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DECAY OF NON-OSCILLATING SOLUTIONS TO THE MAGNETO-HYDRODYNAMIC EQUATIONS

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In this paper we study the decay of solutions to the Magneto-Hydrodynamic equations. We show that if the magnetic energy decays to a limit L; that is, if the energy of the magnetic field is non-oscillating, then the energy of the velocity decays to zero.

1 Introduction

In this paper we study the large time behavior of solutions to the Magneto-Hydrodynamic equations in all of $\mathbf{R}^3$-space

$$\begin{aligned} u_t + u \cdot \nabla u - B \cdot \nabla B + \nabla p &= \Delta u \\ B_t + u \cdot \nabla B - B \cdot \nabla u &= 0 \\ \operatorname{div} u = \operatorname{div} B &= 0 \end{aligned} \tag{1}$$

with the initial conditions

$$(u(x,0), B(x,0)) = (u_0(x), B_0(x)) \in X$$

where X will be described below. We show that if the magnetic energy $\int_{\mathbf{R}^3} |B(x,t)|^2\,dx$ tends to L as time goes to infinity then the energy of the velocity $\int_{\mathbf{R}^3} |u(x,t)|^2\,dx$ tends to zero at the same rate. This establishes a conjecture described in Moffat's paper[3]. We will present the proof for smooth solutions. The proof is also valid for weak solutions for which a sequence of smooth approximating solutions can be constructed which converge weakly in L^2. In the second case one applies the proof we give to the approximating solutions and then passes to the limit. The proof is based on the Fourier splitting method[2,4]. Here we use the modified version which permits to treat the integrated equations[1].

We will use the notation

$$\|u\|_2^2 = \int_{\mathbf{R}^3} |u|^2\,dx,$$

$$\|u\|_{H^m}^2 = \sum_{|\alpha| \le m} \int_{\mathbf{R}^3} |D^\alpha u|^2\,dx.$$

The main theorem we establish in this paper is the following

Theorem 1 *Let $(u(x,t), B(x,t))$ be a smooth solution to (1) with data $(u_0, B_0) \in \left(L^2(\mathbf{R}^3) \cap L^1(\mathbf{R}^3) \cap H^2(\mathbf{R}^3)\right)^2$. Suppose that $\|B(\cdot,t)\|_2 \to L$ for $t \to \infty$. Then $\|u(\cdot,t)\|_2 \to 0$ for $t \to \infty$.*

2 The decay

In this section we establish Theorem 1. As was said in the introduction, the main tool is the Fourier splitting method. The proof we present can be used for solutions in $\mathbf{R}^n$, $n \geq 3$.

In what follows we suppose the existence of smooth solutions. Such solutions should exist at least for small enough data.

Proof of Theorem 1. The first step is to derive an energy inequality which involves the energy of the magnetic field, the velocity and the gradient of the velocity. Multiplying the first equation in (1) by u and the second by B, integrating in space and adding both equations yields after some integration by parts:

$$\frac{d}{dt}\int_{\mathbf{R}^3} \left(|u|^2 + |B|^2\right)\, dx = -\int_{\mathbf{R}^3} |\nabla u|^2\, dx.$$

Hence integrating in time over $[s,t]$ we have

$$\begin{aligned}\int_{\mathbf{R}^3} &\left(|u(x,t)|^2 + |B(x,t)|^2\right)\, dx \\ &= -2\int_s^t |\nabla u|^2\, dx ds + \int_{\mathbf{R}^3} \left(|u(x,s)|^2 + |B(x,s)|^2\right)\, dx.\end{aligned} \tag{2}$$

We first need to estimate the Fourier transform of the solution in a neighborhood of the origin in frequency space.

Lemma 2 *Let $u(x,t)$ be the first component of a solution to the MHD equations (1) satisfying the conditions of the Theorem. Then*

$$|\hat{u}(\xi,t)| \leq (1 + \frac{1}{\xi}).$$

Proof. Taking the Fourier transform of the equation we derive an ordinary differential equation which yields for $\hat{u}(\xi,t)$

$$\hat{u}(\xi,t) = \hat{u}_0 e^{-|\xi|^2 t} - \int_0^t \hat{H}(\xi,s) e^{-|\xi|^2(t-s)}\, ds \tag{3}$$

where

$$\hat{H}(\xi,s) = \widehat{u\nabla u}(\xi,s) - \widehat{B\nabla B}(\xi,s) + \widehat{\nabla p}(\xi,s).$$

solution to (1) with data $(u_0, B_0) \in$
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$(\xi, s) + \widehat{\nabla p}(\xi, s).$

Noting that the pressure p satisfies the elliptic equation

$$\Delta p = -\sum \frac{\partial^2}{\partial x_i \partial x_j}(u_i u_j - B_i B_j),$$

it follows that

$$\widehat{\nabla_k p} = \xi_k \hat{p} = i \sum_{i,j} \xi_k \xi_i \xi_j (\widehat{u_i u_j} - \widehat{B_i B_j})$$

and hence

$$|\widehat{\nabla_k p}| \le C|\xi|(|\widehat{u_i u_j}| + |\widehat{B_i B_j}|).$$

Hence

$$\begin{aligned} |\hat{H}| &\le C_0 |\xi| (|\widehat{u_i u_j}| + |\widehat{B_i B_j}|) \\ &\le C_0 |\xi| (\|u_0\|_2^2 + \|B\|_2^2) \end{aligned}$$

so that

$$\begin{aligned} |\hat{u}(\xi,t)| &\le C + C_0 \int_0^t |\xi| (\|u\|_2^2 + \|B\|_2^2) e^{-|\xi|^2(t-s)} \, ds \\ &\le C + \frac{C_*}{|\xi|}. \end{aligned} \tag{4}$$

Here we used that $u_0 \in L^1$ and hence $\hat{u}_0 \in L^\infty$. □

Returning to the proof of Theorem 1, we use the Fourier splitting method with the modification introduced by Wiegner [1].

Let $v(s) = \|u(s)\|_2^2 + \|B(s)\|_2^2$, then (2) reads

$$v(t) - v(s) \le \int_s^t \int_{\mathbf{R}^3} |\nabla u|^2 \, dx ds.$$

Let $S(t) = \{\xi \,:\, |\xi| \le \left[\frac{g(t)}{2}\right]^{1/2}\}$, where $g(t)$ will be specified below. Hence, by the Fourier splitting method, we have

$$\begin{aligned} &v(t) - v(s) + \int_s^t g(r)^2 \|u(r)\|_2^2 \, dr \\ &\le \int_s^t g(r)^2 \int_{S(r)} |\hat{u}(\xi,r)|^2 \, d\xi dr \le \int_s^t g(r)^2 [C(g(r)^{n/2} + g^{n/2-2})] \, dr \end{aligned} \tag{5}$$

where we need (3) for the last inequality. Now following Wiegner let

$$e(t) = e^{\int_0^t g(r)^2\, dr},$$

hence

$$e(t) - e(t-h) = e(t-h)\int_{t-h}^t g(r)^2\, dr + h\epsilon(h) \tag{6}$$

where $\epsilon(h) \to 0$ as $h \to 0$. Now write

$$\begin{aligned}&e(t)(v(t)-L) - e(t-h)(v(t-h)-L) = \\ &\quad (e(t)-e(t-h))(v(t)-L) + e(t-h)((v(t-h)-L) - (v(t+h)-L))\end{aligned}$$

so that by (6)

$$\begin{aligned}&e(t)(v(t)-L) - e(t-h)(v(t-h)-L) = \\ &e(t-h)\int_{t-h}^t g(r)^2\, dr(v(t)-L) + e(t-h)(v(t)-v(t-h))h\epsilon(h)(v(t)-L) = \\ &e(t-h)\left[\int_{t-h}^t g(r)^2[v(t)-v(r)]\, dr + v(t) - v(t-h)\right. \\ &\left.+\int_{t-h}^t g(r)^2(v(r)-L)\, dr\right] + h\epsilon(h)(v(t)-L).\end{aligned}$$

Writing $v(r) - L = \|u(r)\|_2^2 + (\|B(r)\|_2^2 - L)$, we get

$$\begin{aligned}&e(t)(v(t)-L) - e(t-h)(v(t-h)-L) = \\ &e(t-h)\left[\int_{t-h}^t g(r)^2[v(t)-v(r)]\, dr + v(t) - v(t-h)\right. \\ &\left.+\int_{t-h}^t g(r)^2\|u(r)\|_2^2\, dr\right] + e(t-h)\int_{t-h}^t g(r)^2(\|B(r)\|_2^2 - L)\, dr \\ &+h\epsilon(h)(v(t)-L).\end{aligned} \tag{7}$$

Let $g(t)^2 = \alpha(t+1)^{-1}$, with α sufficiently large. Then

$$e(t) = e^{\alpha\int_0^t \frac{dr}{r+1}} = (t+1)^\alpha.$$

Let T_0 be such that for $t \geq T_0$ we have $\left|\|B(t)\|_2^2 - L\right| < \epsilon$. From (5) and (7) it follows that, for $t \geq T_0$,

$$e(t)[v(t)-L] - e(t-h)[v(t-h)-L]$$

$$\leq e(t-h)\int_{t-h}^{t} g(r)^2[v(t)-v(r)]\,dr + C\int_{t-h}^{t} e(r)(r+1)^{-n/2}\,dr$$
$$+\,\epsilon e(t-h)\int_{t-h}^{t} g(r)^2\,dr + h\epsilon(h)(v(t)-L). \tag{8}$$

Note that

$$|v(t)-v(r)| < O(h) \quad \text{if} \quad |t-r| < h$$

since, by (5),

$$|v(t)-v(t-h)| \leq \int_{t-h}^{t} (r+1)^{-n/2} \leq Ch.$$

Hence summing (8) over intervals of length h it follows that for $t \geq T$

$$e(t)[v(t)-L] - e(T)[v(T)-L]$$
$$\leq O(h)\int_T^t e(r)g(r)^2\,dr + \int_T^t e(r)(r+1)^{-n/2}\,dr + \epsilon\int_T^t g(r)^2 e(r)\,dr$$
$$+\,\epsilon(h)[v(0)-L]$$

Let $h \to 0$, then

$$e(t)[v(t)-L] \leq e(T)[v(T)-L] + C(t+1)^{\alpha-n/2+1} + \epsilon C_0 e(t). \tag{9}$$

Here we used that

$$\int_T^t g(r)^2 e(r)\,dr = C_0\int_T^t (r+1)^{\alpha-1}\,dr \leq C(t+1)^{\alpha}.$$

Dividing by $e(t)$ yields

$$\int |u(t)|^2\,dx + \int |B(t)|^2\,dx - L \leq \frac{e(T)}{e(t)}(v(0)+L) + C(t+1)^{-n/2+1} + \epsilon C_0$$

concluding the proof of Theorem 1.

Acknowledgements

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