^n$, $n \leqq 5$, implies a decay of the higher order norms, e.g. $\|D^\alpha u(t)\|_2 = O(t^{-\mu-|\alpha|/2})$ and $\|u(t)\|_\infty = O(t^{-\mu-n/4})$.

1. Introduction

During the last decade there has been some interest and significant progress in the study of the behaviour of solutions of the Navier–Stokes equations in unbounded domains. In this paper, we shall devote our attention to the decay for large times in higher-order norms in dimensions $n \leqq 5$. Let us mention right at the beginning, that our paper contributes only to the Cauchy problem on the whole space, but it attempts to lead the direction of research also for the case of exterior domains.

So suppose that $\Omega \subset \mathbb{R}^n$ is either an exterior domain (with sufficiently smooth boundary), meaning that $K = \mathbb{R}^n \backslash \Omega$ is the closure of a bounded domain, or Ω is the whole space $\mathbb{R}^n$. There are several known ways **[3, 4, 12, 17]** to construct a weak solution

$$u \in L_\infty([0, \infty), L_{2,\sigma}(\Omega)^n) \cap L_2([0, \infty), \mathring{H}_2^1(\Omega)^n)$$

of the Navier–Stokes equation

$$\begin{aligned} u_t - \Delta u + (u \cdot \nabla)u + \nabla p &= 0, \\ \operatorname{div} u &= 0, \\ u(0) &= a \in L_{2,\sigma}(\Omega)^n. \end{aligned} \tag{1.1}$$

For some constructions and $n \leqq 4$, they also fulfil the so-called *generalised energy inequality*

$$\|u(t)\|_2^2 + 2\int_s^t \|\nabla u(r)\|_2^2\, dr \leqq \|u(s)\|_2^2 \tag{1.2}$$

for $t \geqq 0$, $s = 0$ and almost all $s \leqq t$ (see also the remarks in the appendix of **[18]**).

Here $L_{2,\sigma}(\Omega)^n$ denotes the L_2-closure of the smooth divergence-free vector fields; and if P is the corresponding projection, then $A := -P\Delta$ is called the Stokes operator with domain $D(A) = H_2^2(\Omega)^n \cap \mathring{H}_2^1(\Omega)^n \cap L_{2,\sigma}(\Omega)^n$. Then the results on the asymptotic behaviour of the L_2-norm of u, starting with Schonbek [**13, 14**] and with contributions by Kajikiya and Miyakawa [**7**], Galdi and Maremonti [**3**], Borchers and Miyakawa [**1**] and Wiegner [**18**], may be summarised as follows:

Proposition 1.1. *If u is a suitably constructed weak solution and $\|u_0(t)\|_2 = O(t^{-\alpha})$, with $\alpha > \frac{1}{2}$ for simplicity, where $u_0(t) := e^{-tA}a$, then:*
$\|u(t) - u_0(t)\|_2 = O(t^{-(n+2)/4})$, if $\Omega = \mathbb{R}^n$, $n \geqq 2$ [**18**];
$\|u(t) - u_0(t)\|_2 = O(t^{-n/4})$, if Ω is an exterior domain, $n \geqq 3$ [**1**].

Note the difference between these cases. The case of a two-dimensional exterior domain was settled in 1993 by Kozono and Ogawa [**10**] (under some additional assumption on the initial value); also the boundedness of e^{-tA} was proved recently [**2**].

In particular, we infer from [**18**]:

$$\text{if } \|u_0(t)\|_2 \leqq C_0(1+t)^{-\alpha}, \quad \alpha \geqq 0, \quad \text{then } \|u(t)\|_2 \leqq c(C_0)(1+t)^{-\min\{\alpha,(n+2)/4\}}. \tag{1.3}$$

Furthermore this exponent is generically optimal (see [**15**]).

2. Strong solutions

Let us now consider strong solutions. First we remember two facts. On the one hand, global (unique) strong solutions are known to exist either in two dimensions or for small L_n-data. On the other hand, we also have some information in three and four dimensions. The above-mentioned weak solutions with generalised energy inequality [**12, 17**] become *strong after a finite time*:

$$\text{there is some } T_0(\|a\|_2) > 0, \quad \text{such that } u \in C([T_0,\infty), L_p)$$
$$\text{for} \quad 2 \leqq p \leqq \infty, \quad \nabla u \in C([T_0,\infty), L_p) \quad \text{for}$$
$$2 \leqq p \leqq n \quad \text{and} \quad D^2 u \in C([T_0,\infty), L_2) \quad (\text{see [9]}).$$

This follows from the embedding $\|u\|_n \leqq c\|u\|_2^{\frac{1}{2}}\|\nabla u\|_2^{\frac{1}{2}}$ for $n = 3$ (respectively $\leqq c\|\nabla u\|_2$ for $n = 4$), and the estimate $\int_0^\infty \|\nabla u\|_2^2\,ds \leqq \|a\|_2^2$, which implies that $\|u(T^*)\|_n$ is small for some T^*; then the generalised energy inequality allows the reconstruction of u as a strong solution.

So in what follows we assume either:

(a) u is a weak solution with generalised energy inequality and $n \leqq 4$; or

(b) $a \in L_n(\Omega) \cap L_2(\Omega)$, $\|a\|_n$ small enough, so that u is a global strong solution, where we pose the additional assumption that $n \leqq 5$. The reason for this restriction can be found in Lemma 3.1.

We know from Kato [**8**, Theorem 2], that

$$\|u(t)\|_\infty \leqq C_0(t - T_0)^{-\frac{1}{4}} \quad \text{for some } C_0 > 0. \tag{2.1}$$

In fact, there are sharper results known under suitable conditions on the initial value (see [**8, 16, 19, 20**]) but these will follow as a corollary from our results below.

Let us remark that Iwashita's estimates for the Stokes semigroup [**5**] allow us also to prove estimates for $\|u(t)\|_\infty$ and $\|D^2u\|_2$ in the exterior domain case, see [**9**] for $n=3,4$ and [**10**] for $n=2$.

In this situation, the main result is then without any further assumption on the initial value:

$$\text{if } \|u(t)\|_2^2 = O(t^{-2\mu}), \quad \text{then } \|D^m u(t)\|_2^2 = O(t^{-m-2\mu}) \quad \text{for all } m \in \mathbb{N}.$$

REMARK 2.1. (1) The case $n=2$ was settled under more stringent assumptions on the initial data in [**16**]. For the exterior domain case with $n=3$, this estimate, but only for $m=1$ and 2, can be found in [**11**].

(2) The effect that stability in higher norms is implied by energy stability, is also observed in other situations—for the Boussinesq equation, see [**6**].

The proof will follow from a sequence of lemmata.

3. Proofs

As we are considering solutions which are smooth for $t > T_0$, we know that

$$D_2 u \in C_0\left(\left[T_0 + \frac{3\varepsilon}{4}, \infty\right), L_2(\mathbb{R}^n)^n\right) \quad \text{and} \quad \|u(t)\|_\infty \leqq C_\varepsilon(1+t)^{-\frac{1}{4}}, \quad t \geqq T_0 + \frac{3\varepsilon}{4}$$

(see (2.1)). As a first step, we claim:

LEMMA 3.1. *If $n \leqq 5$, $D^\alpha u$ and $D^\alpha u_t \in L_2(\mathbb{R}^n)^n$ for all multi-indices α, provided $t > T_0$.*

Proof. Let $t_m = T_0 + \varepsilon(1 - 2^{-m})$. We may represent the solution with the help of the Fourier transformation by

$$\widehat{u_i}(t+t_m) = (\delta_{ij} - \xi_i \xi_j |\xi|^{-2})\left(e^{-t|\xi|^2}\widehat{u_j}(t_m) - \int_0^t e^{-(t-s)|\xi|^2} i\xi_k(\widehat{u_j u_k})(s+t_m)\,ds\right). \tag{3.1}$$

As

$$\begin{aligned}\|(1+|\xi|^2)(\widehat{u_j u_k})\|_2 &= \|(I+\Delta)u_j u_k\|_2 \leqq C(\|u\|_2 + \|\nabla^2 u\|_2)\|u\|_\infty, \\ &\leqq C_\varepsilon(t) \quad \text{for } t \geqq t_2,\end{aligned}$$

we get

$$\begin{aligned}\|\hat{u}(t+t_2)\|_1 &\leqq Ct^{-n/2} + c\int_0^t \left(\int_0^\infty \frac{e^{-2(t-s)r^2} r^{n+1}}{(1+r^2)^2}\,dr\right)^{\frac{1}{2}} ds \\ &\leqq Ct^{-n/2} + c\int_0^t (t-s)^{-(n-2)/4}\,ds < \infty \quad \text{for } t > 0 \text{ due to } n \leqq 5.\end{aligned}$$

Now $|\xi|^q|(\widehat{u_j u_k})(\xi)| \leqq 2c_q(|\xi|^q|\hat{u}(\xi)| * |\hat{u}(\xi)|)$ due to $|\xi|^q \leqq c_q(|\xi-\mu|^q + |\mu|^q)$. Hence the estimate for convolutions implies

$$\|\,|\xi|^q|\widehat{u_j u_k}|\,\|_2 \leqq 2c_q\|\hat{u}\|_1\|\,|\xi|^q\hat{u}\|_2.$$

Assuming by induction that $\|\,|\xi|^{m-\frac{1}{2}}\hat{u}(s)\|_2 \leqq C(s, m-\frac{1}{2}) < \infty$ for $s \geqq t_m$, we may multiply (3.1) by $|\xi|^m$ to get the bound

$$\| |\xi|^m \hat{u}(t+t_m) \|_2 \leqq c_m t^{-\frac{1}{2}} + c_m \int_0^t (t-s)^{-\frac{3}{4}}\, ds \leqq C(t,m) < \infty$$

for $t > 0$.

Thus $\| |\xi|^m \hat{u}(t+t_m) \|_2$ is finite for all m and $t > 0$. The same reasoning applies after differentiating (3.1) with respect to t, which proves the claim. $\square$

LEMMA 3.2. *For $m \in \mathbb{N}$, we have the inequality*

$$\frac{d}{dt} \| D^m u \|_2^2 + \tfrac{3}{2} \| D^{m+1} u \|_2^2 \leqq c_m (\| u \|_\infty^2 \| D^m u \|_2^2 + R_m),$$

with

$$R_m = \begin{cases} 0 & \text{for } m = 1, 2, \\ \sum\limits_{1 \leq j \leq m/2} \| D^j u \|_\infty^2 \| D^{m-j} u \|_2^2 & \text{for } m \geqq 3. \end{cases}$$

Proof. Applying D^α to the equation, multiplying by $2D^\alpha u_i$, integrating by parts and summing over all α, $|\alpha| = m$, we get

$$\begin{aligned} \frac{d}{dt} \| D^m u \|_2^2 + 2 \| D^{m+1} u \|_2^2 &\leqq 2 \sum_{|\alpha| = m} \int D^\alpha \left(\frac{\partial u_i}{\partial x_k} u_k \right) D^\alpha u_i \, dx \\ &= -2 \sum_{|\alpha| = m} \int D^\alpha (u_i u_k) D^{\alpha + e_k} u_i \, dx \\ &\leqq \frac{1}{2} \| D^{m+1} u \|_2^2 + c \sum_{\substack{|\beta| \leqq |\alpha|/2 \\ |\alpha| = m}} \int |D^\beta u|^2 |D^{\alpha - \beta} u|^2 \, dx, \end{aligned}$$

hence the claim for $m \geqq 3$.

If $m = 1$, then

$$\begin{aligned} -\int \frac{\partial}{\partial x_l}(u_i u_k) \frac{\partial^2}{\partial x_l \, \partial x_k} u_i \, dx &= -\frac{1}{2} \int \frac{\partial}{\partial x_k} \left(\left(\frac{\partial u_i}{\partial x_l} \right)^2 \right) u_k \, dx - \int u_i \frac{\partial u_k}{\partial x_l} \frac{\partial^2 u_i}{\partial x_l \, \partial x_k} \\ &\leqq c \int |u| \, |Du| \, |D^2 u| \, dx, \end{aligned}$$

as the first term vanishes due to $\operatorname{div} u = 0$.

The same is used for $m = 2$, where we get

$$\begin{aligned} &\int \frac{\partial}{\partial x_k} \left(\frac{\partial^2}{\partial x_l \, \partial x_s} (u_i u_k) \right) \frac{\partial^2 u_i}{\partial x_l \, \partial x_s} \, dx \\ &\quad = \int \frac{\partial}{\partial x_k} \left(\frac{\partial u_i}{\partial x_l} \frac{\partial u_k}{\partial x_s} + \frac{\partial u_i}{\partial x_s} \frac{\partial u_k}{\partial x_l} + u_i \frac{\partial^2 u_k}{\partial x_l \, \partial x_s} \right) \frac{\partial^2 u_i}{\partial x_l \, \partial x_s} \, dx \\ &\quad = \int \left(\frac{\partial^2 u_i}{\partial x_l \, \partial x_k} \frac{\partial u_k}{\partial x_s} + \frac{\partial^2 u_i}{\partial x_k \, \partial x_s} \frac{\partial u_k}{\partial x_l} + \frac{\partial u_i}{\partial x_k} \frac{\partial^2 u_k}{\partial x_l \, \partial x_s} \right) \frac{\partial^2 u_i}{\partial x_l \, \partial x_s} \, dx \end{aligned}$$

and an additional integration by parts gives a bound against

$$c\int |u|\,|D^2u|\,|D^3u|\,dx \leqq \frac{1}{2}\|D^3u\|_2^2 + c\|u\|_\infty^2\|D^2u\|_2^2.$$

This proves the claim. $\square$

The following lemma is based on the Fourier splitting method of M. E. Schonbek.

Lemma 3.3. *Let $m \in \mathbb{N}$, $T_m = T_0 + 1 - 2^{-m}$ and assume*

$$\|D^{m-1}u\|_2^2 \leqq C_{m-1}(t - T_{m-1})^{-\rho_{m-1}} \quad \textit{for } t > T_{m-1}.$$

Suppose

$$\frac{d}{dt}\|D^m u\|_2^2 + \|D^{m+1}u\|_2^2 \leqq c_0(t - T_{m-1})^{-1}\|D^m u\|_2^2 + \sum_{i=1}^{m} c_i(t - T_{m-1})^{-s_i},$$

with $s_i \geqq \rho_{m-1} + 2$. Then

$$\|D^m u\|_2^2 \leqq C_m(t - T_m)^{-\rho_m} \quad \textit{for } t > T_m,$$

with $\rho_m = 1 + \rho_{m-1}$ and some C_m, depending on $C_{m-1}, c_i, s_i, \rho_{m-1}, m$, but not on T_0.

Proof. Let

$$S = \left\{\xi \;\middle|\; |\xi| \leqq \left(\frac{c_0 + a}{t - T_{m-1}}\right)^{\frac{1}{2}}\right\}$$

with $a = \max\{s_i\} + 1$. Then

$$\begin{aligned}
\|D^{m+1}u\|_2^2 &\geqq \int_{\mathbf{R}^n\setminus S} |\xi|^2|\widehat{D^m u}|^2\,d\xi \\
&\geqq \frac{c_0 + a}{t - T_{m-1}}\|D^m u\|_2^2 - \frac{c_0 + a}{t - T_{m-1}}\int_S |\widehat{D^m u}|^2\,d\xi \\
&\geqq \frac{c_0 + a}{t - T_{m-1}}\|D^m u\|_2^2 - \left(\frac{c_0 + a}{t - T_{m-1}}\right)^2\int_S |\widehat{D^{m-1}u}|^2\,d\xi
\end{aligned}$$

and therefore

$$\begin{aligned}
&\frac{d}{dt}\|D^m u\|_2^2 + \frac{a}{(t - T_{m-1})}\|D^m u\|_2^2 \\
&\qquad \leqq C_{m-1}(c_0 + a)^2(t - T_{m-1})^{-(2+\rho_{m-1})} + \sum_{i=1}^{m} c_i\,(t - T_{m-1})^{-s_i}.
\end{aligned}$$

Multiply by $(t - T_{m-1})^a$, integrate and divide again by $(t - T_{m-1})^a$ to get

$$\|D^m u\|_2^2 \leqq \tilde{c}_0(t - T_{m-1})^{-\rho_{m-1}-1} + \sum \tilde{c}_i(t - T_{m-1})^{1-s_i}.$$

Hence on $t \geqq T_m$, this is the claimed estimate, since for $q \geqq q_0$, $t \geqq T_m = T_{m-1} + 2^{-m}$ we have

$$(t - T_{m-1})^{-q} \leqq 2^{m(q-q_0)}(t - T_m)^{-q_0}. \quad \square$$

We are now in a position to prove our main theorem.

THEOREM 3.4. *Suppose* $\|u\|_2^2 \leqq C_0(t+1)^{-2\mu}$, *for* $t \geqq 0$, *with some* $\mu \geqq 0$. *Then for* $m \in \mathbb{N}$, *there is some* $C_m = C_m(\mu, C_0)$, *independent of* T_0, *with* T_0 *given by* (2.1), *such that*

$$\|D^m u\|_2^2 \leqq C_m(t-T_0-1)^{-m-2\mu} \quad \textit{for } t \geqq T_0+1.$$

REMARK 3.5. If $n=2$ or if $\|a\|_n$ is small enough, we have $T_0=0$, while $T_0 = c(n)\|a\|_2^{4/(n-2)}$ is admissible for weak solutions, $n=3$ or 4.

Proof of Theorem 3.4. We want to show by induction the estimate

$$\|D^m u\|_2^2 \leqq C_m(t-T_m)^{-m-2\mu} \quad \text{for } t \geqq T_m = T_0+1-2^{-m}.$$

If $m=1$ or 2, we know by Lemma 3.2 and (2.1), that

$$\frac{d}{dt}\|D^m u\|_2^2 + \|D^{m+1}u\|_2^2 \leqq cK(t-T_0)^{-1}\|D^m u\|_2^2.$$

We may apply Lemma 3.3 to get the claim for $m=1$ or 2.

For $m \geqq 3$, we have the additional term R_m to estimate. We use the interpolation inequality

$$\|D^j u\|_\infty \leqq c\|D^{m+1}u\|_2^{a_j}\|u\|_2^{1-a_j} \quad \text{with } a_j = \frac{j+\frac{n}{2}}{m+1}.$$

Note that

$$\frac{j}{m+1} < a_j < 1 \quad \text{for } j \leqq \frac{m}{2} \quad \text{and} \quad m \geqq n-1$$

and in the case $n=5$, $m=3$, $j=1$. Then

$$\begin{aligned} R_m &\leqq \frac{1}{2}\|D^{m+1}u\|_2^2 + c\|u\|_2^2 \sum_{1\leqq j\leqq m/2}\|D^{m-j}u\|_2^{2/(1-a_j)} \\ &\leqq \frac{1}{2}\|D^{m+1}u\|_2^2 + c\sum_j (t-T_{m-1})^{-s_j}, \end{aligned}$$

with

$$s_j = 2\mu + (m+1)\frac{m-j}{m-j+1-\frac{n}{2}} \geqq 2\mu+m+1,$$

where we used the induction hypothesis (weakened to) $\|D^k u\|_2^2 \leqq C_k(t-T_k)^{-k}$ for $k \leqq m-1$.

Hence Lemma 3.3 may be applied again and proves the claim. □

THEOREM 3.6. *Under the same assumptions, there holds, for* $2 \leqq p \leqq \infty$ *and* $j \in \mathbb{N}_0$,

$$\|D^j u\|_p \leqq c(t-T_0-1)^{-(j/2+n/2(1/2-1/p)+\mu)}$$

(*especially* $\|u\|_\infty \leqq c(t-T_0-1)^{-(\mu+n/4)}$).

Proof. By interpolation,

$$\| D^j u \|_p \leqq c \| D^m u \|_2^a \| u \|_2^{1-a}, \quad \frac{1}{p} = \frac{j}{n} + a\left(\frac{1}{2} - \frac{m}{n}\right) + (1-a)\frac{1}{2}$$

for suitable m. By Theorem 3.4 the claim follows. □

In order to derive estimates also for time derivatives of u, we have to show first a generalisation of Lemma 3.1.

LEMMA 3.7. *For $t > T_0$, there holds*

$$D^\alpha \frac{d^k}{dt^k} u \in L_2(\mathbb{R}^n)^n$$

for all multi-indices α and all $k \in \mathbb{N}$.

Proof. The claim for $k = 0$ and 1 is contained in Lemma 3.1. From the representation (3.1), we get the estimate (with any $T_1 > T_0$)

$$\begin{aligned} |\xi|^m \left| \frac{d^k}{dt^k} \hat{u}(t + T_1) \right| \leqq{} & c|\xi|^{2k+m} e^{-t|\xi|^2} |\hat{u}(T_1)| \\ & + c \sum_{j=0}^{k-1} |\xi|^{2(k-j)-1+m} \left| \frac{d^j}{dt_j} (\widehat{u \cdot u})(t + T_1) \right| \\ & + c \int_0^t e^{-(t-s)|\xi|^2} |\xi|^{2k+1+m} |\widehat{u \cdot u}(s + T_1)| \, ds. \end{aligned}$$

Since we already know from Lemma 3.1 that

$$\|(1 + |\xi|^q)\widehat{u \cdot u}\|_2 < \infty \quad \text{for all } q,$$

we get

$$\| \, |\xi|^m \hat{u}(s) \|_1 \leqq C(s) \quad \text{for } s \geqq T_1 \text{ and all } m.$$

Suppose now that

$$\left\| \, |\xi|^m \frac{d^j}{dt^j} \hat{u}(s) \right\|_q \leqq C_{k-1}(s) \quad \text{for } s \geqq T_1, m \in \mathbb{N}_0, \quad q = 1 \text{ and } 2, \quad \text{and} \quad j \geqq k-1.$$

By induction, with the help of $| \, |\xi|^q \widehat{u \cdot u}| \leqq c(|\xi^q| \hat{u}| * |\hat{v}|) + c(|\xi|^q |\hat{v}| * |\hat{u}|)$ and $\|f * g\|_q \leqq \|f\|_q \cdot \|g\|_1$, it follows also that

$$\left\| \, |\xi|^m \frac{d^k}{dt^k} \hat{u}(s) \right\|_q$$

is finite for $s \geqq T_1$, $q = 1$ and 2. This proves the lemma. □

THEOREM 3.8. *Under the same assumptions as in Theorem 3.4, we get, for $t > T_0 + 2$,*

$$\left\| D^m \frac{d^k}{dt^k} u \right\|_2 \leqq c(t - T_0 - 2)^{-(m/2)-k-\mu}.$$

Proof. By Lemma 3.7, we know that we may apply

$$D^\alpha \frac{d^{k-1}}{dt^{k-1}}$$

to the equation. After scalar multiplication by

$$D^\alpha \frac{d^k}{dt^k} u$$

and an integration by parts, the pressure drops out and we get with Hölder's inequality

$$\left\| D^\alpha \frac{d^k}{dt^k} u \right\|_2 \leqq c \left\| D^\alpha \frac{d^{k-1}}{dt^{k-1}} (\Delta u - u \cdot \nabla u) \right\|.$$

Thus, letting $k = 1$, we get, for $t > T_1 + 1$,

$$\begin{aligned}\left\| D^\alpha \frac{d}{dt} u \right\|_2 &\leqq c \| D^{|\alpha|+2} u \|_2 + c \sum_{j \leqq |\alpha|/2} \| D^j u \|_\infty \| D^{|\alpha|-j+1} u \|_2 \\ &\leqq c(t - T_0 - 1)^{-|\alpha|/2 - 1 - \mu} + c \sum_j (t - T_0 - 1)^{-(j/2 + n/4 + \mu) - ((|\alpha| - j + 1)/2 + \mu)} \\ &\leqq c(t - T_1 - 1)^{-(|\alpha|/2) - 1 - \mu},\end{aligned}$$

where we used Theorem 3.6.

Estimates for

$$\left\| D^\alpha \frac{d}{dt} u \right\|_p, \quad 2 \leqq p \leqq \infty,$$

now follow by interpolation. The case of general k is then a consequence of straightforward induction. □

References

1 W. Borchers and T. Miyakawa. L^2 decay for the Navier–Stokes flows in unbounded domains with application to exterior stationary flows. *Arch. Rational Mech. Anal.* **118** (1992), 273–95.

2 W. Borchers and W. Varnhorn. On the boundedness of the Stokes semigroup in two-dimensional exterior domains. *Math. Z.* **213** (1993), 275–99.

3 G. P. Galdi and P. Maremonti. Monotonic decreasing and asymptotic behaviour of the kinetic energy for weak solutions of the Navier–Stokes equations in exterior domains. *Arch. Rational Mech. Anal.* **94** (1986), 253–66.

4 J. G. Heywood. The Navier–Stokes equations: on the existence, regularity and decay of solutions. *Indiana Univ. Math. J.* **29** (1980), 639–81.

5 H. Iwashita. $L_q - L_r$ estimates for solutions of nonstationary Stokes equations in an exterior domain and the Navier–Stokes initial value problem in L_q spaces. *Math. Ann.* **285** (1989), 265–88.

6 Y. Kagei and W. von Wahl. Stability of higher norms in terms of energy-stability for the Boussinesq-equations. Remarks on the asymptotic behaviour of convection-roll-type solutions. *Differential and Integral Equations* 7 (1994), 921–48.

7 R. Kajikiya and T. Miyakawa. On L^2 decay of weak solutions of the Navier–Stokes equations in $\mathbf{R}^n$. *Math. Z.* **192** (1986), 135–48.

8 T. Kato. Strong L^p-solutions of the Navier–Stokes equation in $\mathbf{R}^m$, with applications to weak solutions. *Math. Z.* **187** (1984), 471–80.

9 H. Kozono, T. Ogawa and H. Sohr. Asymptotic behaviour in L^r for turbulent solutions of the Navier–Stokes equations in exterior domains. *Manusripta Math.* **74** (1992), 253–75.

10 H. Kozono and T. Ogawa. Decay properties of strong solutions for the Navier–Stokes equations in two-dimensional unbounded domains. *Arch. Rational Mech. Anal.* **122** (1993), 1–17.

11 P. Maremonti. On the asymptotic behaviour of the L_2-norm of suitable weak solutions to the Navier–Stokes equations in three-dimensional exterior domains. *Comm. Math. Phys.* **118** (1988), 385–400.

12 T. Miyakawa and H. Sohr. On energy inequality, smoothness and large time behaviour in L^2 for weak solutions of the Navier–Stokes equations in exterior domains. *Math. Z.* **199** (1988), 455–78.

13 M. E. Schonbek. L^2-decay for weak solutions of the Navier–Stokes equations. *Arch. Rational Mech. Anal.* **88** (1985), 209–22.

14 M. E. Schonbek. Large-time behaviour of solutions of the Navier–Stokes equations. *Comm. Partial Differential Equations* **11** (1986), 753–63.

15 M. E. Schonbek. Lower bounds of rates of decay for solutions to the Navier–Stokes equations. *J. Amer. Math. Soc.* **4** (1991), 423–49.

16 M. E. Schonbek. Large time behaviour of solutions to the Navier–Stokes equations in H^m spaces (Preprint, 1993).

17 H. Sohr, W. von Wahl and M. Wiegner. Zur Asymptotik der Gleichungen von Navier–Stokes. *Nachr. Akad. Wiss. Göttingen Math.-Phys. Kl. II* **3** (1986), 49–59.

18 M. Wiegner. Decay results for weak solutions of the Navier–Stokes equations in $\mathbb{R}^n$. *J. London Math. Soc.* **35** (1987), 303–13.

19 M. Wiegner. Decay and stability in L_p for strong solutions of the Cauchy problem for the Navier–Stokes equations. In *The Navier–Stokes Equations, Theory and Numerical Methods*, eds J. G. Heywood *et al.*, Proceedings Oberwolfach, Lecture Notes in Mathematics 1431, 95–9 (Berlin: Springer, 1990).

20 M. Wiegner. Decay of the L_∞-norm of solutions of the Navier–Stokes equations in unbounded domains. In *Proceedings of 'Mathematical problems for Navier–Stokes equations, Cento 1993'*, *Acta Appl. Math.* **37** (1994), 215–19.

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