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Energy Decay for a Weak Solution of the Navier–Stokes Equation with Slowly Varying External Forces

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We consider the Navier–Stokes system with slowly decaying external forces

$$\begin{cases} \dfrac{\partial u}{\partial t} + u \cdot \nabla u = -\nabla p + \Delta u + f & \text{in} \quad t > 0, \quad x \in \mathbf{R}^n \quad (n = 2, 3, 4), \\ \operatorname{div} u = 0 & \\ u(0) = a & \text{in} \quad t > 0, \quad x \in \mathbf{R}^n, \end{cases} \tag{NS}$$

We show that the energy norm of a weak solution has non-uniform decay,

$$\|u(t)\|_2 \to 0 \qquad (t \to \infty),$$

under suitable conditions on the data f and a which make the energy of solution bounded in time. Also, we show the exact rate of the decay (uniform decay) of the energy,

$$\|u(t)\|_2 \leqslant C(1+t)^{-\varepsilon},$$

for external forces with a given explicit decay rate.

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INTRODUCTION

We are concerned with the asymptotic behavior of weak solutions to the Navier–Stokes equation,

$$\begin{cases} \dfrac{\partial u}{\partial t} + u \cdot \nabla u = -\nabla p + \Delta u + f & \text{in} \quad t > 0, \quad x \in \mathbf{R}^n \quad (n = 2, 3, 4), \\ \operatorname{div} u = 0 & \\ u(0) = a, & \text{in} \quad t > 0, \quad x \in \mathbf{R}^n, \end{cases} \tag{NS}$$

where $u = u(x, t) = (u_1(x, t), u_2(x, t), ..., u_n(x, t))$ and $p = p(x, t)$ denote unknown velocity vector and pressure at point $(x, t) \in \mathbf{R}^n \times (0, \infty)$, while $a = a(x) = (a_1(x)\, a_2(x), ..., a_n(x))$ and $f = f(x) = (f_1(x), f_2(x), ..., f_n(x))$ are given initial velocity and external force.

We shall discuss the energy decay problem of weak solutions to (NS):

$$\|u(t)\|_2^2 \to 0 \qquad \text{as} \quad t \to \infty. \tag{D}$$

This problem was originally suggested by Leray in his pioneering papers [13, 14]. For the case $f = 0$, Kato [10] gave the first affirmative answer for strong solutions with small data. For weak solutions, Schonbek [19] obtained algebraic decay for the Cauchy problem in $\mathbf{R}^3$ for large data in $L^1 \cap L^2$. At the same time, Masuda [16] proved the non-uniform decay for weak solutions satisfying the strong energy inequality (see (E) below) in the general domain when $n = 2$ and $f \in L^1(0, \infty; L^2(\mathbf{R}^2))$ or when $n \geqslant 3$ and $f = 0$. When the external force $f = 0$, the method in [19] was extended by Kajikiya and Miyakawa [9] and Wiegner [26] for the case $\mathbf{R}^n$ $(n = 2, 3, 4)$ (see also Schonbek [20]). Other unbounded domain cases were treated by Heywood [6], Borchers and Miyakawa [1], Ukai [25], Maremonti [15], and Kozono and Ogawa [12] (for strong solutions see [3, 8, 11]; see also a recent work by Carpio [29]). All these results are valid even if we consider an external force with sufficiently fast decay. For example, if $\|f(t)\|_2 \leqslant C(1+t)^{-n/2-1}$ and $f \in L^\infty(0, \infty;\ W^{-1,1}(\mathbf{R}^n))$, then the energy decays at the same rate as for $f = 0$ ([19]). On the other hand, Miyakawa and Sohr [17] showed the existence of a weak solution to (NS) in an exterior domain in $\mathbf{R}^n$ $(n = 2, 3, 4)$ which satisfies the strong energy inequality

$$\|u(t)\|_2^2 + 2\int_s^t \|\nabla u(\tau)\|_2^2\, d\tau \leqslant \|u(s)\|_2^2 + \int_s^t |\langle f(\tau), u(\tau)\rangle|\, d\tau \tag{E}$$

for almost all $s > 0$ with $s = 0$ and all $t \geqslant s$. Using this inequality, they extended Masuda's non-uniform decay to the case where the external force $f \in L^1(0, \infty;\ L^2(\mathbf{R}^n))$.

In this paper, we first extend their results to more general external forces and establish non-uniform decay. For that purpose, we construct a weak solution which satisfies a generalized energy inequality (see Proposition 2.3) by a method analogous to that due to Leray [14] and Caffarelli *et al.* [2]. The Fourier splitting method (see [19, 20]) combined with an argument due to Masuda [16] on the generalized energy inequality yields non-uniform decay (D).

Next we show uniform decay of the energy of the solutions,

$$\|u(t)\|_2 \leqslant C(1+t)^{-\varepsilon}, \tag{UD}$$

for slowly decaying external forces. More precisely, we obtain uniform decay (UD) for a class of functions which contain functions f satisfying

$$\|f(t)\|_2 \leqslant C(1+t)^{-1-\varepsilon} \tag{0.1}$$

or

$$\|rf(t)\|_2 \leqslant C(1+t)^{-1/2-2\varepsilon}, \tag{0.2}$$

where $\varepsilon > 0$ is a small constant and $r = |x|$. We also need a condition on the behavior of the external force:

$$f \in L^\infty(0, \infty; W^{-1,1}). \tag{0.3}$$

Interpolating the two conditions (0.1) and (0.2) for the decay of the external force, we obtain a more general assumption on the external force which will also yield a uniform rate of decay (UD) of the solution when $n = 3, 4$.

We also discuss an alternative condition,

$$f \in L^2(0, \infty; L^1),$$

instead of (0.3), which gives us an another aspect for the uniform decay (see Assumptions A.2 and A.3 in Section 1 for the details). We consider the two-dimensional case at the end of Section 4, in which we need a slight modification of the argument to obtain (UD).

Throughout this paper, the following notation will be used.

Let $L^2_\sigma(\mathbf{R}^n)$ and $\dot{H}^1_{0,\sigma}(\mathbf{R}^n)$ denote the completions of $C^\infty_{0,\sigma}$ (C^∞_0 functions with divergence free) in the $L^2(\mathbf{R}^n)$-norm $\|\cdot\|_2$ and the Dirichlet (homogeneous H^1) norm $\|\nabla\cdot\|_2$. We abbreviate them as L^2_σ and $\dot{H}^1_\sigma$. $\mathscr{F}\phi = \hat{\phi}$ and

$\mathscr{F}^{-1}\phi = \check{\phi}$ denote the Fourier and the inverse Fourier transform of ϕ, respectively. We denote

$$L^p(a, b; L^q) = \left\{ f\colon (a, b) \times \mathbf{R}^n \to \mathbf{R}^n;\ \|f\|_{L^p(a, b;\, L^q)} = \left(\int_a^b \|f(\tau)\|_p^q \, d\tau \right)^{1/q} < \infty \right\}.$$

The notation $\|\cdot\|_{L^{p,q}}$ will be used for the norm of $L^p(0, \infty; L^q)$. $H^1_\sigma = \dot{H}^1_\sigma \cap L^2_\sigma$, $H^{-1} = (H^1_\sigma)^*$. The symbol $\langle \cdot, \cdot \rangle$ denotes the inner product in L^2. Various constants are simply denoted by C.

1. RESULTS

We first recall the definition of a weak solution according to Leray and Hopf.

DEFINITION. For $a \in L^2$ and $T > 0$, we call u a weak solution of Leray–Hopf type (Leray–Hopf solution) if and only if

(1) $u \in L^\infty(0, T; L^2_\sigma) \cap L^2(0, T; \dot{H}^1_\sigma)$.

(2) For any $0 \leqslant s < t \leqslant T$, u satisfies

$$\int_s^t \{ -\langle u(\tau), \Phi_t(\tau) \rangle + \langle u(\tau) \cdot \nabla u(\tau), \Phi(\tau) \rangle + \langle \nabla u(\tau), \nabla \Phi(\tau) \rangle \} \, d\tau$$

$$= -\langle u(t), \Phi(t) \rangle + \langle u(s), \Phi(s) \rangle + \int_s^t \langle f(\tau), \Phi(\tau) \rangle \, d\tau \tag{1.1}$$

for all test functions $\Phi \in C^1([0, T]; C^\infty_{0,\sigma})$.

The external force f is assumed to satisfy the following conditions:

A.1. For $x_0 \in \mathbf{R}^n$, let

$$\rho = \rho_{x_0}(x) = \begin{cases} |x - x_0|, & \text{if} \quad n = 3, 4, \\ |x - x_0| \, (1 + \left|\log |x - x_0|\right|), & \text{if} \quad n = 2. \end{cases} \tag{1.2}$$

Suppose that for $0 \leqslant \gamma \leqslant 1$, $2 \leqslant p \leqslant 2n/(n - 2 + 2\gamma)$ ($< \infty$ if $(n = 2)$) and $\theta = 4p/(2p\gamma + np - 2n)$,

$$\rho^\gamma f \in L^{\theta'}(0, \infty; L^{p'}),$$

where p' and θ' are the conjugates of p and θ.

Then the decay result reads:

THEOREM A (Non-uniform Decay). *Let $n = 2, 3, 4$ and $a \in L^2$. Assume that f satisfies* A.1. *Then there exists a weak solution u of (NS) satisfying the strong energy inequality*

$$\|u(t)\|_2^2 + 2\int_s^t \|\nabla u(\tau)\|_2^2 \, d\tau \leqslant \|u(s)\|_2^2 + \int_s^t |\langle f(\tau), u(\tau)\rangle| \, d\tau \tag{1.3}$$

for almost all $s > 0$ with $s = 0$ and all $t \geqslant s$. Also the energy of the solution satisfies

$$\|u(t)\|_2 \to 0 \qquad as \quad t \to \infty. \tag{1.4}$$

Remarks. (1) The restriction on the dimensions is needed to obtain integrability for the nonlinear term in the definition of the weak solutions.

(2) Assumption A.1 includes the following typical cases: $f \in L^1(0, \infty; L^2)$ and $f \in L^2(0, \infty; L^{2n/(n+2)})$ $(n \geqslant 3)$. These conditions ensure that f is in the dual of the space to which the weak solution belongs. On the other hand, if we assume that f belongs to some weighted space, then a weaker condition on the time decay suffices to yield the decay of the solution. For example, one can choose

$$rf \in L^2(0, \infty; L^2).$$

Assumption A.1 is obtained by interpolating the decay conditions on f mentioned above.

(3) The condition on f can be relaxed as follows: For a set of exponents $(\gamma_i, p_i, \theta_i)$ $(i = 1, 2, ..., m)$ satisfying the same conditions as in A.1, f is written as

$$f = \sum_{i=1}^{m} f_i,$$

where each f_i satisfies

$$|x - x_{0,i}|^{\gamma_i} f_i \in L^{\theta_i'}(0, \infty; L^{p_i'}), \qquad i = 1, 2, ..., m,$$

for some $x_{0,i} \in \mathbf{R}^n$. However for simplicity, we only consider the case A.1.

If $n = 2$, $f \in L^1(0, \infty; L^2)$ or if $n \geqslant 3$ and $f = 0$, Masuda [16] showed that any weak solution in a general domain has a non-uniform rate of decay in the energy norm. If $n = 3, 4$ and $f \in L^1(0, \infty; L^2)$, Miyakawa and Sohr [17] constructed weak solutions satisfying the strong energy inequality in exterior domains. Using Masuda's estimate,

$$\int_{t-1}^{t} \|u(\tau)\|_2 \, d\tau \to 0 \qquad \text{as} \quad t \to \infty,$$

they showed that these solutions have a non-uniform rate of decay. Under more general conditions on f, Borchers and Miyakawa [1] considered the energy decay in unbounded domain. If f satisfies

$$\int_0^\infty \tau \, \|f(\tau)\|_2 \, d\tau < \infty,$$

they showed that there is a weak solution for which the energy decays as in Theorem A. Our theorem extends this result to the case of the Cauchy problem (NS) for more general external forces.

For the uniform decay result, a slightly stronger assumption than A.1 is needed. We assume:

A.2. Let γ, p, θ, and ρ be the same as in assumption A.1. For small $\varepsilon > 0$, we suppose that f satisfies

$$(1+t)^{\beta} \rho^{\gamma} f \in L^\infty(0, \infty; L^{p'}),$$

where $\beta = (1/\theta') + \varepsilon(\theta + 2)/\theta$.

By choosing $\gamma = 0$, $p = p' = 2$, and $\beta = 1 + \varepsilon$, we have a typical condition of the external force:

$$\|f(t)\|_2 \leqslant C(1+t)^{-1-\varepsilon}.$$

Also, if we choose $\gamma = 1$, $p = p' = 2$, then $\theta = 2$ and $\beta = 1/2 + 2\varepsilon$, which yields (0.2)

$$\|\rho f(t)\|_2 \leqslant C(1+t)^{-(1/2)-2\varepsilon}.$$

It should also be remarked that the condition A.1 implies A.2.

In order to show uniform decay for the weak solution, we also need to assume proper behavior of the Fourier transform of f near $|\xi| \simeq 0$.

A.3. Let f satisfy either:

(1) f can be written as $f = Dg$, where D is any first order derivative and $g \in L^\infty(0, \infty; L^1)$, or

(2) $f \in L^2(0, \infty; L^1)$.

THEOREM B (Uniform Decay). *Let f satisfy* A.2 *and* A.3 *and the initial data $a \in L^q \cap L^2$, where $q \leqslant 2n/(4\varepsilon + n)$. Then the solution constructed in Theorem A has the following uniform decay*:

$$\|u(t)\|_2 \leqslant \begin{cases} C(1+t)^{-\varepsilon} & \text{if} \quad n = 3, 4, \\ C(\log(e+t))^{-1/2} & \text{if} \quad n = 2. \end{cases}$$

Furthermore, if we assume that $f \in L^1(0, \infty; L^1(\mathbf{R}^2))$ in the 2-dimensional case, then

$$\|u(t)\|_2 \leqslant C(1+t)^{-\varepsilon}.$$

Here the constant C only depends on norms of a and f.

Remark. If there is no restriction on the initial data other than $a \in L^2$, it is known that there is no explicit uniform decay as in Theorem B. (c.f. [20]. In the case of an exterior domain, see Hishida [7].) If $n = 3$ or 4, $q > 2n/(4\varepsilon + n)$, and $a \in L^2 \cap L^q$, then the uniform decay rate in Theorem B is determined by the term involving the initial data and hence becomes slower (c.f., Lemma 2.7 below). On the other hand, if we choose a more rapid decay for f the decay for the solution is determined by the nonlinear term (cf. [19, 20, 27]).

2. EXISTENCE AND GENERALIZED ENERGY INEQUALITY

In this section, we establish the existence of a weak solution which satisfies a certain energy inequality. To this end, we need some apriori estimates for the solution. The following auxiliary lemma is first established.

LEMMA 2.1. *Let $0 \leqslant \gamma \leqslant 1$ and $2 \leqslant p \leqslant 2n/(n-2+2\gamma)$. If $n = 2$ and $\gamma = 0$ then $p < \infty$. Let f satisfy A.1. Then for $u \in L^\infty(0, T; L^2_\sigma) \cap L^2(0, T; \dot{H}^1_\sigma)$ and $0 \leqslant s < t < \infty$, we have*

$$\int_s^t |\langle f, u\rangle| \, d\tau \leqslant C\mathscr{C}^{(1-\lambda)(1-\gamma)}(t) \left(\int_s^t \|\nabla u(\tau)\|_2^2 \, d\tau\right)^{1/\theta} \|\rho(|x - x_0|)^\gamma f\|_{L^{\theta', p'}}, \tag{2.1}$$

where $x_0 \in \mathbf{R}^n$, $\mathscr{E}(t) = \sup_{\tau < t} \|u(\tau)\|_2^2$ and $\lambda = [n/(1-\gamma)](1/2 - 1/p)$. The weight functions ρ and θ are the same as those described in Assumption A.1.

Proof. We first consider the case when $\lambda < 1$. Put $r = |x - x_0|$. Recall that

$$\rho = \rho_{x_0}(r) = \begin{cases} r(1 + |\log r|) & (n=2), \\ r & (n=3,4). \end{cases}$$

By using the Hölder, Gagliardo–Nirenberg, and Hardy inequalities, we have

$$\begin{aligned}
\int_s^t |\langle f, u \rangle| \, d\tau &\leqslant \int_s^t \|\rho^\gamma f\|_{p'} \left\| \frac{u}{\rho^\gamma} \right\|_p d\tau \\
&\leqslant \int_s^t \|\rho^\gamma f\|_{p'} \|u\|_\delta^{1-\gamma} \left\| \frac{u}{\rho} \right\|_2^\gamma d\tau \qquad \left(\text{where} \quad \delta = \frac{2p(1-\gamma)}{2-\gamma p} > 0 \right) \\
&\leqslant C \int_s^t \|\rho^\gamma f\|_{p'} \|u\|_2^{(1-\lambda)(1-\gamma)} \|\nabla u\|_2^{\lambda(1-\gamma)} \left\| \frac{u}{\rho} \right\|_2^\gamma d\tau \\
&\leqslant C \int_s^t \|\rho^\gamma f\|_{p'} \|u\|_2^{(1-\lambda)(1-\gamma)} \|\nabla u\|_2^{\gamma + \lambda(1-\gamma)} d\tau \\
&\leqslant C \mathscr{E}(t)^{(1-\lambda)(1-\gamma)} \|\rho^\gamma f\|_{L^{\theta', p'}} \left(\int_s^t \|\nabla u(\tau)\|_2^2 \, d\tau \right)^{1/\theta}. \qquad (2.2)
\end{aligned}$$

If γ and p satisfy the extremal case $\lambda = 1$, i.e., $n(1/2 - 1/n) = 1 - \gamma$, then $\theta = 2$ and $\delta = 2n/(n-2)$ and it follows that

$$\begin{aligned}
\int_s^t |\langle f, u \rangle| \, d\tau &\leqslant \int_s^t \|\rho^\gamma f\|_{p'} \left\| \frac{u}{\rho} \right\|_p d\tau \\
&\leqslant \int_s^t \|\rho^\gamma f\|_{p'} \|u\|_{2n/(n-2)}^{1-\gamma} \left\| \frac{u}{\rho} \right\|_2^\gamma d\tau \\
&\leqslant C \int_s^t \|\rho^\gamma f\|_{p'} \|\nabla u\|_2 \, d\tau \\
&\leqslant C \|\rho^\gamma f\|_{L^{2,p'}} \left(\int_s^t \|\nabla u(\tau)\|_2^2 \, d\tau \right)^{1/2}. \qquad (2.2')
\end{aligned}$$

Remark. If $n = 2$ and $\gamma = 0$ in Lemma 2.1, the case $p = \infty$ is excluded. In this case, the weak solution becomes bounded and regular. However, a bound like (2.1) seems too difficult to obtain. But if we assume that $f \in L \log L$, a bound similar to the one in Lemma 2.1 can be obtained.

A straightforward consequence of Lemma 2.1 is the following apriori estimate:

PROPOSITION 2.2. *Let $0 \leqslant s \leqslant t < \infty$ and suppose f satisfies Assumption* A.1. *Then for all $u \in L^\infty(0, T; L^2_\sigma) \cap L^2(0, T; \dot{H}^1_\sigma)$ satisfying the strong energy inequality*

$$\|u(t)\|_2^2 + 2\int_s^t \|\nabla u(\tau)\|_2^2 \, d\tau \leqslant \|u(s)\|_2^2 + \int_s^t |\langle f(\tau), u(\tau)\rangle| \, d\tau \tag{2.3}$$

for almost all $s > 0$ with $s = 0$ and all $t \geqslant s$, we have the following apriori estimate on u:

$$\mathscr{E}(t) = \sup_{0 \leqslant \tau \leqslant t} \|u(\tau)\|_2 \leqslant C_1, \tag{2.4}$$

$$\int_s^t \|\nabla u(\tau)\|_2^2 \, d\tau \leqslant C_1. \tag{2.5}$$

Interpolating (2.4) *and* (2.5), *for $2 \leqslant q \leqslant 2n/(n-2)$ and $n/q + 2/\sigma = n/2$,*

$$\|u\|_{L^{\sigma, q}} \leqslant C_1, \tag{2.6}$$

where C_1 is a constant that only depends on $\|a\|_2$ and $\|\rho^\gamma f\|_{L^{\theta', p'}}$.

Proof of Proposition 2.2. Note that $2 \leqslant \theta$ and $(1-\lambda)(1-\gamma) \leqslant 1$ in Lemma 2.1. Hence the apriori estimate (2.4) and (2.5) can be obtained by applying Lemma 2.1 to (2.3).

By Proposition 2.2, we note that the solution is bounded in $L^{2+4/n}(0, \infty : L^{2+4/n})$. It is well known that the exponent of (2.6) in $n \geqslant 3$ has a gap to the condition of the regularity criteria $n/q + 2/\sigma \leqslant 1$ for an L^p weak solution. For the regularity criteria, see Serrin [21] and Giga [4].

We now state the existence result:

PROPOSITION 2.3 (Existence and Generalized Energy Inequality). *Let $a \in L^2$ and f satisfy Assumption* A.1. *Then there exists a weak solution u to* (NS) *in $L^\infty(0, \infty; L^2_\sigma) \cap L^2(0, \infty; \dot{H}^1_\sigma)$ satisfying the strong energy inequality* (E). *For $E(t) \in C^1(\mathbf{R}; \mathbf{R}_+)$ with $E(t) \geqslant 0$ and $\psi \in C^1(\mathbf{R}; C^1 \cap L^2)$, the weak solution satisfies*

$$
\begin{aligned}
E(t)\,\|\psi(t)*u(t)\|_2^2 \leqslant{} & E(s)\,\|\psi(s)*u(s)\|_2^2+\int_s^t E'(\tau)\,\|\psi(\tau)*u(\tau)\|_2^2\,d\tau \\
& +2\int_s^t E(\tau)\,|\langle \psi'(\tau)*u(\tau),\psi(\tau)*u(\tau)\rangle \\
& \qquad -\|\nabla\psi(\tau)*u(\tau)\|_2^2|\,d\tau \\
& +2\int_s^t E(\tau)(|\langle u\cdot\nabla u(\tau),\psi(\tau)*\psi(\tau)*u(\tau)\rangle| \\
& \qquad +|\langle f(\tau),\psi(\tau)*\psi(\tau)*u(\tau)\rangle|)\,d\tau \qquad (2.7)
\end{aligned}
$$

for $0\leqslant s<t<\infty$.

Proof of Proposition 2.3. Following the argument of Leray [14] and Cafarelli *et al.* [2], we consider the approximated equation to (NS). For some fixed $T>0$, let u_k be a solution of

$$
\begin{cases}
\dfrac{\partial u_k}{\partial t}-\Delta u_k+v_k\cdot\nabla u_k+\nabla p_k=f & \text{in } x\in\mathbf{R}^n,\quad 0<t<T,\\
\operatorname{div} u_k=0 & \text{in } x\in\mathbf{R}^n,\quad 0<t<T,\\
u_k(0)=a, &
\end{cases}
\qquad (2.8)
$$

where $v_k=\Psi_k(u_k)$ is the approximation of u_k obtained through a retarded mollifier:

$$
\Psi_k(h)=k^{-1-n}\int_0^t\int_{\mathbf{R}^n}\rho\left(\frac{t-s}{k},\frac{x-y}{k}\right)h(s,y)\,dy\,ds,\qquad k=1,2,....
$$

The construction of a smooth solution to (2.8) is well known (see [17, 22, 24]). Moreover, it is shown in [2] that for small $\delta>0$,

$$
\begin{cases}
u_k\to u & L^2(0,T;L^2)\cap L^{2+4/n-\delta}(0,T;L^{2+4/n-\delta})\\
 & \qquad\text{and weak*-}L^\infty(0,T;L^2),\\
\nabla u_k\to\nabla u & \text{weak-}L^2(0,T;L^2),
\end{cases}
\qquad (2.9)
$$

and

$$
v_k\to u \qquad L^2(0,T;L^2)\cap L^{2+4/n-\delta}(0,T;L^{2+4/n-\delta}).
$$

The limit function u satisfies (NS) in the sense of distributions. To show the energy estimate, we multiply (2.8) by $E(t)\psi * \psi * u_k$ and integrate by parts to get

$$\begin{aligned}\frac{d}{dt} E(t)\,\|\psi * u_k(t)\|_2^2 = E'(t)\,\|\psi * u_k(t)\|_2^2 + 2E(t)\{\langle \psi'(\tau) * u_k(\tau), \psi(\tau) * u_k\rangle \\ - \|\nabla\psi(\tau) * u_k\|_2^2\} - 2E(t)\langle v_k\cdot\nabla u_k, \psi * \psi * u_k\rangle \\ + 2E(t)\langle f, \psi * \psi * u_k\rangle. \end{aligned} \tag{2.10}$$

Integrating (2.10) over (s, t), we have

$$\begin{aligned} & E(t)\,\|\psi * u_k(t)\|_2^2 + 2\int_s^t E(\tau)\,\|\nabla\psi * u_k(\tau)\|_2^2\,d\tau \\ & \leqslant E(s)\,\|\psi * u_k(\tau)\|_2^2 + \int_s^t E'(\tau)\,\|\psi * u_k(\tau)\|_2^2\,d\tau \\ & \quad + 2\int_s^t E(\tau)\{|\langle \psi'(\tau) * u_k(\tau), \psi(\tau) * u_k(\tau)\rangle| \\ & \quad + |\langle v_k\cdot\nabla u_k, \psi * \psi * u_k\rangle| + |\langle f, \psi * \psi * u_k\rangle|\}\,d\tau. \end{aligned} \tag{2.11}$$

We then take the limit as $k \to \infty$. The nonlinear term in (2.11) converges to the one involving the limit function provided $\nabla(\psi * \psi) \in L^\infty(0, \infty; L^2)$. In fact, since $\operatorname{div} v_k = 0$, we have

$$\begin{aligned} & \left| \int_s^t \langle v_k\cdot\nabla u_k, \psi * \psi * u_k\rangle\,ds - \int_s^t \langle u\cdot\nabla u, \psi * \psi * u\rangle\,ds \right| \\ & \leqslant \int_s^t \{|\langle (v_k - u)\cdot\nabla u_k, \psi * \psi * u_k\rangle| + |\langle u\cdot\nabla(u_k - u), \psi * \psi * u_k\rangle| \\ & \quad + |\langle u\cdot\nabla u, \psi * \psi * (u_k - u)\rangle|\}\,d\tau \\ & = \int_s^t |\langle (v_k - u)\cdot\nabla(\psi * \psi) * u_k, u_k\rangle|\,d\tau \\ & \quad + \int_s^t |\langle u\cdot\nabla(\psi * \psi) * u_k, u_k - u\rangle|\,d\tau \\ & \quad + \int_s^t |\langle u\cdot\nabla(\psi * \psi) * (u_k - u), u\rangle|\,d\tau \\ & \equiv I_1 + I_2 + I_3. \end{aligned} \tag{2.12}$$

Then (2.9) and the Hausdorff–Young inequality give us

$$
\begin{aligned}
I_1 &\leqslant \int_s^t \|(v_k - u)\cdot\nabla(\psi * \psi) * u_k\|_2 \, \|u_k\|_2 \, d\tau \\
&\leqslant \int_s^t \|v_k - u\|_2 \, \|\nabla(\psi * \psi) * u_k\|_\infty \, \|u_k\|_2 \, d\tau \\
&\leqslant \int_s^t \|\nabla(\psi * \psi)\|_2 \, \|u_k\|_2^2 \, \|v_k - u\|_2 \, d\tau \\
&\leqslant C \int_s^t \|v_k - u\|_2 \, d\tau \leqslant C(t-s)^{1/2} \left(\int_s^t \|v_k - u\|_2^2 \, d\tau \right)^{1/2} \to 0 \\
&\qquad \text{as} \quad k \to \infty. \qquad (2.13)
\end{aligned}
$$

Similarly

$$
\begin{aligned}
I_2 &\leqslant \int_s^t \|u\|_2 \, \|\nabla(\psi * \psi)\|_2 \, \|u_k - u\|_2 \, \|u_k\|_2 \, d\tau \\
&\leqslant C \int_s^t \|u_k - u\|_2 \, d\tau \leqslant C(t-s)^{1/2} \left(\int_s^t \|u_k - u\|_2^2 \, d\tau \right)^{1/2} \to 0 \\
&\qquad \text{as} \quad k \to \infty, \qquad (2.14)
\end{aligned}
$$

$$
\begin{aligned}
I_3 &\leqslant \int_s^t \|\nabla(\psi * \psi)\|_2 \, \|u_k - u\|_2 \, \|u\|_2^2 \, d\tau \\
&\leqslant C \int_s^t \|u_k - u\|_2 \, d\tau \leqslant C(t-s)^{1/2} \left(\int_s^t \|u_k - u\|_2^2 \, d\tau \right)^{1/2} \to 0 \\
&\qquad \text{as} \quad k \to \infty. \qquad (2.15)
\end{aligned}
$$

Therefore, (2.12)–(2.15) imply

$$\int_s^t \langle v_k \cdot \nabla u_k, \psi * \psi * u_k \rangle \, ds \to \int_s^t \langle u \cdot \nabla u, \psi * \psi * u \rangle \, ds. \qquad (2.16)$$

From (2.9), we have

$$\|\psi * u_k(t)\|_2^2 \to \|\psi * u(t)\|_2^2 \qquad \text{almost every} \quad t \geqslant s, \qquad (2.17)$$

$$\int_s^t E(\tau) \, \|\nabla\psi * u(\tau)\|_2^2 \, d\tau \leqslant \varliminf_{k \to \infty} \int_s^t E(\tau) \, \|\nabla\psi * u_k(\tau)\|_2^2 \, d\tau. \qquad (2.18)$$

Using (2.16)–(2.18) in (2.11), we obtain inequality (2.7) for almost all t and s. Since the solution is weakly continuous in L^2, we obtain the desired inequality.

PROPOSITION 2.4. *Let $E(t) \in C^1(\mathbf{R}_+; \mathbf{R})$ and $\tilde{\psi} \in C^1(0, \infty; L^\infty)$ such that $(1-\tilde{\psi}^2) \in L^\infty(0, \infty; L^2)$ and $\nabla \mathscr{F}^{-1}(1-\tilde{\psi}^2) \in L^\infty(0, \infty; L^2)$. Then the weak solution constructed in Proposition 2.3 also satisfies*

$$
\begin{aligned}
E(t)\, \|\tilde{\psi}(t)\, \hat{u}(t)\|_2^2 \leqslant E(s)\, \|\tilde{\psi}(s)\, \hat{u}(s)\|_2^2 &+ \int_s^t E'(\tau)\, \|\tilde{\psi}(\tau)\, \hat{u}(\tau)\|_2^2\, d\tau \\
&+2 \int_s^t E(\tau)\, |\langle \tilde{\psi}'(\tau)\, \hat{u}(\tau), \tilde{\psi}\hat{u}(\tau)\rangle - \|\xi\tilde{\psi}(\tau)\, \hat{u}(\tau)\|_2^2|\, d\tau \\
&+2 \int_s^t E(\tau)(|\langle \widehat{u\cdot\nabla u}\,(\tau), (1-\tilde{\psi}^2)\, \hat{u}(\tau)\rangle| \\
&+|\langle \hat{f}(\tau), \tilde{\psi}(\tau)^2\, \hat{u}(\tau)\rangle|)\, d\tau, \qquad (2.19)
\end{aligned}
$$

for almost all $s>0$ with $s=0$ and all $t \geqslant s$. In particular, the weak solution satisfies

$$
\begin{aligned}
E(t)\, \|u(t)\|_2^2 \leqslant E(s)\, \|u(s)\|_2^2 &+ \int_s^t E'(\tau)\, \|u(\tau)\|_2^2\, d\tau \\
&-2 \int_s^t E(\tau)\, \|\nabla u(\tau)\|_2^2\, d\tau + 2 \int_s^t E(\tau)\, |\langle f(\tau), u(\tau)\rangle|\, d\tau, \qquad (2.20)
\end{aligned}
$$

for almost all $s \geqslant 0$ and all $t \geqslant s$.

Proof of Proposition 2.4. We consider the approximated equation (2.8) in the proof of Proposition 2.3. Recall that the solution u_k are smooth in L^2, and if necessary, by approximating f by a sequence of functions in L^2, we can consider the Fourier transform of the equation (2.8). Multiplying the Fourier transform of (2.8) by $E(\tau)\, \tilde{\psi}^2\, \hat{u}_k(\tau)$, we get

$$
\begin{aligned}
\frac{d}{dt} E(\tau)\, \|\tilde{\psi}\hat{u}_k(\tau)\|_2^2 = E'(\tau)\, \|\tilde{\psi}\hat{u}_k(\tau)\|_2^2 &+ 2E(\tau)\{\langle \tilde{\psi}'(\tau)\, \hat{u}_k(\tau), \tilde{\psi}(\tau)\, \hat{u}_k\rangle \\
&- \|\xi\tilde{\psi}(\tau)\, \hat{u}_k\|_2^2\} - 2E(\tau)\langle \mathscr{F}(v_k\cdot\nabla u_k), \tilde{\psi}^2\hat{u}_k\rangle \\
&+2E(\tau)\langle \hat{f}, \tilde{\psi}^2\hat{u}_k\rangle. \qquad (2.21)
\end{aligned}
$$

Again integrating (2.21) over (s, t) and noting that $\langle \mathscr{F}(v_k \cdot \nabla u_k), \hat{u}_k \rangle = 0$, we have

$$E(t)\, \|\tilde{\psi}\hat{u}_k(t)\|_2^2 + 2\int_s^t E(\tau)\, \|\xi\tilde{\psi}\hat{u}_k(\tau)\|_2^2\, d\tau$$

$$\leqslant E(s)\, \|\tilde{\psi}\hat{u}_k(s)\|_2^2 + \int_s^t E'(\tau)\, \|\tilde{\psi}\hat{u}_k(\tau)\|_2^2\, d\tau$$

$$+2\int_s^t E(\tau)\, |\langle \mathscr{F}(v_k(\tau)\cdot\nabla u_k)(\tau), (1-\tilde{\psi}^2)\, \hat{u}_k(\tau)\rangle|\, d\tau$$

$$+2\int_s^t E(\tau)\, |\langle \hat{f}(\tau), \tilde{\psi}^2 \hat{u}_k(\tau)\rangle|\, d\tau. \tag{2.22}$$

The conditions $1-\tilde{\psi}^2 \in L^\infty(0, \infty; L^2)$ and $\nabla\mathscr{F}^{-1}(1-\tilde{\psi}^2) \in L^\infty(0, \infty; L^2)$ are necessary in order to obtain the convergence of the nonlinear term, since we estimate it as in Proposition 2.3. Using the same argument as in Proposition 2.3, we see that

$$\int_s^t \langle \mathscr{F}(v_k \cdot \nabla u_k), (1-\tilde{\psi}^2)\, \hat{u}_k\rangle\, ds$$

$$= \int_s^t \langle u_k \cdot \nabla u_k, \mathscr{F}^{-1}(1-\tilde{\psi}^2) * u_k\rangle\, ds$$

$$\to \int_s^t \langle u\cdot\nabla u, \mathscr{F}^{-1}(1-\tilde{\psi}^2) * u\rangle\, ds.$$

Therefore by passing to the limit as $k \to \infty$ in (2.22), (2.19) follows. The second inequality (2.20) is obtained by setting $\tilde{\psi} \equiv 1$ in (2.19).

From the generalized energy inequality (2.7) in Proposition 2.3, it follows that

COROLLARY 2.5. *For a weak solution u and $\phi \in L^2(\mathbf{R}^n)$, we have*

$$\|\check{\phi} * u(t)\|_2^2 \leqslant \|e^{\Delta(t-s)}\check{\phi} * u(s)\|_2^2 + 2\int_s^t (|\langle u\cdot\nabla u, e^{2\Delta(t-\tau)}\check{\phi}^2 * u(\tau)\rangle|$$

$$+ |\langle f, e^{2\Delta(t-\tau)}\check{\phi}^2 * u(\tau)\rangle|)\, d\tau. \tag{2.23}$$

Proof of Corollary 2.5. Take $E(t) = 1$ and $\psi(\tau)$ as

$$\psi(\tau) = \mathscr{F}^{-1} e^{-|\xi|^2(t+\eta-\tau)}\phi(\xi), \qquad \eta > 0$$

in (2.7). Then $\psi(\tau) * u(\tau) = e^{\Delta(t+\eta-\tau)}\check{\phi} * u(\tau)$ and

$$\begin{aligned}\langle \psi'(\tau) * u(\tau), \psi(\tau) * u(\tau)\rangle - \|\nabla\psi(\tau) * u(\tau)\|_2^2 \\ = \langle \Delta e^{\Delta(t+\eta-\tau)}\check{\phi} * u(\tau), e^{\Delta(t+\eta-\tau)}\check{\phi} * u(\tau)\rangle - \|\nabla e^{\Delta(t+\eta-\tau)}\check{\phi} * u(\tau)\|_2^2 \\ = 0.\end{aligned}$$

Hence we have

$$\begin{aligned}\|e^{\Delta\eta}\check{\phi} * u(t)\|_2^2 \leqslant \|e^{\Delta(t+\eta-s)}\check{\phi} * u(s)\|_2^2 \\ +2\int_s^t (|\langle u\cdot\nabla u, e^{2\Delta(t+\eta-\tau)}\check{\phi}^2 * u(\tau)\rangle| \\ +|\langle f, e^{2\Delta(t+\eta-\tau)}\check{\phi}^2 * u(\tau)\rangle|)\, d\tau. \qquad (2.24)\end{aligned}$$

Letting $\eta \to 0$ in (2.24), we obtain (2.23).

Remark. If $\psi \in L^\infty(0, T; L^{2n/(n+2)})$, then the generalized energy inequality (2.6) can be derived from the definition of the weak solution and hence is satisfied by all weak solutions not necessarily satisfying the strong energy inequality. In fact, Masuda [16] obtained an inequality equivalent to (2.21), with $E \equiv 1$ and $\tilde{\psi}(t, \xi) = e^{-(t-\tau)|\xi|^2}(1+|\xi|^2)^{-\alpha/2}$, and used it to obtain the non-uniform decay of the weak solutions.

To show uniform decay, the following expression of the weak solution will be used:

PROPOSITION 2.6. *Let u be a weak solution in* (NS). *Then for almost all t with $0 \leqslant s \leqslant t < \infty$,*

$$u(t) = e^{\Delta(t-s)}u(s) - \int_s^t e^{\Delta(t-\tau)} P(u\cdot\nabla u(\tau) - f(\tau))\, d\tau \qquad (2.25)$$

in L^2_σ, where $P = \mathscr{F}^{-1}(\delta_{ij} - (\xi_i\xi_j/|\xi|^2))\mathscr{F}$ is the projection operator from L^2 to L^2_σ.

Proof of Proposition 2.6. Using $\Phi(t) = Pe^{\Delta(t+\eta-\tau)}\phi$ for all $\phi \in L^2$ in the definition of the weak solution (1.1), we have

$$\begin{aligned}&\langle u(t), e^{\Delta\eta}\phi\rangle - \langle e^{\Delta(t-s)}u(s), e^{\Delta\eta}\phi\rangle \\ &= \int_s^t \{\langle u(\tau), \Delta e^{\Delta(t+\eta-\tau)}\phi\rangle - \langle \nabla u(\tau), \nabla P e^{\Delta(t+\eta-\tau)}\phi\rangle \\ &\quad - \langle P(u\cdot\nabla u(\tau), e^{\Delta(t+\eta-\tau)}\phi\rangle + \langle Pf(\tau), e^{\Delta(t+\eta-\tau)}\phi\rangle\}\, d\tau \\ &= \int_s^t \{-\langle P(u\cdot\nabla u(\tau), e^{\Delta(t+\eta-\tau)}\phi\rangle + \langle Pf(\tau), e^{\Delta(t+\eta-\tau)}\phi\rangle\}\, d\tau.\end{aligned}$$

Then by taking a limit as $\eta \to 0$, $e^{\eta\Delta}\phi \to \phi$ in L^2 and we conclude that

$$\langle u(t), \phi \rangle = \langle e^{\Delta(t-s)}u(s) - \int_s^t e^{\Delta(t-\tau)} P(u \cdot \nabla u(\tau) - f(\tau))\, d\tau, \phi \rangle$$

for all $\phi \in L^2$. This shows (2.24).

Finally, we show a variation of the well-known $L^p - L^q$ type estimate for the linear heat equation (see, e.g., Reed–Simon [18] and Kato [10]).

LEMMA 2.7 (L^p-L^q Estimate). *Let $h(t) \geqslant 0$ be a smooth function of t and define*

$$e^{\Delta h(t)} a \equiv \mathscr{F}^{-1} e^{-h(t)\,|\xi|^2} \hat{a},$$

$$D^\delta a \equiv \mathscr{F}^{-1}\, |\xi|^\delta\, \hat{a}.$$

Then for any $1 \leqslant q \leqslant p \leqslant \infty$ and $0 \leqslant \delta$, we have

$$\|D^\delta e^{\Delta h(t)} a\|_p \leqslant Ch(t)^{-(n/2)(1/q - 1/p) - (\delta/2)}\, \|a\|_q \qquad \text{for} \quad a \in L^q. \tag{2.25}$$

In particular, if $a \in L^q \cap L^2$ $(1 \leqslant q < 2)$, then

$$\|e^{\Delta t} a\|_2 \leqslant C(1+t)^{-(n/2)(1/q - 1/2)}\, (\|a\|_2 + \|a\|_q). \tag{2.26}$$

Proof of Lemma 2.7. Inequality (2.25) is well known if $h(t) = t$. For general $h(t)$, (2.25) is obtained similarly. To obtain (2.26), we set $h(t) = t$ in (2.25) and combine two cases $p = q = 2$ and $q < p = 2$.

3. NON-UNIFORM DECAY

In this section, we prove the non-uniform decay result:

THEOREM 3.1 (Non-uniform Decay). *Let $a \in L^2$ and f satisfy* A.1. *Then the weak solution u to* (NS) *constructed in Proposition* 2.3 *satisfies the energy decay*:

$$\|u(t)\|_2 \to 0 \qquad as \quad t \to \infty.$$

We split the proof into two steps, i.e., the estimates for the low frequency part of the energy and for the high frequency part. In the each step, we use the generalized energy inequality established in Section 2.

Proof of Theorem 3.1. To show the decay of the low frequency part of the energy, we follow an argument due to Masuda [16]. Taking $\phi(\xi)=\exp(-|\xi|^2)$, we have by Corollary 2.5 and Plancherel's identity that

$$\begin{aligned}\|\phi\hat{u}(t)\|^2 &\leqslant \|e^{-|\xi|^2(t-s)}\phi\hat{u}(s)\|^2\\ &\quad+2\int_s^t |\langle u\cdot\nabla u, e^{2\Delta(t-\tau)}\check{\phi}^2 * u\rangle|\,d\tau+2\int_s^t |\langle f, e^{2\Delta(t-\tau)}\check{\phi}^2 * u\rangle|\,d\tau\\ &\leqslant \|e^{-|\xi|^2(t-s)}\phi\hat{u}(s)\|^2+2\int_s^t |\langle\check{\phi}^2 * u\cdot\nabla u, e^{2\Delta(t-\tau)}u\rangle|\,d\tau\\ &\quad+2\int_s^t |\langle f, e^{2\Delta(t-\tau)}\check{\phi} * u\rangle|\,d\tau. \end{aligned}\tag{3.1}$$

Since $\check{\phi}^2$ is a rapidly decreasing function, by the Hausdorff–Young, Hölder, and Sobolev inequalities it follows that for $n\geqslant 3$,

$$\begin{aligned}|\langle\check{\phi}^2 * u\cdot\nabla u, e^{2\Delta(t-\tau)}u\rangle| &\leqslant \|\check{\phi}^2 * u\cdot\nabla u\|_2\,\|u\|_2\\ &\leqslant C\,\|\check{\phi}^2\|_{2n/(n+2)}\,\|u\cdot\nabla u\|_{n/(n-1)}\,\|u\|_2\\ &\leqslant C\,\|\check{\phi}^2\|_{2n/(n+2)}\,\|u\|_2\,\|u\|_{2n/(n-2)}\,\|\nabla u\|_2\\ &\leqslant C(\phi)\,\|u\|_2\,\|\nabla u\|_2^2. \end{aligned}\tag{3.2}$$

Similarly if $n=2$ we have

$$\begin{aligned}|\langle\check{\phi}^2 * u\cdot\nabla u, e^{2\Delta(t-\tau)}u\rangle| &\leqslant |\langle u\cdot\nabla\check{\phi}^2 * e^{2\Delta(t-\tau)}u, u\rangle|\\ &\leqslant \|u\|_4^2\,\|\check{\phi}^2 * e^{2\Delta(t-\tau)}\nabla u\|_2\\ &\leqslant \|\check{\phi}^2\|_\infty\,\|u\|_4^2\,\|\nabla u\|_2\\ &\leqslant C(\phi)\,\|u\|_2\,\|\nabla u\|_2^2. \end{aligned}\tag{3.3}$$

Hence by (3.1)–(3.3) and Lemma 2.1,

$$\begin{aligned}\|\phi\hat{u}(t)\|_2^2 &\leqslant \|e^{-|\xi|^2(t-s)}\phi\hat{u}(s)\|_2^2+2\int_s^t C(\phi)\,\mathscr{E}(\tau)\,\|\nabla u\|_2^2\,d\tau\\ &\quad+C\mathscr{E}^{(1-\lambda)(1-\gamma)}(t)\left(\int_s^t \|\rho^\gamma f\|_{p'}^{\theta'}\,d\tau\right)^{1/\theta'}\left(\int_s^t \|\nabla u\|_2^2\,d\tau\right)^{1/\theta}. \end{aligned}\tag{3.4}$$

Since $\overline{\lim}_{t\to\infty} \|e^{-|\xi|^2(t-s)}\phi\hat{u}(s)\|_2^2 = 0$, we have by taking a limit $t\to\infty$ in (3.4) that

$$\overline{\lim_{t\to\infty}} \|\phi\hat{u}(t)\|_2^2 \leqslant C \sup_t \mathscr{E}(t) \int_s^\infty \|\nabla u\|_2^2\, d\tau$$
$$+ C \sup_t \mathscr{E}^{(1-\lambda)(1-\gamma)}(t) \left(\int_s^\infty \|\rho^\gamma f\|_{p'}^{\theta'}\, d\tau\right)^{1/\theta'} \left(\int_s^\infty \|\nabla u\|_2^2\, d\tau\right)^{1/\theta}. \tag{3.5}$$

By Proposition 2.2, the right hand side of (3.5) converges 0 as $s\to\infty$.

Next, the Fourier splitting method ([19]) is used to estimate the high frequency part of the energy. Choose $\widetilde{\psi} = 1-\exp(-|\xi|^2)$ and let $\chi(t) = \{\xi\in\mathbf{R}^n;\ |\xi|\leqslant G(t)\}$. Then inequality (2.19) yields that

$$\begin{aligned}
E(t)\|(1-\phi)\,\hat{u}(t)\|_2^2 &\leqslant E(s)\|(1-\phi)\hat{u}(s)\|^2 + \int_s^t E'(\tau)\int_{\chi(\tau)} |(1-\phi)\,\hat{u}(\tau)|^2\, d\xi\, d\tau \\
&\quad + \int_s^t E'(\tau)\int_{\mathbf{R}^n\setminus\chi(\tau)} |(1-\phi)\,\hat{u}(\tau)|^2\, d\xi\, d\tau \\
&\quad - 2\int_s^t E(\tau)\, \|\xi(1-\phi)\,\hat{u}(\tau)\|_2^2\, d\tau \\
&\quad + 2\int_s^t E(\tau)(|\langle \mathscr{F}(u\cdot\nabla u), (1-(1-\phi)^2)\,\hat{u}(\tau)\rangle| \\
&\qquad + |\langle \hat{f}, (1-\phi)^2\,\hat{u}(\tau)\rangle|)\, d\tau \\
&\leqslant E(s)\, \|(1-\phi)\,\hat{u}(s)\|_2^2 + \int_s^t E'(\tau)\int_{\chi(\tau)} |(1-\phi)\,\hat{u}(\tau)|^2\, d\xi\, d\tau \\
&\quad + \int_s^t (E'(\tau) - 2E(\tau)\, G^2(\tau))\int_{\mathbf{R}^n\setminus\chi(\tau)} |(1-\phi)\,\hat{u}(\tau)|^2\, d\xi\, d\tau \\
&\quad + 2\int_s^t E(\tau)(|\langle \mathscr{F}(u\cdot\nabla u), (1-(1-\phi)^2)\,\hat{u}(\tau)\rangle| \\
&\qquad + |\langle f, u(\tau)\rangle|)\, d\tau.
\end{aligned} \tag{3.6}$$

Since $\mathscr{F}^{-1}(1-(1-\phi)^2)\equiv\psi$ is a rapidly decreasing function, the fourth term of the right hand side of (3.6) is estimated as follows if $n=3, 4$:

$$\begin{aligned}
&\int_s^t E(\tau)\,|\langle u\cdot\nabla u,\ \psi * u(\tau)\rangle|\,d\tau \\
&\quad\leqslant \int_s^t E(\tau)\,\|\psi\|_{2n/(n+2)}\,\|u\cdot\nabla u\|_{n/(n-1)}\,\|u\|_2\,d\tau \\
&\quad\leqslant C(\phi)\,\mathscr{E}(t)\int_s^t E(\tau)\,\|u\|_{2n/(n-2)}\,\|\nabla u\|_2\,d\tau \\
&\quad\leqslant C(\phi)\,\mathscr{E}(t)\int_s^t E(\tau)\,\|\nabla u\|_2^2\,d\tau. \qquad (3.7)
\end{aligned}$$

If $n=2$, by the Gagliardo–Nirenberg inequality, it follows that

$$\begin{aligned}
&\int_s^t E(\tau)\,|\langle u\cdot\nabla u,\ \psi * u(\tau)\rangle|\,d\tau \\
&\quad= \int_s^t E(\tau)\,|\langle u\cdot\psi * \nabla u(\tau),\ u(\tau)\rangle|\,d\tau \\
&\quad= \int_s^t E(\tau)\,\|\psi\|_\infty\,\|u\|_4^2\,\|\nabla u\|_2\,d\tau \\
&\quad\leqslant C(\psi)\int_s^t E(\tau)\,\|u\|_2\,\|\nabla u\|_2^2\,d\tau \\
&\quad\leqslant C\mathscr{E}(t)\int_s^t E(\tau)\,\|\nabla u\|_2^2\,d\tau. \qquad (3.8)
\end{aligned}$$

Since $E(t)\geqslant 0$, from (3.6)–(3.8) we have

$$\begin{aligned}
\|(1-\phi)\,\hat{u}(t)\|_2^2 \leqslant{}& \frac{E(s)}{E(t)}\,\|(1-\phi)\,\hat{u}(s)\|_2^2 + \frac{1}{E(t)}\int_s^t E'(\tau)\int_{\chi(\tau)} |(1-\phi)\,\hat{u}(\tau)|^2\,d\xi\,d\tau \\
&+\frac{1}{E(t)}\int_s^t (E'(\tau)-2E(\tau)\,G^2(\tau))\int_{\mathbf{R}^n\setminus\chi(\tau)} |(1-\phi)\,\hat{u}(\tau)|^2\,d\xi\,d\tau \\
&+2\,\frac{C(\phi)\,\mathscr{E}(t)}{E(t)}\int_s^t E(\tau)\|\nabla u\|_2^2\,d\tau \\
&+\frac{1}{E(t)}\int_s^t E(\tau)|\langle f, u(\tau)\rangle|\,d\tau. \qquad (3.9)
\end{aligned}$$

We now choose $E(t)=(1+t)^{\alpha}$, $G^2(t)=\alpha/2(1+t)$ in (3.9). Since $E'(t)=2E(t)\,G^2(t)$, it follows from (3.9) and (2.2) that

$$\begin{aligned}
\|(1-\phi)\,\hat{u}(t)\|_2^2 \leqslant & \left(\frac{1+s}{1+t}\right)^{\alpha} \|(1-\phi)\,\hat{u}(s)\|_2^2 + \frac{\alpha}{(1+t)^{\alpha}} \\
& \times \int_s^t (1+\tau)^{\alpha-1} \int_{\chi(\tau)} |(1-\phi)\,\hat{u}|^2 \,d\xi\,d\tau \\
& + C(\phi)\,\frac{1}{(1+t)^{\alpha}}\,\mathscr{E}(t) \int_s^t (1+\tau)^{\alpha}\,\|\nabla u\|_2^2\,d\tau \\
& + C \sup_{t,\,\xi} |(1-\phi)^2| \int_s^t \left(\frac{1+\tau}{1+t}\right)^{\alpha} |\langle \hat{f}, \hat{u}\rangle|\,d\tau \\
\leqslant & \left(\frac{1+s}{1+t}\right)^{\alpha} \|(1-\phi)\,\hat{u}(s)\|_2^2 + \frac{\alpha}{(1+t)^{\alpha}} \int_s^t (1+\tau)^{\alpha-1} \\
& \times \int_{\chi(\tau)} |(1-\phi)\,\hat{u}|^2\,d\xi\,d\tau + C(\phi)\,\mathscr{E}(t) \int_s^t \|\nabla u\|_2^2\,d\tau \\
& + C \sup_t \mathscr{E}^{(1-\lambda)(1-\gamma)}(t) \left(\int_s^t \|\rho^{\gamma} f\|_{p'}^{\theta'}\,d\tau\right)^{1/\theta'} \left(\int_s^t \|\nabla u\|_2^2\,d\tau\right)^{1/\theta},
\end{aligned} \tag{3.10}$$

where γ, p, and θ are defined in Assumption A.1. Observing that $|1-\phi| \leqslant |\xi|^2$ if $|\xi| < 1$, we have

$$\int_{\chi(\tau)} |(1-\phi)\,\hat{u}|^2\,d\xi \leqslant CG(\tau)^4 \int_{\chi(\tau)} |\hat{u}|^2\,d\xi \leqslant C\mathscr{E}(\tau)(1+\tau)^{-2}. \tag{3.11}$$

Therefore by letting $t \to \infty$ in (3.10), (3.10) and (3.11) we get, if $p > 2$,

$$\begin{aligned}
\varlimsup_{t\to\infty} \|(1-\phi)\,\hat{u}(t)\|_2^2 \leqslant & \varlimsup_{t\to\infty} \left(\frac{1+s}{1+t}\right)^{\alpha} \sup_t \mathscr{E}(t) \\
& + C \sup_t \mathscr{E}(t) \varlimsup_{t\to\infty} \left(\frac{1}{(1+t)^{\alpha}} \int_s^t (1+\tau)^{\alpha-3}\,d\tau\right) \\
& + C \sup_t \mathscr{E}(t) \int_s^{\infty} \|\nabla u(\tau)\|_2^2\,d\tau \\
& + C \sup_t \mathscr{E}(t)^{1-\lambda}\,\|\rho^{\gamma} f\|_{L^{\theta',p'}} \left(\int_0^{\infty} \|\nabla u(\tau)\|_2\,d\tau\right)^{1/\theta} \\
\leqslant & C \sup_t \mathscr{E}(t) \int_s^{\infty} \|\nabla u(\tau)\|_2^2\,d\tau \\
& + C \sup_t \mathscr{E}(t)^{1-\lambda}\,\|\rho^{\gamma} f\|_{L^{\theta',p'}} \left(\int_s^{\infty} \|\nabla u(\tau)\|_2\,d\tau\right)^{1/\theta}.
\end{aligned} \tag{3.12}$$

Since the first term vanishes, letting $s \to \infty$, the last two terms on the right-hand side of (3.12) tend to 0. If $p=2$, the last inequality reduces to

$$\begin{aligned}\overline{\lim_{t\to\infty}}\ \|(1-\phi)\,\hat{u}(t)\|_2^2 &\leqslant C \sup_t \mathscr{E}(t) \int_s^\infty \|\nabla u(\tau)\|_2^2\, d\tau\\ &\quad + C \sup_t \mathscr{E}(t) \int_s^\infty \|f(\tau)\|_2\, d\tau,\end{aligned}$$

which also converges to zero as $s \to \infty$. This proves Theorem 3.1.

A simple consequence of Theorem 3.1 leads to the following

COROLLARY 3.2. *Suppose that f satisfies the same conditions as in Theorem* 3.1. *Let u be a solution satisfying the strong energy inequality. Then*

$$\frac{1}{t}\int_0^t \|u(\tau)\|_2\, d\tau \to 0 \qquad \textit{as} \quad t \to \infty. \tag{3.13}$$

Proof. For any $\varepsilon > 0$, we can choose s sufficiently large so that

$$\|u(\tau)\|_2 \leqslant \varepsilon \qquad \text{for} \quad \tau \geqslant s.$$

Then

$$\begin{aligned}\frac{1}{t}\int_0^t \|u(\tau)\|_2\, d\tau &= \frac{1}{t}\int_0^s \|u(\tau)\|_2\, d\tau + \frac{1}{t}\int_s^t \|u(\tau)\|_2\, d\tau\\ &\leqslant \frac{1}{t}\int_0^s \|u(\tau)\|_2\, d\tau + \varepsilon\, \frac{t-s}{t}\\ &\to \varepsilon \qquad \text{as} \quad t \to \infty.\end{aligned}$$

Remark. A similar result was obtained by Kato [10] for small data and no external force. Miyakawa and Sohr [17] showed that the average decay (3.13) implies non-uniform decay. Our result extends the class of functions f for which Miyakawa and Sohr had previously obtained (3.13).

4. UNIFORM DECAY

In this section, we obtain uniform rates of decay for the solutions using the Fourier splitting method under assumptions A.2 and A.3. More precisely, we prove:

THEOREM 4.1 (Uniform Decay). *Let u be the weak solution constructed in Theorem* 1.1. *Suppose that the initial data $a \in L^2 \cap L^q$, with $q \leqslant 2n/(4\varepsilon + n)$.*

(1) *If f satisfies assumptions* A.2 *and* A.3, *then*

$$\|u(t)\|_2 \leqslant \begin{cases} C(1+t)^{-\varepsilon} & \text{if } n=3, 4, \\ C(\log(e+t))^{-1/2} & \text{if } n=2, \end{cases}$$

where the constant C only depends on a and f.

(2) *If we further restrict f by assuming $f \in L^1(0, \infty; L^1)$, then the decay in two dimensions can be improved to yield*

$$\|u(t)\|_2 \leqslant C(t+1)^{-\varepsilon}.$$

Proof of Theorem 4.1. We first treat the case when f satisfies assumption A.3(1). Recall that $\chi(t) = \{\xi \in \mathbf{R}^n;\ |\xi| < G(t)\}$. Let $E(t)$ satisfy $E'(t) = 2G^2(t)\, E(t)$. Using this in (2.20) it follows that

$$E(t)\|u(t)\|_2^2 \leqslant E(0)\|a\|_2^2 + 2\int_0^t E'(\tau)\left[\int_{\chi(\tau)} |\hat{u}|^2\, d\xi\right] d\tau$$
$$+ 2\int_0^t E(\tau)|\langle f(\tau), u(\tau)\rangle|\, d\tau. \tag{4.2}$$

By Proposition 2.6, we see that

$$\begin{aligned} |\mathscr{F}(u(\tau))| &\leqslant |\mathscr{F}(e^{\Delta\tau}a)| + \left|\int_0^\tau e^{-|\xi|^2(\tau-\sigma)}(\mathscr{F}(P(u\cdot\nabla u - f))\, d\sigma\right| \\ &\leqslant |e^{-|\xi|^2\tau}\hat{a}| + \int_0^\tau e^{-|\xi|^2(\tau-\sigma)} \left|\left(\delta_{ij} - \frac{\xi_i\xi_j}{|\xi|^2}\right)\xi_k \mathscr{F}(u_j u_k) - \hat{f}_j\right| d\sigma \\ &\leqslant |e^{-|\xi|^2\tau}\hat{a}| + \int_0^\tau e^{-|\xi|^2(\tau-\sigma)}(|\xi_k||\mathscr{F}(u_j u_k)| + |\xi||\hat{g}|)\, d\sigma \\ &\leqslant |e^{-|\xi|^2\tau}\hat{a}| + \int_0^\tau e^{-|\xi|^2(\tau-\sigma)}\, |\xi|\, d\sigma\, (\sup_t \|\mathscr{F}(u_j u_k)\|_\infty + \|\hat{g}\|_\infty). \end{aligned}$$

Therefore, by Hölder's inequality, the second term of the right hand side of (4.2) is

$$4\int_0^t E'(\tau)\left[\int_{\chi(\tau)} |e^{-|\xi|^2\tau}\hat{a}|^2\, d\xi + (\sup_t \|u_j u_k\|_1 + \|g\|_1)^2\right.$$

$$\left.\times \int_{\chi(\tau)} \tau \int_0^\tau e^{-2|\xi|^2(\tau-\sigma)} |\xi|^2\, d\sigma\, d\xi\right] d\tau$$

$$\leqslant C\int_0^t E'(\tau)[\|e^{\Delta\tau}a\|_2^2 + \sup(\|u_j u_k\|_1 + \|g\|_1)^2\, \tau G(\tau)^n]\, d\tau. \tag{4.3}$$

Combining (4.2) and (4.3), we have

$$E(t)\,\|\hat{u}(t)\|_2^2 \leqslant E(0)\,\|a\|_2^2 + C\int_0^t E'(\tau)\,\|e^{\Delta\tau}a\|_2^2\, d\tau + C(\|u\|_2^4, \|g\|_1)$$

$$\times \int_0^t \tau E'(\tau)\, G(\tau)^n\, d\tau + 2\int_0^t E(\tau)\,|\langle f(\tau), u(\tau)\rangle|\, d\tau. \tag{4.4}$$

Suppose that $n=3$ and 4. Let $E(t)=(1+t)^\alpha$ again. The assumption on the initial data that $a\in L^2\cap L^q$, combined with the $L^p - L^q$ estimate (2.6) for the second term on the right-hand side of (4.4), yields

$$(1+t)^\alpha\,\|\hat{u}(t)\|_2^2 \leqslant C\,\|a\|_2^2 + \int_0^t (1+\tau)^{\alpha-1-n(1/q-1/2)}\,\{\|a\|_2^2 + \|a\|_q^2\}\, d\tau$$

$$+ C\int_0^t (1+\tau)^{\alpha-n/2}\, d\tau + \int_0^t (1+\tau)^\alpha\,|\langle f(\tau), u(\tau)\rangle|\, d\tau$$

$$\equiv I_1 + I_2 + I_3 + I_4. \tag{4.5}$$

Hypothesis A.2 and Proposition 2.2 imply

$$I_4 \leqslant \int_0^t (1+\tau)^\alpha\,\|\rho^\gamma f\|_{p'}\,\|u(\tau)\|_2^{(1-\lambda)(1-\gamma)}\,\|\nabla u(\tau)\|_2^{2/\theta'}\, d\tau$$

$$\leqslant C\mathscr{E}(t)^{(1-\lambda)(1-\gamma)}\int_0^t (1+\tau)^{\alpha-\beta}\,\|\nabla u(\tau)\|_2^{2/\theta'}\, d\tau$$

$$\leqslant C\mathscr{E}(t)^{(1-\lambda)(1-\gamma)}\left(\int_0^t (1+\tau)^{(\alpha-\beta)\theta'}\, d\tau\right)^{1/\theta'}\left(\int_0^t \|\nabla u(\tau)\|_2^2\, d\tau\right)^{1/\theta}$$

$$\leqslant C, \tag{4.6}$$

provided $\alpha < \beta - 1/\theta' = \varepsilon([\theta+2]/\theta)$. On the other hand, I_2 and I_3 are bounded if $\alpha < \min((n/2)-1,\ n(1/q-1/2))$.

Choosing

$$\alpha < \min\left\{\frac{n}{2}-1,\, n\left(\frac{1}{q}-\frac{1}{2}\right),\, \varepsilon\left(\frac{\theta+2}{\theta}\right)\right\} \equiv \alpha_0$$

and combining this with (4.5) and (4.6) we can conclude that

$$\begin{aligned}(1+t)^{\alpha}\,\|u(t)\|_2^2 &\leqslant \|a\|_2^2 + C\{\|a\|_2^2 + \|a\|_p^2 + \sup_t \mathscr{E}(t)\\ &\qquad + \sup_t \mathscr{E}^4(t) + \sup_t \|g\|_1^2 + \sup_t \mathscr{E}(t)^{(1-\gamma)(1-\lambda)}\,\|\nabla u\|_{L^{2,2}}^{2/\theta'}\}\\ &\leqslant C,\end{aligned} \tag{4.7}$$

where the constant C is independent of t. Hence we obtain

$$\|u(t)\|_2 \leqslant C(1+t)^{-\alpha_0}.$$

We assume that ε is small and the minimum value of α_0 is $\varepsilon([\theta+2]/\theta)$. Hence we obtain, by setting $\eta=(\theta+2)/\theta$,

$$\|u(t)\|_2 \leqslant C(1+t)^{-(\varepsilon/2)\eta}. \tag{4.8}$$

Substituting (4.8) into the last term in (4.5) and setting $\kappa=(1-\lambda)(1-\gamma)$, we have

$$\begin{aligned}&\int_0^t (1+\tau)^{\alpha}\,|\langle f(\tau), u(\tau)\rangle|\,d\tau\\ &\leqslant \int_0^t (1+\tau)^{\alpha}\,\|\rho^{\gamma} f\|_{p'}\,\|u(\tau)\|_2^{(1-\lambda)(1-\gamma)}\,\|\nabla u(\tau)\|_2^{2/\theta}\,d\tau\\ &\leqslant \int_0^t (1+\tau)^{\alpha-\varepsilon\kappa\eta/2}\,\|\rho^{\gamma} f\|_{p'}\,\|\nabla u(\tau)\|_2^{2/\theta}\,d\tau\\ &\leqslant C\int_0^t (1+\tau)^{\alpha-\varepsilon\eta\kappa/2-\varepsilon\eta-1/\theta'}\,\|\nabla u(\tau)\|_2^{2/\theta}\,d\tau\\ &\leqslant C\left(\int_0^t (1+\tau)^{\theta'(\alpha-\varepsilon\eta(1+\kappa/2))-1}\,d\tau\right)^{1/\theta'}\left(\int_0^t \|\nabla u(\tau)\|_2^2\,d\tau\right)^{1/\theta}\\ &\leqslant C,\end{aligned} \tag{4.9}$$

if $\alpha < \varepsilon\eta(1+\kappa/2)$. We then obtain

$$\|u(t)\|_2 \leqslant C(1+t)^{-(\varepsilon/2)(1+(\kappa/2))\eta}.$$

Noting $\kappa = 1-\gamma-(n(p-2)/2p) = 1-(2/\theta)$, we continue to iterate the above estimate to obtain

$$\begin{aligned}\|u(t)\|_2 &\leqslant C(1+t)^{-(\varepsilon/2)\eta(1+(\kappa/2)+(\kappa/2)^2+\cdots)}\\ &\leqslant C(1+t)^{-\varepsilon\eta(\theta/(\theta+2))}\\ &\leqslant C(1+t)^{-\varepsilon}.\end{aligned}$$

When $n=2$, we choose $E(t)=(\log(e+t))^\alpha$ $(\alpha>1)$ and $G^2(t)=\alpha(2(e+t)\log(e+t))^{-1}$. From (4.4),

$$\begin{aligned}(\log(e+t))^\alpha \|\hat{u}(t)\|_2^2 \leqslant{} & C\|a\|_2^2+\int_0^t (\log(e+\tau))^\alpha (e+\tau)^{-1-2(1/q-1/2)}\\ & \times\{\|a\|_2^2+\|a\|_q^2\}\,d\tau + C\int_0^t (\log(e+\tau))^{\alpha-1}\\ & \times\{(e+\tau)\log(e+\tau)\}^{-1}\,d\tau+\int_0^t (\log(e+\tau))^\alpha\\ & \times|\langle f(\tau),u(\tau)\rangle|\,d\tau\\ \equiv{} & I_1+I_2+I_3+I_4. \end{aligned} \tag{4.10}$$

Clearly, I_1 and I_2 are bounded, while

$$I_3 = C\int_0^t \frac{\log(e+\tau)^{\alpha-2}}{e+\tau}\,d\tau \leqslant C(\log(e+t))^{\alpha-1}. \tag{4.11}$$

By Assumption A.2 and Proposition 2.2, we have

$$\begin{aligned} I_4 &\leqslant \int_0^t (\log(e+\tau))^\alpha \|\rho^\gamma f\|_{p'} \|u(\tau)\|_2^{(1-\lambda)(1-\gamma)} \|\nabla u(\tau)\|_2^{2/\theta}\,d\tau\\ &\leqslant C\mathscr{E}(t)^{(1-\lambda)(1-\gamma)}\int_0^t \frac{\log(1+\tau)^\alpha}{(e+t)^\beta}\|\nabla u(\tau)\|_2^{2/\theta}\,d\tau\\ &\leqslant C\mathscr{E}(t)^{(1-\lambda)(1-\gamma)}\left(\int_0^t \frac{(\log(e+\tau))^{\alpha\theta'}}{(e+\tau)^{1+\eta\varepsilon\theta'}}\,d\tau'\right)^{1/\theta'}\left(\int_0^t \|\nabla u(\tau)\|_2^2\,d\tau\right)^{1/\theta}\\ &\leqslant C. \end{aligned} \tag{4.12}$$

Therefore from (4.10)–(4.12), it follows that

$$\|u(t)\|_2 \leqslant C \log(e+t)^{-1/2}.$$

Next we discuss the case we assume $f \in L^2(0, \infty; L^1)$. To show the decay in this case, we use inequality (2.20) with $\phi(\xi) = e^{-h(t)|\xi|^2}$, where $h(t)$ will be determined later. We write the second term on the right-hand side of (2.20) as follows:

$$\int_0^t E'(\tau) \int_{\mathbf{R}^n} |\phi(\xi)\, \hat{u}(\xi)|^2 \, d\xi \, d\tau$$

$$+ \int_0^t E'(\tau) \int_{\mathbf{R}^n} |(1-\phi^2(\xi))^{1/2}\, \hat{u}(\xi)|^2 \, d\xi \, d\tau. \tag{4.13}$$

Since $(1-\phi^2)$ behaves like $2h(t)|\xi|^2$ near $|\xi| \simeq 0$, by choosing $E(\tau)$ such that $h(\tau)\, E'(\tau) \leqslant E(\tau)$, it follows from (4.13) and (2.19) that

$$E(t)\|u(t)\|_2^2 \leqslant E(0)\|a\|_2^2 + \int_0^t E'(\tau) \int_{\mathbf{R}^n} |\phi(\xi)\, \hat{u}(\tau)|^2 \, d\xi \, d\tau$$

$$+2 \int_0^t E(\tau) |\langle f(\tau), u(\tau)\rangle| \, d\tau. \tag{4.14}$$

We proceed to estimate the second term of right hand side in (4.14). Using the expression of the solution in Proposition 2.6, we have

$$\int_{\mathbf{R}^n} |\phi(\xi)\, \mathscr{F}(u(\tau))|^2 \, d\xi \leqslant 2 \int_{\mathbf{R}^n} |\phi(\xi)| \left\{ |\mathscr{F}(e^{\Delta \tau} a)|^2 + \left| \int_0^\tau e^{-|\xi|^2(\tau-\sigma)} \right.\right.$$

$$\left.\left. \times \left(\left(\delta_{ij} - \frac{\xi_i \xi_j}{|\xi|^2} \right) \mathscr{F}(u \cdot \nabla u - f) \right) d\sigma \right|^2 \right\} d\xi$$

$$\leqslant 2 \int_{\mathbf{R}^n} \phi(\xi)^2 \, e^{-2\tau |\xi|^2} |\hat{a}|^2 \, d\xi + 2 \int_{\mathbf{R}^n} \phi(\xi)^2 \, \tau$$

$$\times \int_0^\tau e^{-2(\tau-\sigma)|\xi|^2} \{ |\mathscr{F}(u \cdot \nabla u)| + |\hat{f}| \}^2 \, d\sigma \, d\xi$$

$$\leqslant 2\int_{\mathbf{R}^n} \phi(\xi)^2\, |e^{-|\xi|^2\tau}\hat{a}|^2\, d\xi$$

$$+4\tau \int_0^\tau \int_{\mathbf{R}^n} \phi(\xi)^2\, e^{-2(\tau-\sigma)|\xi|^2}\, |\xi|^2\, |\mathscr{F}(u_j u_k)|^2\, d\xi\, d\sigma$$

$$+4\tau \int_0^\tau \int_{\mathbf{R}^n} \phi(\xi)^2\, e^{-2(\tau-\sigma)|\xi|^2}\, |\hat{f}|^2\, d\sigma\, d\xi$$

$$\equiv I_1 + I_2 + I_3.$$

Suppose that $n = 3, 4$. Then we have

$$\begin{aligned} I_1 &\leqslant C \int_{\mathbf{R}^n} e^{-2(\tau+h(\tau))|\xi|^2} |\hat{a}|^2\, d\xi \\ &\leqslant C\, \|e^{(\tau+h(\tau))\Delta} a\|_2^2 \\ &\leqslant C(\tau+h(\tau))^{-n/2}\, \|a\|_1^2, \end{aligned} \tag{4.15}$$

$$\begin{aligned} I_2 &\leqslant C\tau \int_0^\tau \int_{\mathbf{R}^n} e^{-2(\tau-\sigma+h(\tau))|\xi|^2} |\xi|^2\, |\mathscr{F}(u_j u_k)|^2\, d\xi\, d\sigma \\ &\leqslant C\tau \sup_\tau \|\mathscr{F}(u_j u_k)\|_\infty^2 \int_0^\tau \int_{\mathbf{R}^n} e^{-2(\tau-\sigma+h(\tau))|\xi|^2} |\xi|^2\, d\xi\, d\sigma \\ &\leqslant C\tau \sup_\tau \|u\|_2^4 \int_0^\tau (h(\tau)+\tau-\sigma)^{-(n/2)-1}\, d\sigma, \end{aligned} \tag{4.16}$$

$$\begin{aligned} I_3 &\leqslant C\tau \int_0^\tau \int_{\mathbf{R}^n} e^{-2(\tau-\sigma+h(\tau))|\xi|^2} |\hat{f}|^2\, d\xi\, d\sigma \\ &\leqslant C\tau \int_0^\tau \|e^{\Delta(\tau-\sigma+h(\tau))} f\|_2^2\, d\sigma \\ &\leqslant C\tau \int_0^\tau (\tau-\sigma+h(\tau))^{-n/2}\, \|f(\sigma)\|_1^2\, d\sigma. \end{aligned} \tag{4.17}$$

We choose $h(\tau) = (1+\tau)$ and $E(\tau) = (1+\tau)^\alpha$; then from (4.15)–(4.17) we get

$$\begin{aligned} I_1 + I_2 + I_3 &\leqslant C(1+2\tau)^{-n/2}\, \|a\|_1^2 + C\tau(1+\tau)^{-n/2} \sup_\tau \|u\|_2^4 \\ &\quad + C\tau(1+\tau)^{-n/2} \int_0^\tau \|f(\sigma)\|_1^2\, d\sigma. \end{aligned} \tag{4.18}$$

Hence it follows from (4.13), (4.14), and (4.18) that

$$
\begin{aligned}
(1+t)^{\alpha}\,\|\hat{u}(t)\|_2^2 \leqslant C\,\|a\|_2^2 + C\int_0^t (1+\tau)^{\alpha-1}\bigg[\,(1+\tau)^{-n/2}\,\|a\|_1^2 + \tau(1+\tau)^{-n/2} \\
\times\left\{\sup_{\tau}\,\|u\|_2^4 + \int_0^{\tau}\|f(\sigma)\|_1^2\,d\sigma\right\}\bigg]\,d\tau \\
+\int_0^t (1+\tau)^{\alpha}\,|\langle f(\tau), u(\tau)\rangle|\,d\tau \\
\leqslant C\,\|a\|_2^2 + C\,\|a\|_1^2\int_0^t (1+\tau)^{\alpha-1-(n/2)}\,d\tau \\
+C\left\{\sup_{\tau}\,\|u\|_2^4 + \int_0^{\infty}\|f(\sigma)\|_1^2\,d\sigma\right\}\int_0^t (1+\tau)^{\alpha-(n/2)}\,d\tau \\
+C\int_0^t (1+\tau)^{\alpha}\,\|u\|_2^{(1-\lambda)(1-\gamma)}\,\|\rho^{\gamma} f\|_{p'}\,\|\nabla u\|_2^{2/\theta}\,d\tau \\
\leqslant C_0 + C_1\int_0^t (1+\tau)^{\alpha-1-(n/2)}\,d\tau + C_2\int_0^t (1+\tau)^{\alpha-(n/2)}\,d\tau \\
+C_3\mathscr{E}(t)^{(1-\lambda)(1-\gamma)}\left(\int_0^t (1+\tau)^{(\alpha-\varepsilon\eta)\theta'-1}\,d\tau\right)^{1/\theta'} \\
\leqslant C \qquad (4.19)
\end{aligned}
$$

if $\alpha < \min(n/2,\ n/2-1,\ \varepsilon\eta)$. We assume, for ε sufficiently small,

$$\|u(t)\|_2 \leqslant C(1+t)^{-(\varepsilon/2)\eta}. \qquad (4.20)$$

Substituting (4.20) into (4.19), we obtain a better rate of decay:

$$\|u(t)\|_2 \leqslant C(1+t)^{-(\varepsilon/2)\,\eta(1+(\kappa/2))}.$$

Iterating this argument, the desired estimate is obtained.

When $n=2$, we choose $h(\tau)=(1+\tau)(\log(e+\tau))$ and $E(\tau)=(\log(e+\tau))^{\alpha}$ with $\alpha>1$. Then by (4.15)–(4.17), it follows that

$$
\begin{aligned}
\int_{\mathbf{R}^n} |\phi(\xi)\, \mathscr{F}(u(\tau))|^2\, d\xi \leqslant I_1 + I_2 + I_3 \\
&\leqslant C(1+\tau+\tau\log(e+\tau))^{-1}\, \|a\|_1^2 \\
&\quad + C(\log(e+\tau))^{-2} \sup_\tau \|u\|_2^4 \\
&\quad + C(\log(e+\tau))^{-1} \int_0^\infty \|f(\sigma)\|_1^2\, d\sigma. \qquad (4.21)
\end{aligned}
$$

Hence from (4.14) and (4.21) we have

$$
\begin{aligned}
(\log(e+\tau))^\alpha\, \|\hat{u}(t)\|_2^2 &\leqslant \|a\|_2^2 + C \int_0^t (\log(e+\tau))^{a-1}\, (e+\tau)^{-1} \\
&\quad \times \bigg[(1+\tau+\tau\log(e+\tau))^{-1}\, \|a\|_1^2 + (\log(e+\tau))^{-2} \\
&\quad \times \sup_\tau \|u\|_2^4 + (\log(e+\tau))^{-1} \int_0^\tau \|f(\sigma)\|_1^2\, d\sigma \bigg]\, d\tau \\
&\quad + \int_0^t (\log(e+\tau))^\alpha\, |\langle f(\tau), u(\tau)\rangle|\, d\tau \\
&\leqslant \|a\|_2^2 + C\, \|a\|_1^2 \int_0^t \frac{\log(e+\tau)^{\alpha-2}}{(e+\tau)^2}\, d\tau + C \sup_\tau \|u\|_2^4 \\
&\quad \times \int_0^t \frac{\log(e+\tau)^{\alpha-3}}{e+\tau}\, d\tau + C\, \|f\|_{2,1}^2 \int_0^\tau \frac{\log(e+\tau)^{\alpha-2}}{e+\tau}\, d\tau \\
&\quad + C \int_0^\tau (\log(e+\tau))^\alpha\, \|u\|_2^{(1-\lambda)(1-\gamma)}\, \|\rho^\gamma f\|_{p'}\, \|\nabla u\|_2^{2/\theta}\, d\tau \\
&\leqslant C + C(\log(e+t))^{\alpha-1}, \qquad (4.22)
\end{aligned}
$$

if $\alpha \geqslant 1$. Thus,

$$
\|u(t)\|_2 \leqslant C(\log(e+t))^{-1/2}. \qquad (4.23)
$$

To prove the second part of the theorem, we start the inequality (4.2) again (cf. Zhang [28]).

From (2.25) in Proposition 2.6 and the Hölder inequality, it follows that

$$
\begin{aligned}
\int_{\chi(\tau)} |\mathscr{F}(u(t))|\, d\xi \leqslant{} & 2 \int_{\chi(\tau)} |e^{-|\xi|^2 \tau} \hat{a}|^2\, d\xi \\
& + 2 \int_{\chi(\tau)} \left\{ \int_0^\tau e^{-2|\xi|^2(\tau-\sigma)} (|\xi|\, \|\mathscr{F}(u_j u_k)\|_\infty + \|f\|_\infty)\, d\tau \right\}^2 d\xi \\
\leqslant{} & 2\, \|a\|_1^2 \int_{\chi(\tau)} d\xi + 2 \int_{\chi(\tau)} |\xi|^2\, d\xi \left(\int_0^\tau \|u(\sigma)\|_2^2\, d\sigma \right)^2 + 2 \int_{\chi(\tau)} d\xi \\
& \times \left(\int_0^\tau \|f(\sigma)\|_1\, d\sigma \right)^2 \\
\leqslant{} & 2\, |\chi(\tau)| \left\{ \|a\|_1^2 + \left(\int_0^\tau \|f(\sigma)\|_1\, d\sigma \right)^2 \right\} \\
& + 2\tau \int_{\chi(\tau)} |\xi|^2\, d\xi \left(\int_0^\tau \|u(\sigma)\|_2^4\, d\sigma \right).
\end{aligned} \tag{4.24}
$$

Set $E(t) = (1+t)^2$ and $G^2(t) = 2/(1+t)$. Using the logarithmic decay (4.23) and estimate (4.24) and plugging in (4.2), we have

$$
\begin{aligned}
(1+t)^2\, \|u(t)\|_2^2 \leqslant{} & \|a\|_1^2 + C \int_0^t (1+\tau) \left[(1+\tau)^{-1} \left\{ \|a\|_1^2 + \left(\int_0^\tau \|f(\sigma)\|_1\, d\sigma \right)^2 \right\} \right. \\
& \left. + 2\tau (1+\tau)^{-2} \int_0^\tau \|u(\sigma)\|_2^2\, (\log(e+\sigma))^{-1/2}\, d\sigma \right] d\tau \\
& + 2 \int (1+\tau)^2\, |\langle f(\tau), u(\tau) \rangle|\, d\tau \\
\leqslant{} & C_0 + C_1(1+t) \left(\|a\|_1^2 + \left(\int_0^t \|f(\sigma)\|_1\, d\sigma \right)^2 \right. \\
& + C(1+t) \int_0^t \|u(\sigma)\|_2^2\, (\log(e+\sigma))^{-1/2}\, d\sigma \\
& + C\mathscr{E}(t)^{(1-\lambda)(1-\gamma)} \left(\int_0^t (1+\tau)^{(2-\varepsilon\eta)\theta' - 1}\, d\tau \right)^{1/\theta'} \\
& \times \left(\int_0^t \|\nabla u\|_2^2\, d\tau \right)^{1/\theta} \\
\leqslant{} & C_0 + C_1(\|a\|_1^2, \|f\|_{1,1}^2)(1+t) + C_2(1+t) \\
& \times \int_0^t \|u(\sigma)\|_2^2\, (\log(e+\sigma))^{-1/2}\, d\sigma + C_3(1+t)^{2-\varepsilon\eta}.
\end{aligned} \tag{4.25}
$$

Hence we have

$$(1+t)\|u(t)\|_2^2 \leqslant C_0(1+t)^{-1} + C_1 + C_2 \int_0^t \|u(\sigma)\|_2^2 (\log(e+\sigma))^{-1}\, d\sigma$$
$$+ C_3(1+t)^{1-\varepsilon\eta}. \tag{4.26}$$

Applying the Gronwall inequality, (4.26) implies that

$$(1+t)\|u(t)\|_2^2 \leqslant C \log(e+t) + C(\varepsilon) \log(e+t)^{2m} \int_0^t \frac{d\tau}{(e+\tau)^{\varepsilon\eta} \log(e+\tau)^{2m}}$$
$$\leqslant C \log(e+t) + C(\varepsilon)(e+t)^{1-\varepsilon\eta} \log(e+t)^{2m},$$

where m is some integer which depends on the constant C_2 in (4.26). Hence we obtain

$$\|u(t)\|_2 \leqslant C(1+t)^{-\varepsilon\eta/2} \log(e+t)^m. \tag{4.27}$$

Here the constant C depends on the norm of the initial data a and the external force f in the conditions A.2 and A.3(2).

We reiterate by substituting this decay (4.27) into (4.25). As in (4.9), we see that

$$\int_0^t (1+\tau)^2 |\langle f(\tau), u(\tau)\rangle|\, d\tau$$
$$\leqslant \int_0^t (1+\tau)^2 \|\rho^\gamma f\|_{p'} \|u\|_2^\kappa \|\nabla u\|_2^{2/\theta}\, d\tau$$
$$\leqslant \int_0^t (1+\tau)^{2-\beta-\varepsilon\eta\kappa/2} \log(1+\tau)^{m\kappa/2} \|\nabla u\|_2^{2/\theta}\, d\tau$$
$$\leqslant C\left(\int_0^t (1+\tau)^{\theta'(2-\varepsilon\eta(1+\kappa/2))-1} \log(e+\tau)^{m\kappa\theta'/2}\, d\tau\right)^{1/\theta'}$$
$$\leqslant C(1+t)^{2-\varepsilon\eta(1+\kappa/2)} \log(e+t)^{m\kappa/2}. \tag{4.28}$$

Noting that there is a constant C such that $(1+t)^{-\varepsilon\eta/4} \log(1+t)^m \leqslant C$, we have from (4.25), (4.27), and (4.28) that

$$(1+t)\|u(t)\|_2^2 \leqslant C_0(1+t)^{-1} + C_1$$
$$+ C_2 \int_0^t (1+\tau)\|u(\tau)\|_2 (1+\tau)^{-1-\varepsilon\eta/4}\, d\tau$$
$$+ C_3(1+t)^{1-\varepsilon\eta(1+\kappa/2)} \log(1+t)^{m\kappa/2}.$$

We again solve this inequality by using Gronwall's lemma and it follows that

$$\begin{aligned}(1+t)\|u(t)\|_2^2 &\leqslant Ce^{-(C/\sigma)(1+t)^{-\sigma}} + Ce^{-(C/\sigma)(1+t)^{-\sigma}} \\ &\quad \times \int_0^t (1+\tau)^{-\delta} \log(1+\tau)^l\, e^{(C/\sigma)(1+\tau)^{-\sigma}}\, d\tau \\ &\leqslant C + C(1+t)^{1-\varepsilon\eta(1+(\kappa/2))} \log(1+t)^{m(\kappa/2)}, \end{aligned} \tag{4.29}$$

where we have put $\sigma = \varepsilon\eta/4$, $\delta = \varepsilon\eta(1+\kappa/2)$, and $l = m\kappa/2$. Hence

$$\|u(t)\|_2^2 \leqslant C(1+t)^{-\varepsilon\eta(1+(\kappa/2))} \log(1+t)^{m(\kappa/2)}. \tag{4.30}$$

Repeating this procedure l times, we have

$$\|u(t)\|_2^2 \leqslant C(1+t)^{-\varepsilon\eta(1+(\kappa/2)+(\kappa/2)^2+\cdots+(\kappa/2)^l)} \log(1+t)^{m(\kappa/2)^l}.$$

Since $\kappa = (1-\lambda)(1-\gamma) \leqslant 1$, we conclude by taking the limit as $l \to \infty$ that

$$\|u(t)\|_2 \leqslant C(1+t)^{-\varepsilon}. \tag{4.31}$$

This proves the last part of the theorem.

Finally we should remark that the condition A.3 on f can be generalized in the following form:

A.3′. f is expressed as $f = D^\delta g \equiv \mathscr{F}^{-1}|\xi|^\delta \hat{g}$, where $g \in L^v(0, \infty; L^r)$ and the exponents δ, v and r satisfy

$$\frac{2}{v} + n\left(\frac{1}{r} - \frac{1}{2}\right) + \delta > \varepsilon + 1. \tag{4.32}$$

Under the above general condition, the estimate from the term including g in (4.5) or the estimate for the term I_3 in (4.17) goes way as in (4.17)–(4.19). By using Lemma 2.7,

$$\begin{aligned} I_3(t) &\leqslant 4\tau \int_0^\tau \int_{\mathbf{R}^n} e^{-2(\tau-\sigma+h(\tau))|\xi|^2} |\hat{f}|^2\, d\xi\, d\sigma \\ &\leqslant C\tau \int_0^\tau \|e^{\Delta(1+2\tau-\sigma)} |\nabla|^\delta g\|_2^2\, d\sigma \\ &\leqslant C\tau \int_0^\tau (1+2\tau-\sigma)^{-n(1/r-1/2)-\delta}\, \|g\|_r^2\, d\sigma \end{aligned}$$

$$\leqslant C\tau(1+\tau)^{-n(1/r-1/2)-\delta}\left(\int_0^\tau d\sigma\right)^{(v-2)/v}\left(\int_0^\tau \|g\|_r^v\,d\sigma\right)^{1/v}$$

$$\leqslant C(1+\tau)^{2-(2/v)-n(1/r-1/2)-\delta}\left(\int_0^\tau \|g\|_r^v\,d\sigma\right)^{1/v}.$$

Hence the sufficient condition on the exponent is (4.32)

PROPOSITION 4.2. *Suppose that u is a solution to the Navier–Stokes equation with the condition that the external force f satisfies assumption* A.3′. *Then the solution u has the same decay rate as in* (4.31).

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