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On Possible Singular Solutions to the Navier–Stokes Equations

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Abstract. In this paper, we exclude the possibility of existence of a singular solution of the selfsimilar type proposed by Jean Leray. More precisely, using a slightly stronger hypothesis we give a simpler proof to the analogous result established by J. Nečas, M. Růžička and V. Šverák. We also discuss the possible existence of a singular solution of pseudo-selfsimilar type.

1. Introduction

In their recent paper, Nečas, Růžička and Šverák [5] excluded the possibility of existence of non-zero singular solutions to the Navier–Stokes equations in the form

$$
\begin{aligned}
\mathbf{u}(t,\mathbf{x}) &= \tfrac{1}{\sqrt{2a(T-t)}}\mathbf{U}\left(\tfrac{\mathbf{x}}{\sqrt{2a(T-t)}}\right) \\
p(t,\mathbf{x}) &= \tfrac{1}{2a(T-t)}P\left(\tfrac{\mathbf{x}}{\sqrt{2a(T-t)}}\right)
\end{aligned}
\qquad a>0\,, \tag{1.1}
$$

for $t \in (-\infty, T)$ and $\mathbf{x} \in \mathbb{R}^3$. Solutions of this type would blow up at the time $T \in \mathbb{R}$. The construction of a singular solution of the type (1.1) was proposed originally by Leray [4]. Since that time it is known that if the system

$$
\begin{aligned}
\operatorname{div}\mathbf{U} &= 0 \\
a\mathbf{U} + ay_k\frac{\partial \mathbf{U}}{\partial y_k} - \nu\Delta\mathbf{U} &+ U_j\frac{\partial \mathbf{U}}{\partial y_j} + \nabla P = \mathbf{0}
\end{aligned} \tag{1.2}
$$

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had a non-trivial solution $\mathbf{U} = (U_1, U_2, U_3)$ (in the class of weak solutions) then $\mathbf{u} = (u_1, u_2, u_3)$ of the form $(1.1)_1$, extended by zero for $t > T$ would be a weak solution[1] to the Navier–Stokes equations

$$
\begin{aligned}
\operatorname{div} \mathbf{u} &= 0 \\
\frac{\partial \mathbf{u}}{\partial t} + u_j \frac{\partial \mathbf{u}}{\partial x_j} - \nu \Delta \mathbf{u} &+ \nabla p = \mathbf{0}\,,
\end{aligned}
\tag{1.3}
$$

for which

$$\lim_{t \to T^-} \|\nabla \mathbf{u}\|_2 = \infty\,.$$

In other words, if $\mathbf{U}$ had been a non-zero solution to (1.2), then $\mathbf{u}$ of the form $(1.1)_1$ would be a counter-example to the possible global existence of the strong (and consequently smooth) solution of (1.3).

Nečas, Růžička and Šverák [5] proved that under the assumption

$$\mathbf{U} \in \mathbf{L}^3(\mathrm{I\!R}^3) \cap \mathbf{W}^{1,2}_{\mathrm{loc}}(\mathrm{I\!R}^3)\,, \tag{1.4}$$

(1.2) has only the trivial solution.

The objective of this paper is twofold. First, we will present an alternative proof of $\mathbf{U}$ being identically zero under the stronger but natural[2] condition

$$\mathbf{U} \in \mathbf{W}^{1,2}(\mathrm{I\!R}^3)\,. \tag{1.5}$$

More precisely, condition (1.5) allows us to present a simpler (and detailed) proof here. The proof consists of three parts: in Section 2 it is shown that (1.5) implies (even in more general situations) $\mathbf{U} \in \mathbf{W}^{2,2}(\mathrm{I\!R}^3) \cap \mathbf{L}^\infty(\mathrm{I\!R}^3)$ and some further regularity properties for P and $\mathbf{U}$. In Section 3 a maximum principle for the quantity $\frac{|\mathbf{U}|^2}{2} + P + aU_i y_i$ is derived. This is the crucial step to prove that $\mathbf{U} \equiv \mathbf{0}$ as shown finally in Section 4.

The second aim of the paper is to discuss the possible existence of singular weak solutions in the more general form

$$
\begin{aligned}
\mathbf{u}(t, \mathbf{x}) &= \mu(t) \mathbf{U}(\lambda(t) \mathbf{x})\,, \\
p(t, \mathbf{x}) &= \mu^2(t) P(\lambda(t) \mathbf{x})\,.
\end{aligned}
\tag{1.6}
$$

In fact, if instead of $(1.6)_2$ it is assumed that $p(t, \mathbf{x})$ has the form

$$p(t, \mathbf{x}) = \kappa(t) P(\lambda(t) \mathbf{x}) \tag{1.7}$$

it is easy to observe that κ coincides with μ^2 up to a constant multiplier.

Hereafter, the solutions of the type (1.6) are called *pseudo-selfsimilar* solutions to (1.3). This is motivated by the fact that if $\lambda = \mu$ then (1.6) reduces to the selfsimilar solution (1.1), as will be clarified in Section 5.

Assuming μ, $\lambda \in C^1(-\infty, T)$, we will also show in Section 5 that the following possibilities can happen:

[1] For the both systems, the functions p and P are eliminated by divergence–free test functions.

[2] Condition (1.5) holds provided for example that $\mathbf{u} \in L^2(I; \mathbf{W}^{1,2}(\mathrm{I\!R}^3))$, which is the usual space of functions where weak solutions to (1.3) exist.

1. $\mathbf{u}$ has the Leray form $(1.1)_1$ (and as mentioned above, $\mathbf{U}$ must be identically zero);

2. $\mathbf{U} \equiv \mathbf{0}$ due to some trivial arguments;

3. $\mathbf{u}$ is a non-singular selfsimilar solution. The solutions of this form have been recently studied by Cannone, Meyer, Planchon and Barraza, see CANNONE [2] for details and further references;

4. The Fourier transform of $\mathbf{U}$ has a special form in spherical coordinates.

Before ending this introductory part, we recall the notations of some function spaces. The symbol $\mathcal{D}(\mathbb{R}^3)$ is used to denote the space of smooth vector-valued functions having compact support in $\mathbb{R}^3$. Let $p > 1$ and $k \in \mathbb{N}$, then $(L^p(\Omega), \|\cdot\|_p)$ denotes the Lebesgue spaces while $(W^{k,p}(\Omega), \|\cdot\|_{k,p})$ denotes the Sobolev spaces of scalar functions. We write vectors, vector-valued functions and their corresponding function spaces in boldface in order to avoid confusion.

The space of divergence-free vector functions belonging to $\mathbf{W}^{1,2}(\mathbb{R}^3)$ is denoted by $\mathbf{V}$, while $\mathbf{V}^*$ denotes its dual space.

As seen above we use also the summation convention.

2. Regularity results for the Leray system

The aim of the present section is to improve the regularity conditions for $\mathbf{U}$ and P under the assumptions that $\mathbf{U} \in \mathbf{W}^{1,2}(\mathbb{R}^3)$ and $(\mathbf{U}, P)$ is a weak solution to (1.2). Let us note that we will only prove the regularity needed later on. However, both quantities belong to $\mathbf{W}^{k,2}(\mathbb{R}^3)$ for all $k \in \mathbb{N}$.

The idea of the proof is very simple: using the transformation (1.1) we can easily observe that

$$\mathbf{u} \in C([t_1, t_2]; \mathbf{W}^{1,2}(\mathbb{R}^3)) \tag{2.1}$$

for all t_1, t_2 such that $-\infty < t_1 < t_2 < T$. It is well known that (2.1) is certainly a sufficient condition to get full regularity of the solution of (1.3) on (t_1, t_2). Using this and (1.1) (now, reading the properties of $\mathbf{U}$ from $\mathbf{u}$) we get easily the expected regularity for $\mathbf{U}$ and consequently for P.

Let us remark that the proof does not change if (1.1) is replaced by (1.6) assuming that μ, $\lambda \in C(-\infty, T)$. Then, again, $\mathbf{U} \in \mathbf{W}^{1,2}(\mathbb{R}^3)$ implies (2.1).

Let us finally recall that a function $\mathbf{u} \in L^\infty(t_1, T; \mathbf{L}^2(\mathbb{R}^3)) \cap L^2(t_1, T; \mathbf{V})$ with $\frac{\partial \mathbf{u}}{\partial t} \in L^{4/3}(t_1, T; \mathbf{V}^*)$ is a weak solution to (1.3) if

$$\langle \frac{\partial \mathbf{u}(t)}{\partial t}, \boldsymbol{\varphi} \rangle_{V^*,V} + \int_{\mathbb{R}^3} u_k(t) \frac{\partial \mathbf{u}(t)}{\partial x_k} \cdot \boldsymbol{\varphi}\, dx + \nu(\nabla \mathbf{u}(t), \nabla \boldsymbol{\varphi}) = 0 \tag{2.2}$$

holds for all $\boldsymbol{\varphi} \in \mathcal{D}(\mathbb{R}^3)$ satisfying $\operatorname{div} \boldsymbol{\varphi} = 0$, and a.a. $t \in (t_1, T)$.

If (2.1) holds, it is easy to check that $\frac{\partial \mathbf{u}}{\partial t} \in L^2(t_1, T; \mathbf{V}^*)$. Thus, (2.2) is valid for all $\boldsymbol{\varphi} \in \mathbf{V}$ and we can use $\mathbf{u}(t)$ as a test function in (2.2).

We are ready to prove the following statement.

Lemma 2.1. *Assume that* $\mathbf{U} \in \mathbf{W}^{1,2}(\mathbb{R}^3)$ *is a weak solution to (1.2). Let* $\mathbf{u}$ *be given either by (1.1) or by (1.6), where we assume* λ, $\mu \in C^1(-\infty, T)$. *Then*

$$\mathbf{U} \in \mathbf{W}^{2,2}(\mathbb{R}^3) \cap \mathbf{L}^\infty(\mathbb{R}^3) \quad \text{and} \quad P \in W^{1,2}(\mathbb{R}^3)\,. \tag{2.3}$$

Proof. Let us take t_1, t_2 such that $-\infty < t_1 < t_2 < T$ and set $I \equiv (t_1, t_2)$. By the assumptions on λ and μ we see that

$$\mathbf{u} \in C(\overline{I}; \mathbf{W}^{1,2}(\mathbb{R}^3))\,. \tag{2.4}$$

Let us denote (for a $\mathbf{z}(t) \in \mathbf{W}^{1,2}(\mathbb{R}^3)$)

$$\Delta_r^h \mathbf{z}(t) \equiv \frac{\mathbf{z}(t, \mathbf{x} + h\boldsymbol{e}^r) - \mathbf{z}(t,\mathbf{x})}{h}\,, \quad r = 1,2,3,$$

where $\boldsymbol{e}^r$, $r = 1,2,3$, are unit vectors $(1,0,0)$, $(0,1,0)$, $(0,0,1)$. From the weak formulation (2.2) we get

$$\begin{aligned} &\langle \frac{\partial}{\partial t} \Delta_r^h \mathbf{u}(t), \boldsymbol{\varphi} \rangle_{V^*,V} + \nu(\Delta_r^h \nabla \mathbf{u}(t), \nabla \boldsymbol{\varphi}) \\ &+ \frac{1}{h} \int_{\mathbb{R}^3} \left[(u_k \frac{\partial u_i}{\partial x_k})(t, \mathbf{x} + h\boldsymbol{e}^r) - (u_k \frac{\partial u_i}{\partial x_k})(t, \mathbf{x}) \right] \varphi_i(\mathbf{x}) d\mathbf{x} = 0\,. \end{aligned} \tag{2.5}$$

Taking $\Delta_r^h \mathbf{u}(t)$ in (2.5) instead of $\boldsymbol{\varphi}$, we obtain (for simplicity we write $\Delta_r^h \mathbf{u}$ instead of $\Delta_r^h \mathbf{u}(t)$)

$$\frac{1}{2} \frac{d}{dt} \|\Delta_r^h \mathbf{u}\|_2^2 + \nu \|\Delta_r^h \nabla \mathbf{u}\|_2^2 = - \int_{\mathbb{R}^3} \Delta_r^h u_k \frac{\partial u_i(t, \mathbf{x} + h\boldsymbol{e}^r)}{\partial x_k} \Delta_r^h u_i d\mathbf{x} \equiv Y\,,$$

where we used the fact that $\int_{\mathbb{R}^3} u_k \frac{\partial(\Delta_r^h u_i)}{\partial x_k} \Delta_r^h u_i d\mathbf{x} = \frac{1}{2} \int_{\mathbb{R}^3} u_k \frac{\partial |\Delta_r^h \mathbf{u}|^2}{\partial x_k} = 0$, which can be checked after integration by parts since $\mathbf{u}$ satisfies (2.4) and $C(\overline{I}; \mathcal{D}(\mathbb{R}^3))$ is dense in $C(\overline{I}; \mathbf{W}^{1,2}(\mathbb{R}^3))$.

Further, using the interpolation inequality $\|z\|_4 \le \|z\|_2^{\frac{1}{4}} \|z\|_6^{\frac{3}{4}}$, the continuous imbedding of $\mathbf{W}^{1,2}(\mathbb{R}^3)$ into $\mathbf{L}^6(\mathbb{R}^3)$, and the Young inequality, we obtain

$$\begin{aligned} |Y| &\le \|\Delta_r^h \mathbf{u}\|_4^2 \|\nabla \mathbf{u}\|_2 \le \|\Delta_r^h \mathbf{u}\|_2^{\frac{1}{2}} \|\nabla \mathbf{u}\|_2 \|\Delta_r^h \mathbf{u}\|_6^{\frac{3}{2}} \\ &\le c \|\nabla \Delta_r^h \mathbf{u}\|_2^{\frac{3}{2}} \|\Delta_r^h \mathbf{u}\|_2^{\frac{1}{2}} \|\nabla \mathbf{u}\|_2 \le \frac{\nu}{2} \|\Delta_r^h \nabla \mathbf{u}\|_2^2 + c \|\nabla \mathbf{u}\|_2^4 \|\Delta_r^h \mathbf{u}\|_2^2\,. \end{aligned}$$

Hence

$$\frac{d}{dt} \|\Delta_r^h \mathbf{u}\|_2^2 + \nu \|\Delta_r^h \nabla \mathbf{u}\|_2^2 \le c \|\nabla \mathbf{u}\|_2^4 \|\Delta_r^h \mathbf{u}\|_2^2\,.$$

Since $\|\nabla \mathbf{u}\|_2^4 \in L^1(I)$ due to (2.4), we obtain by the Gronwall inequality

$$\mathbf{u} \in L^2(I; \mathbf{W}^{2,2}(\mathbb{R}^3))\,. \tag{2.6}$$

Analogously, applying the difference quotient method with respect to t, we get from (2.2) (formally by testing by $\frac{\partial \mathbf{u}}{\partial t}$)

$$\frac{\partial \mathbf{u}}{\partial t} \in L^2(I; \mathbf{L}^2(\mathbb{R}^3)) . \tag{2.7}$$

Hence, using (1.3), (2.6) and (2.7) we have

$$\nabla p \in L^2(I; \mathbf{L}^2(\mathbb{R}^3)) . \tag{2.8}$$

Taking an arbitrary $t \in I$ and using the formulas (1.1) respectively (1.6) (in fact we express $\mathbf{U}$ and P by means of $\mathbf{u}$ and p) we can easily conclude that (2.6) and (2.8) imply $\mathbf{U} \in \mathbf{W}^{2,2}(\mathbb{R}^3)$ and $P \in W^{1,2}(\mathbb{R}^3)$. Applying finally the Agmon inequality (Agmon [1]) $\|\mathbf{U}\|_\infty \le \|\mathbf{U}\|_{1,2}^{1/2} \|\mathbf{U}\|_{2,2}^{1/2}$ we see that (2.3) is valid. The proof of Lemma 2.1 is complete. □

Next, using the Riesz transformation, some more regularity properties for the self-similar pressure P are obtained.

Lemma 2.2. *Let $\mathbf{U}$ and P be solutions to (1.2) satisfying (2.3). Then*

$$\Delta P = -\frac{\partial U_j}{\partial y_i} \frac{\partial U_i}{\partial y_j} , \tag{2.9}$$

$$P \in W^{2,2}(\mathbb{R}^3) \cap L^q(\mathbb{R}^3) \ \textit{for all} \ q \in (1, \infty] ; \tag{2.10}$$

and denoting $r = |\mathbf{y}|$ we have

$$\int_{\mathbb{R}^3} r^2 \left| \frac{\partial \mathbf{U}}{\partial r} \right|^2 d\mathbf{y} < \infty . \tag{2.11}$$

Proof. Taking the divergence of (1.2), equation (2.9) follows immediately thanks to the regularity of $\mathbf{U}$. Hence

$$P = R_i R_j (U_i U_j) ,$$

where R_j is the Riesz transformation. (Let us recall that the Riesz transformation R_j is the singular integral operator given by the Fourier multiplier $\frac{-i\xi_j}{|\boldsymbol{\xi}|}$, see [6] for details.) As $\mathbf{U} \in \mathbf{L}^2(\mathbb{R}^3) \cap \mathbf{L}^\infty(\mathbb{R}^3)$ we obtain

$$P \in L^q(\mathbb{R}^3) \quad \text{for all} \ \ q \in (1, \infty)$$

due to the Marcinkiewicz multiplier theorem (see e.g. Stein [6]). ¿From (2.9) and (2.3), $\Delta P \in L^2(\mathbb{R}^3)$ and therefore by means of the Agmon inequality above we get $P \in L^\infty(\mathbb{R}^3)$. Finally, as $y_k \frac{\partial \mathbf{U}}{\partial y_k} = r \frac{\partial \mathbf{U}}{\partial r}$ and all the other terms in (1.2) are square integrable, we get (2.11). □

3. Maximum principle

The objective of this section is to show that the quantity[3]

$$X \equiv \frac{|\mathbf{U}|^2}{2} + P + aU_i y_i \tag{3.1}$$

is (i) either a positive constant on $\mathbb{R}^3$;
(ii) or a nonpositive function on $\mathbb{R}^3$.

The main tool is the maximum principle for X. The following observation plays an important role.

Lemma 3.1. *The quantity X defined by (3.1) satisfies*

$$U_j \frac{\partial X}{\partial y_j} - \nu \Delta X + a y_k \frac{\partial X}{\partial y_k} + \nu \left(|\nabla \mathbf{U}|^2 - \frac{\partial U_i}{\partial y_j} \frac{\partial U_j}{\partial y_i} \right) = 0 \, . \tag{3.2}$$

Proof. Equation (3.2) is obtained by the sum of three identities. The first one is obtained by multiplication of $(1.2)_2$ scalarly by $\mathbf{U}$, the second one is derived by the multiplication of $(1.2)_2$ by $a\mathbf{y}$ and the third identity is the pressure equation (2.9) multiplied by ν. □

Lemma 3.2. *There are two possibilities: either $X \leq 0$ on $\mathbb{R}^3$ or X is a positive constant.*

Proof. Let us set

$$X_\beta(\mathbf{y}) \equiv X(\mathbf{y}) \mathrm{e}^{-\beta |\mathbf{y}|^2} \, , \beta > 0 \, . \tag{3.3}$$

Multiplying (3.2) by $\mathrm{e}^{-\beta|\mathbf{y}|^2}$ we get the following equation for X_β:

$$\begin{aligned} -\nu \Delta X_\beta \quad &+ \quad 2(a|\mathbf{y}|^2 - 2\beta\nu|\mathbf{y}|^2 + U_j y_j - 3\nu)\beta X_\beta + U_j \frac{\partial X_\beta}{\partial y_j} \\ &+ \quad (a - 4\beta\nu) y_j \frac{\partial X_\beta}{\partial y_j} + \nu \left(|\nabla \mathbf{U}|^2 - \frac{\partial U_i}{\partial y_j} \frac{\partial U_j}{\partial y_i} \right) \mathrm{e}^{-\beta|\mathbf{y}|^2} = 0 \, . \end{aligned} \tag{3.4}$$

For $R > 0$ large enough there exists a β_0 such that

$$(a - 2\beta\nu)|\mathbf{y}|^2 + U_j y_j - 3\nu > 0 \tag{3.5}$$

for all $\beta \in (0, \beta_0)$ and $|\mathbf{y}| \geq R$. The last term in (3.4) is non-negative, thus, by (3.5) it follows

$$- \nu \Delta X_\beta + b_j(\mathbf{y}) \frac{\partial X_\beta}{\partial y_j} + b(\mathbf{y}) X_\beta \leq 0 \, , \tag{3.6}$$

where $b_j(\mathbf{y}) = U_j(\mathbf{y}) + (a - 4\beta\nu) y_j$ and $b(\mathbf{y}) = 2\beta(a|\mathbf{y}|^2 - 2\beta\nu|\mathbf{y}|^2 + U_j y_j - 3\nu) > 0$ for $|\mathbf{y}| \geq R$ and $\beta \in (0, \beta_0)$.

[3] Recall that a is a positive constant occuring in (1.1) and (1.2).

Let R and β be such that (3.5) holds. Let us further denote $M \equiv \max_{|\mathbf{y}|=R} X$. Then two possibilities can happen: either M is positive or M is not positive. Assume first $M > 0$. Then there is an $R_\beta > R$ such that

$$X_\beta < \frac{M}{2} \quad \text{for } |\mathbf{y}| = R_\beta \quad (|X(\mathbf{y})| \leq c_1|\mathbf{y}| + c_2)\,. \tag{3.7}$$

We can apply the maximum principle to the inequality (3.6) on $B_{R_\beta} \setminus B_R$. As (3.5) holds and $-\nu < 0$, all the signs are correct (see Gilbarg, Trudinger [3]) and then for all $\varrho \in [R, R_\beta]$ it follows that

$$\max_{|\mathbf{y}|=\varrho} X_\beta \leq M\, e^{-\beta R^2}\,. \tag{3.8}$$

Fix some arbitrary ϱ. Letting $\beta \to 0^+$ we observe that (R_β can tend to infinity) $\max_{|\mathbf{y}|=\varrho} X \leq M$. Hence

$$\max_{|\mathbf{y}|=\varrho} X \leq M = \max_{|\mathbf{y}|=R} X \leq \max_{|\mathbf{y}|\leq R} X\,. \tag{3.9}$$

On the other hand, using the maximum principle for (3.2) (rewritten to a similar inequality as (3.6)) on B_ϱ, we see that

$$\max_{|\mathbf{y}|\leq R} X \leq \max_{|\mathbf{y}|=\varrho} X\,. \tag{3.10}$$

Thus, (3.9) and (3.10) yield

$$\max_{|\mathbf{y}|=\varrho} X = M \quad \text{for all } \varrho \geq R\,. \tag{3.11}$$

However (3.9) implies that the maximum is attained inside of B_ϱ. The stronger version of the maximum principle (see Gilbarg, Trudinger [3]) then gives

$$X \equiv M(\text{const.}) > 0 \text{ on } \mathbb{R}^3\,, \tag{3.12}$$

which verifies the first statement of the lemma.

Secondly, if $M \leq 0$ then clearly

$$\sup_{|\mathbf{y}|\geq R} X_\beta \leq 0\,. \tag{3.13}$$

Indeed, for a given $\varepsilon > 0$, let us find an $\tilde{R} > R$ such that $\sup_{|\mathbf{y}|\geq \tilde{R}} X_\beta < \varepsilon$. (such $\tilde{R}$ certainly exists since $|X(\mathbf{y})| \leq c_1|\mathbf{y}| + c_2$). Then applying the maximum principle on $B_{\tilde{R}} \setminus B_R$ for X_β we get $\sup_{\mathbf{y}\in B_{\tilde{R}}\setminus B_R} X_\beta < \max(M, \varepsilon) = \varepsilon$. Since $\varepsilon > 0$ is arbitrary, (3.13) follows.

Inequality (3.13) immediately yields

$$\sup_{|\mathbf{y}|\geq R} X \leq 0\,. \tag{3.14}$$

Applying again the maximum principle to the equation (3.2) on B_R we obtain

$$\max_{|\mathbf{y}|\leq R} X \leq \max_{|\mathbf{y}|=R} X = M \leq 0\,. \tag{3.15}$$

Therefore $X \leq 0$ on $\mathbb{R}^3$ and the proof of Lemma 3.2 is complete. □

4. Main Theorem

Theorem 4.1. *Let* $\mathbf{U}$, P *be a weak solution to the Leray system (1.2). Then* $\mathbf{U} \equiv \mathbf{0}$.

Proof. The proof uses the results of the previous section. Consider X defined in (3.1). By Lemma 3.3 X can be either a positive constant or a nonpositive function. If $X = \ const.$, then (3.2) gives

$$|\nabla \mathbf{U}|^2 - \frac{\partial U_i}{\partial y_j}\frac{\partial U_j}{\partial y_i} = 0\,. \tag{4.1}$$

Integrating (4.1) over $\mathbb{R}^3$ and using the integration by parts justified below we obtain

$$\int_{\mathbb{R}^3} \frac{\partial U_i}{\partial y_j}\frac{\partial U_j}{\partial y_i} d\mathbf{y} = -\int_{\mathbb{R}^3} \frac{\partial^2 U_i}{\partial y_j \partial y_i} U_j \, d\mathbf{y}\,. \tag{4.2}$$

Hence

$$\int_{\mathbb{R}^3} |\nabla \mathbf{U}|^2 d\mathbf{y} = -\int_{\mathbb{R}^3} \frac{\partial}{\partial y_j}(\operatorname{div}\mathbf{U}) U_j \, d\mathbf{y} = 0\,. \tag{4.3}$$

This implies $\mathbf{U} \equiv const.$ Since $\mathbf{U} \in \mathbf{L}^2(\mathbb{R}^3)$ it follows that $\mathbf{U} \equiv \mathbf{0}$. To verify (4.2) one can argue for example by the density argument as follows: Let $\mathbf{U}^n \in \mathcal{D}(\mathbb{R}^3)$ be such that $\|\mathbf{U}^n - \mathbf{U}\|_{2,2} \to 0$ as $n \to \infty$. Clearly (4.2) is valid for all $\mathbf{U}^n$, $n \in \mathbb{N}$. Letting $n \to \infty$ we easily check (4.2) for $\mathbf{U}$.

It remains to show that $\mathbf{U} \equiv \mathbf{0}$ provided that $X \leq 0$.

For the readers convenience, we present first a formal proof: Multiplying $(1.2)_2$ scalarly by $\mathbf{y}$ and integrating over $\mathbb{R}^3$ yields

$$\begin{aligned} a\int_{\mathbb{R}^3} U_i y_i \, d\mathbf{y} \quad + \quad & a\int_{\mathbb{R}^3} y_k \frac{\partial U_i}{\partial y_k} y_i \, d\mathbf{y} + \int_{\mathbb{R}^3} U_k \frac{\partial U_i}{\partial y_k} y_i \, d\mathbf{y} \\ & -\nu \int_{\mathbb{R}^3} y_i \Delta U_i \, d\mathbf{y} + \int_{\mathbb{R}^3} y_i \frac{\partial P}{\partial y_i} d\mathbf{y} = 0\,. \end{aligned}$$

Now, we use formally the integration by parts, i.e. we assume that all "boundary terms" vanish. Then one obtains the equality

$$-3a\int_{\mathbb{R}^3} U_i y_i \, d\mathbf{y} - 3\int_{\mathbb{R}^3} P \, d\mathbf{y} - \int_{\mathbb{R}^3} |\mathbf{U}|^2 \, d\mathbf{y} = 0\,,$$

which can be rewritten as

$$-3\int_{\mathbb{R}^3} X \, d\mathbf{y} + \frac{1}{2}\int_{\mathbb{R}^3} |\mathbf{U}|^2 \, d\mathbf{y} = 0\,.$$

Since $X \leq 0$ on $\mathbb{R}^3$, we see that $\mathbf{U}$ must be identically equal to zero.

In order to justify these formal arguments rigorously we multiply $(1.2)_2$ scalarly by

$$\frac{\mathbf{y}}{(1+\varepsilon r)^\alpha} \qquad \text{where } \ \varepsilon > 0 \ \text{ and } r = |\mathbf{y}|\,.$$

¿From the regularity result proved in Section 2, all terms in $(1.2)_2$ belong to $\mathbf{L}^2(\mathbb{R}^3)$. Evidently, $y_k \frac{\partial \mathbf{U}}{\partial y_k} = r \frac{\partial \mathbf{U}}{\partial r}$. If $\mathbf{z} \in \mathbf{L}^2(\mathbb{R}^3)$ then

$$\int_{\mathbb{R}^3} \frac{z_i y_i}{(1+\varepsilon r)^\alpha} d\mathbf{y} \le \|\mathbf{z}\|_2 \left(\int_{\mathbb{R}^3} \frac{r^2}{(1+\varepsilon r)^{2\alpha}} d\mathbf{y} \right)^{1/2} < \infty$$

provided that $\alpha > \frac{5}{2}$.

Consider therefore $\alpha \in (\frac{5}{2}, 3)$; the choice of the upper bound will be clear later on. Multiplying $(1.2)_2$ by $\frac{\mathbf{y}}{(1+\varepsilon r)^\alpha}$, integrating over $\mathbb{R}^3$ and using the density argument (as explained above), we obtain

$$\int_{\mathbb{R}^3} \frac{\partial U_i}{\partial y_k} \frac{U_k y_i}{(1+\varepsilon r)^\alpha} d\mathbf{y} = -\int_{\mathbb{R}^3} \frac{|\mathbf{U}|^2}{(1+\varepsilon r)^\alpha} d\mathbf{y} + \int_{\mathbb{R}^3} \frac{\alpha\varepsilon (U_i y_i)^2}{r(1+\varepsilon r)^{\alpha+1}} d\mathbf{y}\,, \tag{4.4}$$

$$-\int_{\mathbb{R}^3} \frac{\nu(\Delta U_i) y_i}{(1+\varepsilon r)^\alpha} d\mathbf{y} = \int_{\mathbb{R}^3} \frac{\nu U_i y_i}{(1+\varepsilon r)^\alpha} \left[\frac{3\varepsilon\alpha}{r(1+\varepsilon r)} - \frac{\varepsilon^2 \alpha(\alpha+1)}{(1+\varepsilon r)^2} \right] d\mathbf{y}\,, \tag{4.5}$$

$$\int_{\mathbb{R}^3} \frac{\partial P}{\partial y_i} \frac{y_i}{(1+\varepsilon r)^\alpha} d\mathbf{y} = -\int_{\mathbb{R}^3} \frac{3P}{(1+\varepsilon r)^\alpha} d\mathbf{y} + \int_{\mathbb{R}^3} \frac{\alpha\varepsilon r P}{(1+\varepsilon r)^{\alpha+1}} d\mathbf{y}\,. \tag{4.6}$$

A finer argument is necessary to justify the integration by parts in the integral with the term $y_k \frac{\partial U_i}{\partial y_k}$. It is possible to show (see Lemma 4.2 below) that

$$\int_{\mathbb{R}^3} y_k \frac{\partial U_i}{\partial y_k} \frac{y_i}{(1+\varepsilon r)^\alpha} d\mathbf{y} = -4 \int_{\mathbb{R}^3} \frac{U_i y_i}{(1+\varepsilon r)^\alpha} d\mathbf{y} + \alpha\varepsilon \int_{\mathbb{R}^3} \frac{U_i y_i r}{(1+\varepsilon r)^{\alpha+1}} d\mathbf{y}\,.$$

Hence

$$\begin{aligned} & a \int_{\mathbb{R}^3} \left(U_i + y_k \frac{\partial U_i}{\partial y_k} \right) \frac{y_i}{(1+\varepsilon r)^\alpha} d\mathbf{y} \\ & = -3a \int_{\mathbb{R}^3} \frac{U_i y_i}{(1+\varepsilon r)^\alpha} d\mathbf{y} + \alpha\varepsilon a \int_{\mathbb{R}^3} \frac{U_i y_i r}{(1+\varepsilon r)^{\alpha+1}} d\mathbf{y}\,. \end{aligned} \tag{4.7}$$

¿From (4.4)–(4.7) we finally obtain (X is defined in (3.1))

$$\begin{aligned} & \int_{\mathbb{R}^3} \frac{X}{(1+\varepsilon r)^\alpha} \left[-3 + \frac{\alpha\varepsilon r}{(1+\varepsilon r)} + \frac{\nu}{ar} \frac{3\alpha\varepsilon}{(1+\varepsilon r)} - \frac{\nu}{a} \frac{\varepsilon^2 \alpha(\alpha+1)}{(1+\varepsilon r)^2} \right] d\mathbf{y} \\ & \qquad + \int_{\mathbb{R}^3} \frac{P}{(1+\varepsilon r)^\alpha} \left[\frac{\nu}{a} \frac{\varepsilon^2 \alpha(\alpha+1)}{(1+\varepsilon r)^2} - \frac{\nu}{ra} \frac{3\alpha\varepsilon}{(1+\varepsilon r)} \right] d\mathbf{y} \\ & + \frac{1}{2} \int_{\mathbb{R}^3} \frac{|\mathbf{U}|^2}{(1+\varepsilon r)^\alpha} \left[1 - \frac{\alpha\varepsilon r}{(1+\varepsilon r)} - \frac{\nu}{ra} \frac{3\alpha\varepsilon}{(1+\varepsilon r)} + \frac{\nu}{a} \frac{\varepsilon^2 \alpha(\alpha+1)}{(1+\varepsilon r)^2} \right] d\mathbf{y} \quad \le 0\,. \end{aligned} \tag{4.8}$$

The sign of the inequality in (4.8) is due to omitting the last integral from (4.4) which is positive.

Since $\alpha < 3$, $\frac{\varepsilon r}{(1+\varepsilon r)} \le 1$ and $\frac{\varepsilon}{r(1+\varepsilon r)} \le \frac{1}{r^2}$, we see that there exists an r_1 large enough such that

$$Z \equiv -3 + \frac{\alpha\varepsilon r}{(1+\varepsilon r)} + \frac{\nu}{ar} \frac{3\alpha\varepsilon}{(1+\varepsilon r)} - \frac{\nu}{a} \frac{\varepsilon^2 \alpha(\alpha+1)}{(1+\varepsilon r)^2} \le 0$$

for all $r \geq r_1$. Rewrite (4.8) as

$$
\begin{aligned}
\int_{B_{r_1}} \frac{X}{(1+\varepsilon r)^\alpha} Z d\mathbf{y} &+ \int_{\mathrm{I\!R}^3 \setminus \overline{B}_{r_1}} \frac{X}{(1+\varepsilon r)^\alpha} Z d\mathbf{y} \\
&+ \int_{\mathrm{I\!R}^3} \frac{|\mathbf{U}|^2}{(1+\varepsilon r)^\alpha} (-\frac{1}{2} Z - 1) d\mathbf{y} \\
&+ \frac{\alpha \nu}{a} \int_{\mathrm{I\!R}^3} \frac{P}{(1+\varepsilon r)^\alpha} \left[\frac{\varepsilon^2 (\alpha+1)}{(1+\varepsilon r)^2} - \frac{3\varepsilon}{r(1+\varepsilon r)} \right] d\mathbf{y} \leq 0 \,. \qquad (4.9)
\end{aligned}
$$

Now, using the fact that $\mathbf{U} \in \mathbf{L}^2(\mathrm{I\!R}^3) \cap \mathbf{L}^\infty(\mathrm{I\!R}^3)$ and $P \in L^q(\mathrm{I\!R}^3)$ for all $q > 1$, it is not difficult to check (using the different arguments close to the origin and far of it) that the integrals

$$
I_1 \equiv \int_{\mathrm{I\!R}^3} \frac{1}{r} \frac{|P|}{(1+\varepsilon r)^{\alpha+1}} d\mathbf{y} \ , I_2 \equiv \int_{\mathrm{I\!R}^3} \frac{1}{r^2} \frac{|P|}{(1+\varepsilon r)^{\alpha}} d\mathbf{y} \ , I_3 \equiv \int_{\mathrm{I\!R}^3} \frac{1}{r} \frac{|\mathbf{U}|^2}{(1+\varepsilon r)^{\alpha+1}} d\mathbf{y}
$$

are bounded independently of ε. To be more precise, we verify the uniform estimate of I_1. On B_1, as $(1+\varepsilon r)^{-(1+\alpha)} \leq 1$, we have

$$
\int_{B_1} \frac{|P|}{r} d\mathbf{y} \leq \|P\|_q \left(\int_{B_1} \frac{1}{r^{q/(q-1)}} d\mathbf{y} \right)^{\frac{q-1}{q}} \leq c \|P\|_q \left(\int_0^1 \frac{1}{r^{-2+q/(q-1)}} dr \right)^{\frac{q-1}{q}} ,
$$

and the last term is finite for $q > \frac{3}{2}$. On $\mathrm{I\!R}^3 \setminus B_1$, the Hölder inequality gives

$$
I_1 \leq \|P\|_2 \left(\int_{\mathrm{I\!R}^3 \setminus B_1} \frac{1}{r^2 (1+\varepsilon r)^{2\alpha}} d\mathbf{y} \right)^{1/2} \leq c \|P\|_2 \left(\int_1^\infty \frac{1}{(1+\varepsilon r)^{2\alpha}} dr \right)^{1/2} ,
$$

which is certainly finite for $\alpha > \frac{5}{2}$. The estimates of I_2 and I_3 and the first integral in (4.9) are similar and we can pass to the limit $\varepsilon \to 0$. Finally, since $XZ \geq 0$ on $\mathrm{I\!R}^3 \setminus \overline{B}_{r_1}$ the Fatou Lemma justifies the limit process in the second integral of (4.9). Thus we observe first that

$$
- \int_{\mathrm{I\!R}^3} X d\mathbf{y} < \infty \,,
$$

and then (passing to the limit in (4.9) once more) one has

$$
-3 \int_{\mathrm{I\!R}^3} X d\mathbf{y} + \frac{1}{2} \int_{\mathrm{I\!R}^3} |\mathbf{U}|^2 d\mathbf{y} \leq 0 \,.
$$

However, $X \leq 0$ on $\mathrm{I\!R}^3$ by the assumption, so $\mathbf{U} \equiv \mathbf{0}$. The proof of Theorem 4.1 is now complete. □

Lemma 4.2. *Assume that* $y_k \frac{\partial \mathbf{U}}{\partial y_k} \in \mathbf{L}^2(\mathrm{I\!R}^3)$ *and* $\mathbf{U} \in \mathbf{L}^2(\mathrm{I\!R}^3) \cap \mathbf{W}^{2,2}(\mathrm{I\!R}^3)$. *Let* $\alpha \in (\frac{5}{2}, 3)$ *and* $r \equiv |\mathbf{y}|$. *Then it holds*

$$
\int_{\mathrm{I\!R}^3} y_k \frac{\partial U_i}{\partial y_k} \frac{y_i}{(1+\varepsilon r)^\alpha} d\mathbf{y} = -4 \int_{\mathrm{I\!R}^3} \frac{U_i y_i}{(1+\varepsilon r)^\alpha} d\mathbf{y} + \alpha \varepsilon \int_{\mathrm{I\!R}^3} \frac{U_i y_i r}{(1+\varepsilon r)^{\alpha+1}} d\mathbf{y} \,. \qquad (4.10)
$$

Proof. We have

$$\int_{\mathbb{R}^3} y_k \frac{\partial U_i}{\partial y_k} \frac{y_i}{(1+\varepsilon r)^\alpha} d\mathbf{y} = \lim_{R\to\infty} \int_{B_R} y_k \frac{\partial U_i}{\partial y_k} \frac{y_i}{(1+\varepsilon r)^\alpha} d\mathbf{y}\,,$$

where B_R is a ball with the diameter R. Using the Green formula we get

$$\begin{aligned}\lim_{R\to\infty} \int_{B_R} y_k \frac{\partial U_i}{\partial y_k} \frac{y_i}{(1+\varepsilon r)^\alpha} d\mathbf{y} &= \lim_{R\to\infty} \int_{\partial B_R} \frac{y_k y_i U_i}{(1+\varepsilon R)^\alpha} \frac{y_k}{R} dS \\ - \lim_{R\to\infty} \left[4 \int_{B_R} \frac{U_i y_i}{(1+\varepsilon r)^\alpha} d\mathbf{y} - \alpha\varepsilon \int_{B_R} \frac{U_i y_i y_k}{(1+\varepsilon r)^{\alpha+1}} \frac{y_k}{r} d\mathbf{y} \right]. \end{aligned} \tag{4.11}$$

All limits of the volume integrals in (4.11) exist and give exactly (4.10). In order to finish the proof of this lemma it remains to show that the surface integral in (4.11) tends to zero as R tends to infinity.

Let us take α_1 such that $3 > \alpha > \alpha_1 > \frac{5}{2}$. Using spherical coordinates, we fix the angles φ and ϑ for a moment. For any $R > 1$ and $i = 1, 2, 3$

$$\begin{aligned} R^{4-\alpha_1} |U_i(R,\varphi,\vartheta)| &= |U_i(1,\varphi,\vartheta)| + \int_1^R \frac{\partial}{\partial r} \left| r^{4-\alpha_1} U_i(r,\varphi,\vartheta) \right| dr \\ \le |U_i(1,\varphi,\vartheta)| &+ \int_1^R \left| \frac{\partial U_i}{\partial r}(r,\varphi,\vartheta) \right| r^{4-\alpha_1} dr + (4-\alpha_1) \int_1^R |U_i(r,\varphi,\vartheta)| r^{3-\alpha_1} dr\,. \end{aligned}$$

Integrating over φ and ϑ and using the Hölder inequality we obtain

$$\begin{aligned} R^{4-\alpha_1} & \int_0^{2\pi} \int_0^\pi |U_i(R,\varphi,\vartheta)| \sin\vartheta\, d\vartheta\, d\varphi \le \int_{\partial B_1} |U_i| dS \\ + & \Big(\int_{B_R\setminus B_1} \left| \frac{\partial U_i}{\partial r} \right|^2 r^2 d\mathbf{y} \Big)^{\frac12} \Big(4\pi \int_1^R r^{4-2\alpha_1} dr \Big)^{\frac12} \\ + & (4-\alpha_1) \Big(\int_{B_R\setminus B_1} |U_i|^2 d\mathbf{y} \Big)^{\frac12} \Big(4\pi \int_1^R r^{4-2\alpha_1} dr \Big)^{\frac12}. \end{aligned} \tag{4.12}$$

Evidently, all terms in the right-hand side of (4.12) are bounded independently of R. Thus

$$\begin{aligned} \Big| \int_{\partial B_R} \frac{y_k y_i U_i}{(1+\varepsilon R)^\alpha} \frac{y_k}{R} dS \Big| &\le c R^{4-\alpha} \int_0^{2\pi} \int_0^\pi |\mathbf{U}(R,\varphi,\vartheta)| \sin\vartheta\, d\vartheta\, d\varphi \le \\ \le c R^{4-\alpha_1} \left(\int_0^{2\pi} \int_0^\pi |\mathbf{U}(R,\varphi,\vartheta)| \sin\vartheta\, d\vartheta\, d\varphi \right) & R^{\alpha_1-\alpha} \le c_1 R^{\alpha_1-\alpha} \end{aligned}$$

Finally, letting $R \to \infty$ yields

$$\Big| \lim_{R\to\infty} \int_{\partial B_R} \frac{y_k y_i U_i}{(1+\varepsilon R)^\alpha} \frac{y_k}{R} dS \Big| \le \lim_{R\to\infty} c_1 R^{\alpha_1-\alpha} = 0\,.$$

The proof of Lemma 4.2 is complete. □

5. Pseudo-selfsimilar solutions

In this section singular pseudo-selfsimilar solutions to (1.3) in the form (1.6) are investigated under the supposition that λ, $\mu \in C^1(-\infty, T)$. It is not difficult to see that λ and μ cannot change sign. Indeed, if $\lambda(t_0) = 0$ (or $\mu(t_0) = 0$) for some $t_0 < T$, then $\mathbf{u}(t, \cdot)$ has to be necessarily zero for all $t \geq t_0$ and no singularity can occur as $t \to T_-$. Hence we can assume that λ is positive without loss of generality.

Assume for a moment that $\mathbf{u}$ is given by $(1.6)_1$, p by (1.7) and $\mathbf{U} \in \mathbf{W}^{1,2}(\mathrm{I\!R}^3)$. By Lemma 2.1, we have $\mathbf{U} \in \mathbf{W}^{2,2}(\mathrm{I\!R}^3) \cap \mathbf{L}^\infty(\mathrm{I\!R}^3)$ and $P \in W^{1,2}(\mathrm{I\!R}^3)$. Inserting (1.6) into (1.3) we obtain

$$\begin{aligned} \operatorname{div}\mathbf{U} &= 0 \\ \mu'\mathbf{U} + \frac{\mu}{\lambda}\lambda' y_k \frac{\partial \mathbf{U}}{\partial y_k} - \nu\mu\lambda^2\Delta\mathbf{U} &+ \mu^2\lambda U_j \frac{\partial \mathbf{U}}{\partial y_j} + \kappa\lambda\nabla P = \mathbf{0}\,. \end{aligned} \tag{5.1}$$

Taking div of $(5.1)_2$, we obtain

$$\frac{\kappa}{\mu^2}\Delta P = -\frac{\partial U_j}{\partial y_i}\frac{\partial U_i}{\partial y_j}\,.$$

Since the right-hand side of the last equation is independent of t we see that $\frac{\kappa}{\mu^2}$ must be a constant function. Moreover, redefining P if necessary we can assume that $\frac{\kappa}{\mu^2} = 1$. Thus, p reduces to $(1.6)_2$. Multiplying $(5.1)_2$ by $\frac{1}{\mu^2\lambda}$ yields

$$\begin{aligned} \operatorname{div}\mathbf{U} &= 0 \\ \frac{\mu'}{\mu^2\lambda}\mathbf{U} + \frac{\lambda'}{\mu\lambda^2} y_k \frac{\partial \mathbf{U}}{\partial y_k} &- \nu\frac{\lambda}{\mu}\Delta\mathbf{U} + U_j\frac{\partial \mathbf{U}}{\partial y_j} + \nabla P = \mathbf{0}\,. \end{aligned} \tag{5.2}$$

A natural question appears: Is it possible to construct pseudo-selfsimilar solutions to the Navier–Stokes equations which are singular and which do not have the Leray form (1.1)?

Partial answers to this question are contained in the following exposition.

Assume that $\mathbf{u}(t, \mathbf{x})$ is a singular Leray-Hopf weak solution of (1.3). This means that $\mathbf{u} \in L^\infty(t_1, T; \mathbf{L}^2(\mathrm{I\!R}^3)) \cap L^2(t_1, T; \mathbf{W}^{1,2}(\mathrm{I\!R}^3))$ for $t_1 \in (-\infty, T)$, and $\lim_{t\to T^-} \|\nabla\mathbf{u}\|_2 = \infty$. Using the relations $\|\mathbf{u}\|_2^2 = \frac{\mu^2(t)}{\lambda^3(t)}\|\mathbf{U}\|_2^2$ and $\|\nabla\mathbf{u}\|_2^2 = \frac{\mu^2(t)}{\lambda(t)}\|\nabla\mathbf{U}\|_2^2$, we obtain the following restrictions on μ and λ:

$$\frac{\mu^2(t)}{\lambda^3(t)} \leq \text{ const on } (t_1, T) \qquad (\mathbf{u} \in L^\infty(t_1, T; \mathbf{L}^2(\mathrm{I\!R}^3)))\,, \tag{5.3}$$

$$\int_{t_1}^T \frac{\mu^2(t)}{\lambda(t)}dt < \infty \qquad (\mathbf{u} \in L^2(t_1, T; \mathbf{W}^{1,2}(\mathrm{I\!R}^3)))\,, \tag{5.4}$$

$$\lim_{t\to T^-} \frac{\mu^2(t)}{\lambda(t)} = \infty \qquad (\lim_{t\to T^-} \|\nabla\mathbf{u}\|_2 = \infty)\,. \tag{5.5}$$

Finally, multiplying (5.2) by $\mathbf{U}$, integrating over $\mathrm{I\!R}^3$, integrating by parts and using $\operatorname{div}\mathbf{U} = 0$, we obtain

$$\frac{\mu'}{\mu^2\lambda} - \frac{3}{2}\frac{\lambda'}{\mu\lambda^2} = -\frac{\lambda}{\mu}K_3\nu\,, \tag{5.6}$$

where $K_3 \equiv \frac{\|\nabla \mathbf{U}\|_2^2}{\|\mathbf{U}\|_2^2}$ must be a positive constant, otherwise $\mathbf{U} \equiv \mathbf{0}$.

Now, if $\frac{\lambda(t)}{\mu(t)} \equiv$ *const* then (5.6) leads to the Leray solution (1.1). Indeed, let $\frac{\lambda(t)}{\mu(t)} = \pm A$, where $A > 0$. Equation (5.6) then reduces to $\frac{\mu'(t)}{\mu^3(t)} = a$ with $a \equiv 2A^2 K_3 \nu$. Since $\mu^2(t) \to \infty$ as $t \to T-$ (otherwise we would reach a contradiction with (5.3) and (5.5)), we have $\mu(t) = \frac{1}{\sqrt{2a(T-t)}}$. Since $\lambda(t) = \pm A\mu(t)$ and $\mathbf{u}(t, \mathbf{x}) = \mu(t)\mathbf{U}(\lambda(t)\mathbf{x})$, we obtain the Leray form (1.1) by changing scale from $\mathbf{x}$ to $\pm\frac{1}{A}\mathbf{x}$ if necessary[4] .

In order to have the possibility to construct singular pseudo-selfsimilar solutions which are not of the Leray type one can define the space H by

$$H \equiv \left\{ \boldsymbol{\Phi} \in \mathbf{W}^{1,2}(\mathbb{R}^3) \,;\, \int_{\mathbb{R}^3} U_i \Phi_i \, d\mathbf{y} = \int_{\mathbb{R}^3} \frac{\partial U_i}{\partial y_k} y_k \Phi_i \, d\mathbf{y} = 0 \right\}$$

and consider the following condition:

(5.7) $\qquad$ There exists some $\boldsymbol{\Phi} \in H$ satisfying $\displaystyle\int_{\mathbb{R}^3} \nabla \Phi_i \nabla U_i \, d\mathbf{y} \neq 0$.

If (5.7) holds, then multiplying $(5.2)_2$ by $\boldsymbol{\Phi}$ and integrating over $\mathbb{R}^3$ we obtain

$$\nu \frac{\lambda}{\mu} \int_{\mathbb{R}^3} \nabla U_i \nabla \Phi_i \, d\mathbf{y} + \int_{\mathbb{R}^3} \left(U_j \frac{\partial U_i}{\partial y_j} \Phi_i + \frac{\partial P}{\partial y_i} \Phi_i \right) d\mathbf{y} = 0 \,,$$

which implies

$$\frac{\lambda(t)}{\mu(t)} \equiv \text{ const} \,.$$

If $\frac{\lambda(t)}{\mu(t)}$ is not constant, then (5.7) does not hold, i.e.

(5.8) $\qquad$ for all $\boldsymbol{\Phi} \in H$ necessarily $\displaystyle\int_{\mathbb{R}^3} \nabla \Phi_i \nabla U_i \, d\mathbf{y} = 0$.

We then define two functionals

$$\begin{aligned} \langle F^1, \boldsymbol{\Phi} \rangle &= \int_{\mathbb{R}^3} U_i \Phi_i \, d\mathbf{y} \\ \langle F^2, \boldsymbol{\Phi} \rangle &= \int_{\mathbb{R}^3} \frac{\partial U_i}{\partial y_k} y_k \Phi_i \, d\mathbf{y} \,. \end{aligned} \tag{5.9}$$

Let us remark that thanks to (2.11) $\frac{\partial \mathbf{U}}{\partial y_k} y_k \in \mathbf{L}^2(\mathbb{R}^3)$, and therefore $(5.9)_2$ is well defined. We can rewrite H as

$$H = \left\{ \boldsymbol{\Phi} \in \mathbf{W}^{1,2}(\mathbb{R}^3) \,;\, \langle F^1, \boldsymbol{\Phi} \rangle = \langle F^2, \boldsymbol{\Phi} \rangle = 0 \right\} \,.$$

---

[4] If $\frac{\lambda(t)}{\mu(t)} \equiv$ const, but the energy equation (5.6) is not attainable (this happens for example if we would assume $\mathbf{U} \in \mathbf{W}^{1,2}_{loc}(\mathbb{R}^3) \cap \mathbf{L}^3(\mathbb{R}^3)$, then it follows directly from (5.2) that $\frac{\mu'(t)}{\mu^3(t)} \equiv B$, where B is a constant. If $B > 0$, then the calculations just presented lead to the Leray form (1.1) of a singular solution. If $B < 0$, then one observes that there are solutions only for $t > T$ and they are of the form $\mu(t) = \frac{1}{\sqrt{2B(t-T)}}$. Note that $\lim_{t \to T+} \|\nabla \mathbf{u}\|_2 = \infty$ for such solutions. If T=0, one obtains self-similar solutions studied in Cannone [2] and related papers cited therein.

By the Riesz theorem we have

$$\begin{aligned} \langle F^1, \boldsymbol{\psi}\rangle &= (\mathbf{U}^1, \boldsymbol{\psi}) + (\nabla \mathbf{U}^1, \nabla \boldsymbol{\psi}) \\ \langle F^2, \boldsymbol{\psi}\rangle &= (\mathbf{U}^2, \boldsymbol{\psi}) + (\nabla \mathbf{U}^2, \nabla \boldsymbol{\psi}) \end{aligned} \tag{5.10}$$

for certain $\mathbf{U}^1$ and $\mathbf{U}^2 \in \mathbf{W}^{1,2}(\mathbb{R}^3)$ and all $\boldsymbol{\psi} \in \mathbf{W}^{1,2}(\mathbb{R}^3)$. Because of the regularity of $\mathbf{U}$ and (5.9), equality (5.10) can be rewritten as

$$\begin{aligned} \mathbf{U}^1 - \Delta \mathbf{U}^1 &= \mathbf{U} \\ \mathbf{U}^2 - \Delta \mathbf{U}^2 &= \frac{\partial \mathbf{U}}{\partial y_k} y_k \,. \end{aligned}$$

¿From (5.8) follows that $\mathbf{U}$ belongs to the orthogonal complement of H in $\mathbf{W}^{1,2}(\mathbb{R}^3)$; therefore $\mathbf{U}$ must be a linear combination of $\mathbf{U}^1$ and $\mathbf{U}^2$, i.e.

$$\mathbf{U} - \Delta \mathbf{U} = c_1 \mathbf{U} + c_2 \frac{\partial \mathbf{U}}{\partial y_k} y_k \,. \tag{5.11}$$

Taking the Fourier transform of (5.11) (in the sense of $\mathbf{L}^2(\mathbb{R}^3)$) we have

$$\widehat{\mathbf{U}} + |\boldsymbol{\xi}|^2 \widehat{\mathbf{U}} = c_1 \widehat{\mathbf{U}} - c_2 (3\widehat{\mathbf{U}} + \xi_k \frac{\partial \widehat{\mathbf{U}}}{\partial \xi_k}) \,. \tag{5.12}$$

Considering $\widehat{\mathbf{U}}$ in spherical coordinates, i.e. $\widehat{\mathbf{U}} = \widehat{\mathbf{U}}(r, \frac{\boldsymbol{\xi}}{r})$, (where $r = |\boldsymbol{\xi}|$ and $\frac{\boldsymbol{\xi}}{r}$ denotes angle variables), and denoting $\beta = 1 - c_1 + 3c_2$ we get a differential equation

$$\widehat{\mathbf{U}}(\beta + r^2) + c_2 r \frac{\partial \widehat{\mathbf{U}}}{\partial r} = \mathbf{0} \,,$$

the solution of which has the form

$$\widehat{U_i} = S_i(\frac{\boldsymbol{\xi}}{r}) r^{-\frac{\beta}{c_2}} \mathrm{e}^{-\frac{r^2}{2c_2}} \,. \tag{5.13}$$

Since $\int_{\mathbb{R}^3} \widehat{\mathbf{U}}(\boldsymbol{\xi}) d\boldsymbol{\xi} = \int_0^\infty \int_{\partial B_1(0)} \widehat{\mathbf{U}}(r, \frac{\boldsymbol{\xi}}{r}) \; dS(\frac{\boldsymbol{\xi}}{r}) \, dr$ and $\mathbf{U} \in \mathbf{L}^2(\mathbb{R}^3)$, we see that

$$c_2 > 0 \,, \quad \frac{\beta}{c_2} < \frac{3}{2} \quad \text{and} \quad \gamma \equiv \int_{\partial B_1(0)} S_i(\frac{\boldsymbol{\xi}}{r}) S_i(\frac{\boldsymbol{\xi}}{r}) \; dS(\frac{\boldsymbol{\xi}}{r}) < \infty \,. \tag{5.14}$$

In addition, due to the divergence-free condition $\widehat{U_i}\xi_i = 0$, we have

$$S_i(\frac{\boldsymbol{\xi}}{r}) \xi_i = 0 \,. \tag{5.15}$$

The Fourier transform of (5.2) gives

$$\widehat{U_j \frac{\partial \mathbf{U}}{\partial y_j}} + i\boldsymbol{\xi}\widehat{P} + \frac{\mu'}{\mu^2 \lambda} \widehat{\mathbf{U}} - \frac{\lambda'}{\mu \lambda^2} \left[3\widehat{\mathbf{U}} + \xi_k \frac{\partial \widehat{\mathbf{U}}}{\partial \xi_k} \right] + \nu \frac{\lambda}{\mu} |\boldsymbol{\xi}|^2 \widehat{\mathbf{U}} = \mathbf{0} \,, \tag{5.16}$$

here i denotes the complex unit. We continue as follows: we multiply (5.16) scalarly by $\mathbf{S}$, integrate over $\partial B_1(0)$, use (5.15) and finally multiply the obtained equation by $\frac{1}{\gamma} r^{\frac{\beta}{c_2}} \mathrm{e}^{\frac{r^2}{2c_2}}$. Thus, denoting

$$J(r) \equiv -\frac{1}{\gamma} r^{\frac{\beta}{c_2}} \mathrm{e}^{\frac{r^2}{2c_2}} \int_{\partial B_1(0)} U_j \widehat{\frac{\partial U_i}{\partial y_j}}(r, \frac{\boldsymbol{\xi}}{r}) S_i(\frac{\boldsymbol{\xi}}{r})\, dS(\frac{\boldsymbol{\xi}}{r})$$

we obtain

$$\left(\frac{\mu'}{\mu^2\lambda} - 3\frac{\lambda'}{\mu\lambda^2} + \frac{\lambda'}{\mu\lambda^2}\frac{\beta}{c_2}\right) + \left(\nu\frac{\lambda}{\mu} + \frac{\lambda'}{\mu\lambda^2}\frac{1}{c_2}\right) r^2 = J(r)\,. \tag{5.17}$$

For all $r \in [0,1]$ (and $t \in (-\infty, T)$) the left-hand side of (5.17) is bounded and

$$\lim_{r\to 0+} (\text{LHS of } (5.17)) = \frac{\mu'}{\mu^2\lambda} - 3\frac{\lambda'}{\mu\lambda^2} + \frac{\lambda'}{\mu\lambda^2}\frac{\beta}{c_2}\,.$$

Consequently $\lim\limits_{r\to 0+} J(r)$ exists and is equal to some constant K_1. Therefore,

$$\frac{\mu'}{\mu^2\lambda} - 3\frac{\lambda'}{\mu\lambda^2} + \frac{\lambda'}{\mu\lambda^2}\frac{\beta}{c_2} = K_1\,. \tag{5.18}$$

Multiplying (5.17) by r^{-2}, and arguing analogously (now for $r \to \infty$) we obtain

$$\nu\frac{\lambda}{\mu} + \frac{\lambda'}{\mu\lambda^2}\frac{1}{c_2} = K_2\,. \tag{5.19}$$

Equations (5.6), (5.18) and (5.19) are linearly dependent. To see this, we first multiply (5.12) by $\widehat{\mathbf{U}}$ and integrate over $\mathbb{R}^3$. After the integration by parts we obtain

$$K_3 = \frac{\|\nabla \mathbf{U}\|_2^2}{\|\mathbf{U}\|_2^2} = \frac{\|\,|\boldsymbol{\xi}|\widehat{\mathbf{U}}\|_2^2}{\|\widehat{\mathbf{U}}\|_2^2} = -(1 - c_1 + \frac{3}{2}c_2) = -\beta + \frac{3}{2}c_2\,.$$

Adding (5.18), and (5.19) multiplied by $\frac{3}{2}c_2 - \beta$ leads to

$$K_1 + K_2(\frac{3}{2}c_2 - \beta) = 0\,.$$

If $K_1 = K_2 = 0$, then a singularity for $t \to T-$ cannot occur. Indeed, if $K_2 = 0$ then (5.19) yields $\lambda' = -c_2\nu\lambda^3$. Hence

$$-\frac{1}{2\lambda^2(t)} = -c_2\nu t + const. \tag{5.20}$$

We can look at the solution of (5.20) in two different ways. Firstly, assuming $\lambda \to \infty$ as $t \to T-$ (cf. (5.3) and (5.5)), we have $const = c_2\nu T$ and $\lambda^2(t) = 1/(2c_2\nu(t-T))$, which makes sense for $t > T$. Thus, we come again to the solutions studied by Cannone [2] (cf. footnote 4 before).

Secondly, we fix $const$ in (5.20) at some time $t_0 < T$, i.e. $const = -c_0 + c_2\nu t_0$, where $c_0 = 1/(2\lambda^2(t_0))$. Hence, $\lambda^2(t) = 1/(2(c_2\nu(t - t_0) + c_0))$. Thus, $\lambda(t)$ is bounded as $t \to T-$ and the corresponding $\mathbf{u}$ would not develop a singularity as $t \to T-$.

We finish this section by presenting two results:

(i) we will show that if $\beta > -c_2$ then $K_1 = 0$,

(ii) assuming that $\lambda(t) = (T-t)^{-\gamma}$, $\gamma > 0$, we will prove that γ must be $1/2$, λ coincides with μ and $\mathbf{u}$ is of the form (1.1).

Ad (i). We investigate the behavior of $J(r)$ as r tends to zero. Note that $J(r)$ can be rewritten as

$$J(r) = -\frac{1}{\gamma} r^{\frac{\beta}{c_2}} e^{\frac{r^2}{2c_2}} \int_{\partial B_1(0)} r E_j(\frac{\boldsymbol{\xi}}{r}) \widehat{U_j U_i}(r, \frac{\boldsymbol{\xi}}{r}) S_i(\frac{\boldsymbol{\xi}}{r})\, dS(\frac{\boldsymbol{\xi}}{r}),$$

where $\boldsymbol{\xi} = r\mathbf{E}(\frac{\boldsymbol{\xi}}{r})$ and $|\mathbf{E}| = 1$. Hence

$$|J(r)| \leq \frac{1}{\sqrt{\gamma}} r^{1+\frac{\beta}{c_2}} e^{\frac{r^2}{2c_2}} \left(\int_{\partial B_1(0)} \sum_{i,j=1}^{3} |\widehat{U_j U_i}(r, \frac{\boldsymbol{\xi}}{r})|^2 dS(\frac{\boldsymbol{\xi}}{r}) \right)^{1/2}.$$

Since $\|\widehat{U_j U_i}\|_\infty \leq c\|\mathbf{U}\|_2^2$ we have $J(r) \to 0$ as $r \to 0$ if $\beta > -c_2$. Thus $K_1 = 0$, the situation that was discussed above.

Ad (ii). Let $t_1 \in (-\infty, T)$, then solving (5.6) on (t_1, t) we obtain

$$\mu(t) = L\lambda^{\frac{3}{2}}(t) \mathrm{e}^{-K_3\nu \int_{t_1}^t \lambda^2(\tau) d\tau}, \tag{5.21}$$

where $L = \frac{\mu(t_1)}{\lambda^{\frac{3}{2}}(t_1)}$. Substituting (5.6) and (5.21) into (5.18) yields

$$\left(\frac{\beta}{c_2} - \frac{3}{2}\right) \frac{\lambda'(t)}{\lambda^2(t)} = K_3\nu\lambda(t) + K_1 L\lambda^{\frac{3}{2}}(t) \mathrm{e}^{-K_3\nu \int_{t_1}^t \lambda^2(\tau) d\tau}.$$

¿From here we can calculate $\mathrm{e}^{-K_3\nu \int_{t_1}^t \lambda^2(\tau) d\tau}$ and insert it into (5.21). Then we see that

$$\mu(t) = L_1\lambda(t) + L_2\lambda^{\frac{3}{2}}(t) \int_{t_1}^t \lambda^{\frac{3}{2}}(\tau) d\tau + L_3\lambda^{\frac{3}{2}}(t), \tag{5.22}$$

where

$$L_1 = \Big(\frac{2\beta}{c_2} - 3\Big)\frac{K_3\nu}{K_1}, \quad L_2 = \frac{(K_3\nu)^2}{K_1}, \quad L_3 = L\Big(1 - \frac{\Big(\frac{2\beta}{c_2} - 3\Big)K_3\nu}{K_1 L\lambda^{\frac{1}{2}}(0)}\Big).$$

It can be seen from (5.3), (5.5), (5.21) and (5.22) that

$$\lim_{t \to T^-} \lambda(t) \mathrm{e}^{-K_3\nu \int_{t_1}^t \lambda^2(\tau) d\tau} = \infty \quad \text{and} \quad \int_{t_1}^T \lambda^{\frac{3}{2}}(\tau) d\tau < \infty.$$

Now, putting $\lambda(t)$ in the particular form

$$\lambda(t) = (T-t)^{-\gamma} \quad \text{with } \gamma > 0$$

into (5.21) and (5.22) we see that $\mu(t)$ has to be of the form

$$\mu(t) = \frac{\nu}{K_2}(T-t)^{-\gamma} + \frac{\gamma}{K_2 c_2}(T-t)^{\gamma-1}$$

and

$$\mu(t) = L(T-t)^{-\frac{3\gamma}{2}} \mathrm{e}^{-K_3\nu \int_0^t (T-\tau)^{-2\gamma}\, d\tau},$$

which is possible only if $\gamma = \frac{1}{2}$. Thus the Leray form (1.1) is obtained.

We can conclude our observations:

(1) It is not possible to construct any singular solution to (1.3) in the Leray form (1.1) provided that $\mathbf{U}$, as a solution of (1.2), belongs either to $\mathbf{W}^{1,2}(\mathbb{R}^3)$ or to $\mathbf{L}^3(\mathbb{R}^3) \cap \mathbf{W}^{1,2}_{loc}(\mathbb{R}^3)$ (cf. [5]).

(2) A singular solution $\mathbf{u}$ to (1.1) in the pseudo-selfsimilar form

$$\mathbf{u}(t, \mathbf{x}) = \mu(t)\mathbf{U}(\lambda(t)x)$$

is not excluded: the Fourier transform of $\mathbf{U}$ must satisfy

$$\widehat{\mathbf{U}}(r, \frac{\xi}{|\boldsymbol{\xi}|}) = |\boldsymbol{\xi}|^{-\frac{\beta}{c_2}} e^{-\frac{|\boldsymbol{\xi}|^2}{2c_2}} \mathbf{S}(\frac{\xi}{|\boldsymbol{\xi}|}),$$

where $c_2 > 0$ and $\beta \leq -c_2$. A function $\mathbf{S}$ depending only on angles is square integrable on the unit sphere in $\mathbb{R}^3$ and satisfies $S_i(\frac{\xi}{|\boldsymbol{\xi}|})\xi_i = 0$. The functions λ and μ must solve equations (5.6) and (5.19) with caveat that λ cannot be in a simple form $\lambda(t) = (T-t)^{-\gamma}$, $\gamma > 0$.

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