Asymptotic Decay for a Generalized Boussinesq System

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Abstract. In this paper we study the large time behaviour of solutions to a generalization of the Boussinesq system of equations in $n \geq 2$ spatial dimensions. We establish existence and algebraic decay of the L_2-norm of the solution.

1. Introduction

In this paper we study the large time behaviour of solutions to a generalized Boussinesq system of equations with dissipation of the form

$$
\begin{cases} \omega_t + \sum_{i=1}^{n} (\omega u)_{x_i} \; = \; \beta \Delta \omega, \\ u_t + \sum_{i=1}^{n} (\omega + \frac{u^2}{2})_{x_i} \; = \; (-1)^{p+1} \sum_{|\alpha|=2p} D^\alpha u_t + \beta \Delta u + \Delta^{p+1} u, \end{cases} \tag{1.1}
$$

where $2p > n$ and $(x,\, t) \in I\!R^n \times R_+$. As $\beta \to 0$ our decay estimates will not work, so without loss of generality we can assume that $\beta \equiv 1$. Notice that when $n = 1$ and the dissipative terms expressed by the Laplacian of ω or of u and the biharmonic of u are not included, (1.1) represents the model proposed by Boussinesq for the two way propagation of long surface waves in a channel of constant depth. More precisely, if we let $\rho = \omega - 1$ then (1.1) is the usual Boussinesq system given by

$$
\begin{aligned} \rho_t + u_x + (u\rho)_x \; &= \; 0, \\ u_t + \rho_x + u u_x - u_{xxt} \; &= \; 0. \end{aligned}
$$

If we do not add the term Δu in the second equation the decay will be of order $(t+1)^{-\frac{n}{4}}$ instead of $(t+1)^{-\frac{n}{2}}$. Here $u(x,t)$ represents the velocity and $\omega(x,t) \; = \; 1 + \rho(x,t)$ is the height of the free surface of the fluid above the bottom. Formally the Boussinesq system can be viewed as a perturbation of the one-dimensional wave equation in which dispersive and non-linear effects are of the same order. We obtain the system (1.1) by generalizing the one-dimensional model to higher dimensions by considering a natural extension for the nonlinear terms so that their structure is same as the nonlinear terms in the compressible Navier Stokes equations.

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The purpose of this paper is to establish algebraic bounds for the rate of decay for the energy of the system (1.1). More precisely we obtain L_2 rates of decay for $(\omega - \bar{\omega},\, u - \bar{u})$ where $(\bar{\omega},\, \bar{u})$ are the equlibria to which the data $(\omega_0,\, u_0)$ tend to as $|x| \longrightarrow \infty$. Let

$$\mathcal{X}_\beta \;=\; \left\{(u_0,\, \omega_0) \,:\, |u_0(t) - \bar{u}|^2_{L_2} \le\, C(t+1)^{-\beta},\; |\omega_0(t) - \bar{\omega}|^2_{L_2} \le\, C(t+1)^{-\beta}\,\right\}$$

where $\omega_0(t)$ is the solution to the heat equation with data ω_0 and $u_0(t)$ is the solution to the linearized regularized long-wave equation with data u_0.

We show that for data in $L_2 \cap \mathcal{X}_\beta(I\!\!R^n)$ and $n \,>\, 2$ the solution of (1.1) satisfies

$$\|u(t) - \bar{u}\|_2^2 + \|D^p u(t)\|_2^2 + \|\omega(t) - \bar{\omega}\|_2^2 \,\le\, C(t+1)^{-\frac{1}{2}}. \tag{1.2}$$

If $n \,=\, 2$, we show that the solution decays at a logarithmic rate. More precisely, we show that

$$\|u(t) - \bar{u}\|_2^2 + \|D^p u(t)\|_2^2 + \|\omega(t) - \bar{\omega}\|_2^2 \,\le\, C[\ln(t+e)]^m \tag{1.3}$$

for all $m \,\ge\, 3$. One can construct solutions to the linear part of the equations decaying at any rate $(t+1)^{-\beta}$. As in the case of the heat equation, this depends on the order of the zero at the origin for the Fourier transform of the data. In particular, if $(\omega_0 - \bar{\omega},\, u - \bar{u}) \,\in\, L^1 \cap L^2$ then $\beta \,=\, \frac{n}{2}$. This will follow using the Fourier Splitting method [5], [6].

The method that we present here seems to have some shortcomings in the one-dimensional case. A different approach is needed in this case and the problem will be considered in a forthcoming paper.

To establish the decay (1.2) we first need to construct a positive convex entropy η for the hyperbolic part of the equation. We recall that the underlying hyperbolic system admits an entropy [7] (in the sense of Lax [3]) of the form $\tilde{\eta}(u,\, \omega) \,=\, \frac{u^2}{2} + \omega \ln \omega$. This entropy is not positive but yields a convex positive one if we subtract the linear part that is, by defining $\eta \,=\, \tilde{\eta}$- the linear part at $(\bar{\omega},\, \bar{u})$. Hence formally, standard multiplier methods then yield

$$\frac{d}{dt}\left[\int_{I\!\!R^n} \eta(u,\, \omega)(\cdot, t) dx \,+\, \int_{I\!\!R^n} |D^p u(\cdot, t)|^2 dx\right] \,=\, -\int_{I\!\!R^n} \Big(|\nabla u|^2 + \sigma_0''|\nabla \omega|^2 + |\nabla D^p u|^2\,\Big)\, dx. \tag{1.4}$$

Here, to simplify the notation we have used $|D^j u|^2 \,=\, \sum_{|\alpha|=j} |D^\alpha u(\cdot, t)|^2$ and $|\nabla D^j u|^2 \,=\, \nabla \sum_{|\alpha|=j} |D^\alpha u(\cdot, t)|^2$. Once (1.4) has been obtained, we estimate the term $\sigma_0''|\nabla\omega|^2$ so as to reduce it a form to which the Fourier splitting method can be applied, first developed by one of the authors (M.E.S) to obtain upper bounds for the rate of decay for solutions to parabolic conservation laws as well as for obtaining upper and lower bounds of decay for the solutions to the Navier-Stokes equations. We will apply the method to approximated solutions and then pass to the limit to derive the decay for the weak solutions.

In order to establish the existence of weak solutions to (1.1) we construct a sequence of approximate solutions. For this we smooth out the nonlinear terms in the equation by using a "retarded mollifier." The construction of such approximations was used earlier by Caffarelli, Kohn and Nirenberg [1] to establish the existence of a weak solution for the Navier-Stokes equations. A similar construction can be found in the work of Leray [4] as well, again, for weak solutions of the Navier-Stokes equations.

The plan of the paper is as follows. In section 2, the existence of the weak solutions is established. This is done using the approach mentioned above via approximations which are obtained by using a retarded mollifier to "linearize the equations." The entropy corresponding to the underlying hyperbolic part of the equation yields uniform bounds for the approximate solutions and hence passing to the limit weak solutions are obtained in the appropriate Sobolev spaces.

In section 3 the entropy inequality (1.3) is used to obtain an inequality (1.4) relating the L_2-norms of the solution with the L_2-norms of the gradient. Then the Fourier splitting method will yield the decay of the solution in the energy norm. We recall that to use the Fourier splitting method [5] such an inequality of the type (1.4) is necessary combined with the knowledge of the behaviour of the Fourier transform solution in a neighbourhood of the origin. More precisely it is necessary that the solution in frequency space does not grow too fast or preferably, remains bounded as time goes to infinity. We have such an estimate when $n \geq 2$.

2 Existence of Weak Solutions

In this section we construct a weak solution to the generalized Boussinesq system of equations given by,

$$\tag{2.1} \begin{cases} \omega_t + \sum_{i=1}^{n} (\omega u)_{x_i} \; = \; \Delta\omega, \\ u_t + \sum_{i=1}^{n} (\omega + \frac{u^2}{2})_{x_i} \; = \; (-1)^{p+1} \sum_{|\alpha|=2p} D^\alpha u_t + \Delta u + \Delta^{p+1} u \end{cases}$$

with data $(u_0(x),\, \omega_0(x))$.

We will first construct solutions to approximations to the above system. This will be done using the "retarded mollifier" technique, developed by Caffarelli, Kohn and Nirenberg [1] to construct weak solutions to the Navier-Stokes equations. This technique is similar to the one introduced by Leray [4]. In what follows we assume

$$\begin{aligned} u_0(x) \; &\longrightarrow \; \bar{u} \quad \text{as} \quad x \to \infty \quad \text{and} \\ \omega_0(x) \; &\longrightarrow \; \bar{\omega} \quad \text{as} \quad x \to \infty. \end{aligned}$$

For reasons that will be clear below we suppose that $\bar{u} = 0$ and $\bar{\omega} \neq 0$.

To establish the existence of solutions to (2.1) we will linearize and regularize the equations. To linearize we use a "retarded mollifier" [1]. We next recall the construction of such a mollifier. Let $\psi(x,t) \in C^\infty(I\!R^{n+1})$ be such that $\psi \geq 0$ and $\int\int_{I\!R^{n+1}} \psi dx dt = 1$. As in [1] we also choose ψ such that

$$supp(\psi) \subseteq \left\{ (x,t) \,|\, |x|^2 < t, \; 1 < t < 2 \right\}.$$

Define

$$\psi_\delta(f)(x,t) = \frac{1}{\delta^{n+1}} \int\int_{I\!R^{n+1}} \psi\left(\frac{y}{\delta}, \frac{\tau}{\delta}\right) \tilde{f}(x-y, t-\tau) dy d\tau$$

where $\tilde{f}$ is defined by

$$\tilde{f}(x,t) = \begin{cases} f(x,t) \; if \; (x,t) \in I\!R^n \times (0,T), \\ \quad 0 \quad otherwise. \end{cases}$$

Note that the values of $\psi_\delta(\tilde{f})$ at time t depend only on the values of $f(x,t)$ at times $\tau \in (t - 2\delta, t - \delta)$.

We want to obtain a solution for the approximate system

$$(2.2) \qquad \begin{cases} \omega_t + \sum_{i=1}^{n} (\psi_\delta(u)\omega)_{x_i} = \Delta\omega \\ u_t + \sum_{i=1}^{n} (\omega + \psi_\delta(u)u)_{x_i} = \sum_{|\alpha|=2p} D^\alpha u_t + \Delta u + \Delta^{p+1} u. \end{cases}$$

We notice that we could also have used (2.2) and obtained a solution via a Faedo-Galerkin method. We prefer to use the parabolic nature of the equations. We first need to establish the existence of solutions to a system of the form

$$(2.3) \qquad \begin{cases} \omega_t + \sum_{i=1}^{n} (z_1\omega)_{x_i} = \Delta\omega \\ u_t + \sum_{i=1}^{n} (\omega + z_2 u)_{x_i} = \sum_{|\alpha|=2p} D^\alpha u_t + \Delta u + \Delta^{p+1} u \end{cases}$$

where z_1 and z_2 are given smooth functions. Once a solution of (2.3) is obtained the solutions to (2.2) can be easily constructed.

Theorem 2.1.

Let $(\omega_0 - \bar{\omega},\, u_0) \in \left(C_0^2\right)^2$ *and* $u_0 \in W^{p,2}$ *where* $0 = \lim_{|x_i| \to \infty} u_0(x)$ *and* $\bar{\omega} = \lim_{|x_i|\to\infty} \omega_0(x)$ *and* $\bar{\omega} \neq 0$. *Let* $(\omega,\, u)$ *be a solution to the system (2.3) with data* $(\omega_0,\, u_0)$ *with given* $(z_1,\, z_2) \in C^\infty(\bar{D},\, I\!R^n)$ *where* $\bar{D} = I\!R^n \times (0,\, T)$. *Then there exists a unique solution*

$$(\omega^\delta - \bar{\omega},\, u^\delta) \in L^\infty\left(0,\, T;\, C^2\right)^2.$$

Proof.

Existence, uniqueness and regularity of solutions to (2.3) follow from the standard theory for uniformly parabolic systems [See Friedman [2], Chapter 9]. □

Theorem 2.2.

Let $(\omega_0,\, u_0)$, $(\bar{\omega}, 0)$ *be as in Theorem 2.1. Then there exists a unique solution* $(\omega^\delta, u^\delta)$ *of (2.2) with data* (ω_0, u_0) *such that*

$$(\omega^\delta - \bar{\omega}, u^\delta) \in L^\infty\left(0,\, T;\, C^2\right)^2.$$

Proof.

Solutions $(\omega^\delta,\, u^\delta)$ are constructed in successive intervals of length δ. For $t \in [0, \delta]$ note that $\psi_\delta(u^\delta) = \psi_\delta(\omega^\delta) = 0$, hence we have to solve a uniformly parabolic system to obtain u. The solution in the next interval can now be constructed since $\psi_\delta(u^\delta)$, $\psi_\delta(\omega^\delta)$ are defined for values of $(\omega^\delta,\, u^\delta)$ on the interval $(t - 2\delta,\, t - \delta)$ and by construction satisfy the properties of the functions z_1, z_2 of Theorem 2.1. Since this process can be repeated we obtain a solution in the interval $[0,\, T]$ satisfying (2.2). □

To obtain necessary bounds for solutions to (2.2) we need several estimates for the retarded mollifier.

Lemma 2.3

For any $u \in L^\infty(0, T;\, H) \cap L^2(0, T;\, V) \cap L^\infty(0, T;\, L^1)$ *we have*

$$sup_{0 \le t \le T} \int_{I\!R^n} |\psi_\delta(u)|^2 dx \le C sup_{0\le t \le T} \int_{I\!R^n} |u|^2 dx,$$
$$\int\int_D |\nabla \psi_\delta(u)|^2 dx dt \le \int\int_D |\nabla u|^2 dx dt,$$
$$sup_{0 \le t \le T} \int_{I\!R^n} |\psi_\delta(u)| dx \le C sup_{0\le t \le T} \int_{I\!R^n} |u| dx.$$

Proof.
The proof follows by construction of the retarded mollifier. (See [1], Lemma [A 8]) $\square$

In order to obtain a solution of (2.2) we will take $\delta = \frac{T}{N}$ (as was done in [1]) and let $N \to \infty$. To show that we converge to a weak solution we need to establish some bounds on the solutions $(\omega^\delta,\, u^\delta)$ which are independent of δ. This is done using the entropy-entropy flux pair corresponding to the underlying hyperbolic system [7]. The natural entropy was defined by

$$\tilde{\eta}(u,\,\omega) \;=\; \frac{u^2}{2} \,+\, \omega \ln \omega \;=\; \frac{u^2}{2} \,+\, \sigma(\omega).$$

We need an entropy which is positive so that we can get estimates in some good Orlicz space. This entropy given above is strictly convex but not positive since $\sigma(\omega)$ is not positive. Hence we construct a new entropy which is positive by subtracting the linear part i.e., we define a positive entropy η by

$$\begin{aligned}
\eta(u,\,\omega) \;&=\; \tilde{\eta}(u,\,\omega) - \tilde{\eta}(\bar{u},\,\bar{\omega}) - \nabla\tilde{\eta}(\bar{u},\,\bar{\omega})\,((u,\,\omega) - (\bar{u},\,\bar{\omega}))\\
&=\; \frac{(u-\bar{u})^2}{2} + \omega\,(\ln\omega - \ln\bar{\omega}) + \bar{\omega} - \omega\\
&=\; \frac{(u-\bar{u})^2}{2} + \sigma_0(\omega,\,\bar{\omega})
\end{aligned}$$

Recall that $q(u,\,\omega)$ is such that

$$\nabla\eta\cdot\nabla f \;=\; \nabla q$$

with $f \,=\, (\omega u,\, \omega + \frac{u^2}{2})$. In what follows we use the notation $H =$ Closure of C_0^∞ in $L^2(I\!R^n)$ and $V =$ Closure of C_0^∞ in the norm $(\int_{I\!R^n} |u|^2)^{\frac{1}{2}}$. The entropy-entropy flux pair (η, q) that we have just constructed is for the underlying hyperbolic system for the one-dimensional equations, i.e, for

$$U_t + F(U)_x \;=\; 0$$

where $U \;=\; \begin{pmatrix} \omega \\ u \end{pmatrix}$ and $\nabla F \;=\; \begin{pmatrix} u & \omega \\ 1 & \frac{u}{2} \end{pmatrix}$. We note that this pair $(\eta,\, q)$ can also be used for the extension to higher dimensions. In other words it will still give us the necessary apriori bounds. Recall that our new underlying hyperbolic system is

$$U_t + \sum_{i=1}^{n} F(U)_{x_i} \;=\; 0.$$

Since $\nabla\eta\nabla F \;=\; \nabla q$ we have the new conservation law

$$\eta_t + \sum_{i=1}^{n} q_{x_i} \;=\; 0.$$

Thus integrating we have by supposing that $\lim_{|x_i|\to\infty} q(x) = 0$

$$\frac{d}{dt}\int_{I\!R^n} \eta dx \le 0.$$

We also need to show that $\omega^\delta(x,t) \ge 0$. This is to ensure that $\omega^\delta \ln \omega^\delta$ is well defined. For this we first note that standard parabolic techniques yield bounds for $|u^\delta|_\infty$ and hence it follows that $|\psi_\delta(u)|_\infty \le C$. Now the usual parabolic methods will show that $\omega^\delta \ge 0$. More precisely,

Corollary 2.4

Let $\omega_0(x) \ge 0$ and $\bar\omega > 0$. Let $\omega_0(x)$ and $u_0(x)$ be as in theorem 2.1. Let $(\omega^\delta,\, u^\delta)$ be the solution to (2.3). Then for all δ

$$\omega^\delta(x,t) > 0 \quad in \quad I\!R^n \times [0,\, T], \quad T > 0.$$

Proof.

See Friedman [2] Chapter 2. Section 4 Lemma 5.

Corollary 2.5

Let $(\omega_0,\, u_0)$ be as in Corollary 2.4 with $\bar\omega > 0$. Let $(\omega^\delta,\, u^\delta)$ be a solution to (2.2). Then for all δ

$$\omega^\delta(x,t) > 0 \quad in \quad I\!R^n \times [0,\, T], \quad T > 0.$$

Theorem 2.6.

Let $(\omega_0 - \bar\omega,\, u_0) \in \left(C_0^2 \cap L^2 \cap L^1(I\!R^n)\right)^2$, $u_0 \in W^{p,2}$ and $\lim_{|x|\to\infty}(\omega_0,\, u_0) = (\bar\omega,\, 0)$ and $\bar\omega \ne 0$. Let $(\omega^\delta,\, u^\delta)$ be a solution of the system (2.2) with data (ω_0, u_0). Then

$$(\omega^\delta - \bar\omega,\, u^\delta) \in L^\infty(0,T;\, H) \cap L^2(0,T;\, V).$$

Moreover,

$$\begin{aligned} &\frac{d}{dt}\left[\frac{1}{2}\int_{I\!R^n} |u^\delta|^2 dx + \frac{1}{2}\int_{I\!R^n} |D^p u^\delta|^2 dx + \int_{I\!R^n} \sigma_0(\omega^\delta, \bar\omega) dx\right] \\ &\quad + \int_{I\!R^n} |\nabla u^\delta|^2 dx + \int_{I\!R^n} |\nabla D^p u^\delta|^2 dx + \int_{I\!R^n} \sigma_0'' |\nabla \omega^\delta|^2 dx \\ &\quad \le \epsilon(\delta) t \end{aligned} \tag{2.4}$$

with $\epsilon(\delta) \to 0$ as $\delta \to 0$. *Here* $\sigma_0(\omega^\delta, \bar\omega) = \omega^\delta\left(\ln\omega^\delta - \ln\bar\omega\right) + \bar\omega - \omega^\delta$.

Proof. Multiply equation (2.2) by $\nabla\eta$ and integrate in space and time to get

$$\begin{aligned}
&\int_{I\!R^n} \eta_t dx + \sum_{i=1}^n \int_{I\!R^n} \left(\ln\omega^\delta - \ln\bar\omega\right)\left(\psi_\delta(u^\delta)\omega^\delta_{x_i} + \bar\omega u^\delta_{x_i}\right) dx \\
&\qquad + \int_{I\!R^n} |D^p u^\delta|^2 dx + \sum_{i=1}^n \int_{I\!R^n} (u^\delta - \bar u)\psi_\delta(u^\delta) u^\delta_{x_i} dx \\
&\qquad\qquad = -\int_{I\!R^n} \sigma_0'' |\nabla\omega^\delta|^2 dx - \int_{I\!R^n} |\nabla u^\delta|^2 dx - \int_{I\!R^n} |\nabla D^p u^\delta|^2 dx.
\end{aligned} \tag{2.5}$$

Hence we need to show that

$$\sum_{i=1}^n \int_{I\!R^n} \left(\ln\omega^\delta - \ln\bar\omega\right)(\psi_\delta(u^\delta\omega^\delta)_{x_i} + \sum_{i=1}^n \int_{I\!R^n} u^\delta \psi_\delta(\omega^\delta)_{x_i} = I + II = 0 \tag{2.6}$$

and that

$$III = \sum \int (u^\delta)\psi_\delta(u^\delta) u^\delta_{x_i} dx = 0. \tag{2.7}$$

Note that in (2.5) we have used the fact that

$$\begin{aligned}
&\frac{d}{dt}\left[\int_{I\!R^n} |u^\delta|^2 + \int_{I\!R^n} \sigma_0(\omega^\delta, \bar\omega) dx + \int_{I\!R^n} |D^p u^\delta|^2 dx\right] \\
&\qquad = \int_{I\!R^n} u^\delta u^\delta_t + \int_{I\!R^n} \sigma_{0\,t}(\omega^\delta, \bar\omega) dx + \int_{I\!R^n} D^p u^\delta D^p u^\delta_t.
\end{aligned} \tag{2.8}$$

Here $III = 0$ is a consequence of $u^\delta\psi_\delta(u^\delta)u^\delta_{x_i}$ being an exact derivative i.e., let

$$G(u^\delta(x), t) = \int_{\bar u}^{u^\delta(x)} s\psi_\delta(s) ds.$$

Then $\lim_{|x|\to\infty} G(u(x), t) = 0$ since $\lim_{|x|\to\infty} u^\delta = 0$ and $\frac{\partial}{\partial x_i} G = (u^\delta)\psi_\delta(u^\delta)u^\delta_{x_i}$. Note that since $\lim_{|x|\to\infty} u_0 = 0$ by construction and by the Lebesgue dominated convergence theorem the solution satisfies $\lim_{|x|\to\infty} u^\delta(x, t) = 0$. We need to estimate $I + II$. Using the definition of I and integrating by parts yields

$$\begin{aligned}
I &= \int_{I\!R^n} \left(\ln\omega^\delta - \ln\bar\omega\right)\left(\sum_{i=1}^n (\psi_\delta(u^\delta)\omega^\delta\right)_{x_i} dx \\
&= -\sum_{i=1}^n \int_{I\!R^n} \frac{\omega^\delta_{x_i}}{\omega^\delta} \psi_\delta(u^\delta)\omega^\delta dx \\
&= -\sum_{i=1}^n \int_0^t \int_{I\!R^n} \omega^\delta_{x_i} \psi_\delta(u^\delta) dx.
\end{aligned} \tag{2.9}$$

The boundary terms vanish by hypothesis. Since we chose ψ to be symmetric by hypothesis

$$II \;=\; \sum_{i=1}^{n} \int_{I\!R^n} \omega^{\delta}_{x_i} u^{\delta} dx. \tag{2.10}$$

Hence

$$\begin{aligned} I \,+\, II \,&\leq\, \sum_{i=1}^{n} \int_{I\!R^n} |\omega^{\delta}_{x_i}||\psi_{\delta}(u^{\delta}) - u^{\delta}| dx \\ &\leq\, \frac{1}{2} \int_{I\!R^n} |\nabla \omega^{\delta}|^2 dx + \frac{1}{2} \int_{I\!R^n} |\psi_{\delta}(u^{\delta}) - u^{\delta}|^2 dx ds \\ &\leq\, \frac{1}{2} \int_0^t \int_{I\!R^n} |\nabla \omega^{\delta}|^2 dx ds + \epsilon(\delta) t \end{aligned}$$

where $\|\psi_{\delta}(u^{\delta}) - u^{\delta}\|_2^2 \,\leq\, \epsilon(\delta) \,\to\, 0$ as $\delta \to 0$. Hence (2.4) follows. Note that one can always choose δ so small that $\epsilon(\delta)T \,<\, 1$. In particular our final estimates will be independent of $\epsilon(\delta)T$ since as $\delta \to 0$ this term vanishes. $\square$

Corollary 2.7

Let $p \geq \frac{n}{2}$ and $(\omega_0,\, u_0)$ be as in Theorem 2.6. Then $(\omega^{\delta} - \bar{\omega},\, u^{\delta}) \,\in\, H^{2p} \times L_2$ uniformly in δ and

$$\begin{aligned} &i)\ |u^{\delta}|_{\infty} + |\omega^{\delta}|_{\infty} \,\leq\, C_0, \\ &ii)\ \int_{I\!R^n} |u^{\delta}|^2 dx + \sum_{|\alpha|=p} \int_{I\!R^n} |D^{\alpha} u^{\delta}|^2 dx \,\leq\, C_0, \\ &iii)\ \int_{I\!R^n} |u^{\delta}|^2 dx + \int_{I\!R^n} |\omega^{\delta} - \bar{\omega}|^2 dx \,<\, \infty, \\ &iv)\ \frac{d}{dt}\left[\|u^{\delta}(t)\|_2^2 + \|D^p u^{\delta}(t)\|_2^2 + \int_{I\!R^n} \sigma_0(\omega^{\delta},\, \bar{\omega})(t) dx\right] \\ &\qquad\qquad \leq\, -\|\nabla u^{\delta}(t)\|_2^2 - \|\nabla D^p u^{\delta}(t)\|_2^2 - \int_{I\!R^n} \sigma_0'' |\nabla \omega^{\delta}(t)|^2 dx + \epsilon(\delta) t \end{aligned}$$

where C_0 is a constant independent of δ.

Proof.

The H^p estimate for u follows from (2.4). Standard Sobolev estimatesyield

$$|u^{\delta}|_{\infty} \,\leq\, C|D^p u^{\delta}|_{L_2} \,\leq\, C_0$$

where $2p \,>\, \frac{n}{2}$. Note also that $|Du^{\delta}|_{\infty} \,\leq\, C$. This is obtained by multiplying the derivated equation by Du^{δ} and integrating by parts. Hence it follows that

$$|\psi_{\delta}(u^{\delta})|_{\infty} \,\leq\, C_0.$$

Next, note that the first equation in (2.2) is a parabolic one with nice coefficients bounded in L_∞ independent of δ. Standard arguments for parabolic equations (cf. Friedman [2]) shows that

$$|\omega^\delta|_\infty \le C_0.$$

Hence it follows that

$$\left|\frac{1}{\omega^\delta(\cdot, t)}\right| \ge \frac{1}{C_0}.$$

Recall that we have from Theorem 2.1 that

$$\int \sigma_0(\omega^\delta, \bar{\omega}) \le C_0.$$

But,

$$\sigma_0(\omega^\delta, \bar{\omega}) = \omega^\delta \left(\ln \omega^\delta - \ln \bar{\omega}\right) + \bar{\omega} - \omega.$$

Since $\sigma_0(\bar{\omega}, \bar{\omega}) = 0$ and $\sigma_{0\omega}(\bar{\omega}, \bar{\omega}) = 0$ we have

$$\sigma_0(\omega^\delta, \bar{\omega}) = \frac{1}{\omega^\star}|\omega^\delta - \bar{\omega}|^2.$$

Note that $|\omega^\delta| \le C_0$ implies that $\frac{1}{C_0} \le \frac{1}{|\omega^\delta|}$. In particular, $\frac{1}{C_0} \le |\frac{1}{\omega^\star}| = \frac{1}{\omega^\star}$. Hence it follows that

$$\frac{1}{C_0}\int |\omega^\delta - \bar{\omega}|^2 dx \le \int \frac{1}{\omega^\star}|\omega^\delta - \bar{\omega}|^2 dx = \int \sigma(\omega^\delta, \bar{\omega}) dx.$$

Thus we have

$$\int_{\mathbb{R}^n} |\omega^\delta - \bar{\omega}|^2 dx \le C_0 \int_{\mathbb{R}^n} \sigma_0(\omega^\delta, \bar{\omega})$$

and the bound (iii) follows. We note that the proof of (iv) follows from (ii) and (iii). Note that (2.4) can be expressed as

$$\begin{aligned}
&\frac{d}{dt}\left[\|u^\delta(t)\|_2^2 + \|D^p u^\delta(t)\|_2^2 + 2\int_{\mathbb{R}^n} \sigma_0(\omega^\delta, \bar{\omega})(t) dx\right] \\
&\qquad + 2\int_{\mathbb{R}^n} |\nabla u^\delta|^2 dx + 2\int_{\mathbb{R}^n} |\nabla D^p u^\delta|^2 dx + 2\int_{\mathbb{R}^n} \sigma_0''|\nabla \omega^\delta|^2 dx \\
&\qquad \le 0.
\end{aligned}$$

By (i) of Corollary 2.7 it follows that

$$\frac{1}{\omega^\star} = \left|\frac{1}{\omega^\star}\right| \ge \frac{1}{C_0},$$

i.e ω^δ admits a lower bound. We need to show that

$$\left|\frac{1}{\omega^\star}\right| \le \frac{1}{K}.$$

Note that $\bar{\omega} < \omega^\star < \omega^\delta$ implies that

$$|\frac{1}{\omega^\star}| \le \frac{1}{\bar{\omega}}$$

and we are done. If $\omega^\delta < \omega^\star < \bar{\omega}$, there are two cases to consider. First, if $\frac{\bar{\omega}}{2} < \omega^\star < \bar{\omega}$ then it follows that

$$\frac{1}{\omega^\star} = |\frac{1}{\omega^\star}| < |\frac{2}{\bar{\omega}}| = \frac{2}{\bar{\omega}}$$

and we are done. In the second case if $\omega^\delta < \omega^\star < \frac{\bar{\omega}}{2}$ we have

$$\begin{aligned}\sigma_0(\omega^\delta, \bar{\omega}) &= \omega^\delta(\ln \omega^\delta - \ln \bar{\omega}) + \bar{\omega} - \omega^\delta \\ &= \frac{1}{\omega^\star}(\omega - \bar{\omega})^2\end{aligned}$$

Since σ_0 is a convex function with minimum at $\omega = \bar{\omega}$ and $0 \le \omega^\delta \le k$

$$\begin{aligned}\sigma_0 &\le max\,\{\sigma_0(0, \bar{\omega}),\ \sigma_0(k, \bar{\omega})\} \\ &\le max\,\{\bar{\omega},\ k(\ln k + \ln \bar{\omega}) + \bar{\omega} + k)\} \\ &\le K_0.\end{aligned}$$

Hence

$$|\frac{1}{\omega^\star}|(\omega^\delta - \bar{\omega})^2 \le K_0.$$

Since $\omega^\delta < \frac{\bar{\omega}}{2}$ we obtain $(\omega - \bar{\omega})^2 \ge \left[\frac{\bar{\omega}}{2}\right]^2$. Thus we can conclude that

$$\frac{1}{\omega^\star} \le \frac{4K_0}{\bar{\omega}^2}$$

and hence,

$$|\sigma_0(\omega^\delta, \bar{\omega})| = |\frac{1}{\omega^\star}||\omega^\delta - \bar{\omega}|^2 \le \frac{4K_0}{\bar{\omega}^2}|\omega^\delta - \bar{\omega}|^2$$

and (iv) follows. □

Theorem 2.8.

Let (ω_0, u_0) and $(\bar{\omega}, \bar{u})$ be as in Theorem 2.6. Then there exists a weak solution to (1.1) with data (ω_0, u_0) satisfying

$$\begin{aligned}\frac{d}{dt}&\left[\int_{\mathbb{R}^n} |u|^2 dx + 2\int_{\mathbb{R}^n} \sigma_0(\omega, \bar{\omega}) dx + \int_{\mathbb{R}^n} |D^p u|^2 dx\right] \\ &+ 2\int_{\mathbb{R}^n} |\nabla u|^2 dx + 2\int_{\mathbb{R}^n} |\sigma_0''||\nabla \omega|^2 dx + 2\int_{\mathbb{R}^n} |\nabla D^p u|^2 dx \\ &\le 0.\end{aligned} \tag{2.11}$$

Here C_0 and C depend on the norms of the data.

Proof.

The bounds of (2.4) together with Rellich's Lemma ensure that $(\omega^\delta - \bar{\omega},\, u^\delta)$ have a weak limit. We consider the weak-formulation of the system (2.3) with $\phi \in C_0^\infty(I\!R^n \times (0,\,\infty))$. This yields

$$
(2.12)\qquad \begin{cases} \langle \phi,\, \omega_t^\delta \rangle \,+\, \langle \phi,\, \sum_{i=1}^n \big(\psi_\delta(\tilde{u})\omega^\delta\big)_{x_i} \rangle \,=\, \langle \phi,\, \Delta\omega^\delta \rangle \\ \langle \phi,\, u_t^\delta \rangle \,+\, \langle \phi,\, \sum_{i=1}^n \big(\omega_{x_i}^\delta + \psi_\delta(u) u_{x_i}^\delta\big) \,=\, \langle \phi,\, \sum_{|\alpha|=2p} D^\alpha u_t^\delta \rangle \\ \qquad\qquad + \,\langle \phi,\, \Delta u^\delta \rangle \,+\, \langle \phi,\, \Delta^{p+1} u^\delta \rangle. \end{cases}
$$

Note that by definition,

$$
\psi_\delta(\tilde{u}) \,\longrightarrow\, u \;\text{ as }\; \delta \,\to\, 0,
$$

$$
\psi_\delta(\tilde{\omega}) \,\longrightarrow\, \omega \;\text{ as }\; \delta \,\to\, 0.
$$

We also have

$$
\|\omega - \bar{\omega}\|_2 \,=\, \|\psi_\delta(\omega - \bar{\omega})\|_2 \;\text{ and }\; \|u\|_2 \,=\, \|\psi_\delta(u)\|_2.
$$

and from (2.12) it follows that

$$
(2.13)\qquad \begin{cases} \langle \phi_t,\, \omega^\delta \rangle \,+\, \langle \sum_{i=1}^n \phi_{x_i},\, \psi_\delta(u)\omega^\delta \rangle \,=\, -\langle \Delta\phi,\, \omega^\delta \rangle \quad \text{and} \\ \langle \phi_t,\, u^\delta \rangle \,-\, \sum_{i=1}^n (\omega_{x_i}^\delta + \psi_\delta(u) u_{x_i}^\delta \rangle \,=\, \langle \sum_{|\alpha|=2p} D^\alpha \phi_t,\, u^\delta \rangle \\ \qquad\qquad - \,\langle \Delta\phi,\, u^\delta \rangle \,+\, \langle \Delta^{p+1}\phi,\, u^\delta \rangle. \end{cases}
$$

Note that as $\delta \to 0$ we have

$$
(2.14)\qquad \omega^\delta \to \omega \quad \textit{weakly},
$$

$$
(2.15)\qquad u^\delta \to u \quad \textit{strongly} \quad \text{and}
$$

$$
(2.16)\qquad \psi_\delta(u) \to u \quad \textit{strongly}.
$$

Thus it is clear that the linear terms in (2.13) converge to the inner product of the appropriate derivative of ϕ with the function u or ω as the case may be. That is,

$$
(2.17)\qquad \begin{aligned} &\langle \phi_t,\, \omega^\delta \rangle \,+\, \langle \Delta\phi,\, \omega^\delta \rangle \,\longrightarrow\, \langle \phi_t,\, \omega \rangle \,+\, \langle \Delta\phi,\, \omega \rangle \quad \text{and} \\ &\langle \phi_t,\, u^\delta \rangle \,+\, \langle \Delta\phi,\, u^\delta \rangle \,-\, \langle \sum_{|\alpha|=2p} D^\alpha \phi_t,\, u^\delta \rangle \,+\, \langle \Delta^{p+1}\phi,\, u^\delta \rangle \\ &\qquad\qquad \longrightarrow\, \langle\, \phi_t,\, u \rangle \,+\, \langle \Delta\phi,\, u \rangle \,-\, \langle \sum_{|\alpha|=2p} D^\alpha \phi_t,\, u \rangle + \langle \Delta^{p+1}\phi,\, u \rangle. \end{aligned}
$$

Thus we only need to show that the nonlinear terms converge to the correct terms. From (2.14), (2.15) and (2.16) we note that

$$\psi_\delta(u)\omega^\delta \longrightarrow u\omega$$

weakly as $\delta \to 0$. Thus we get

$$\langle \sum_{i=1}^n \phi_{x_i},\, \psi_\delta(u)\omega^\delta\rangle \longrightarrow \langle \sum_{i=1}^n \phi_{x_i},\, u\omega\rangle, \tag{2.18}$$

and

$$\langle \phi,\, \sum_{i=1}^n \omega^\delta_{x_i}\rangle = -\langle \sum_{i=1}^n \phi_{x_i},\, \omega^\delta\rangle \longrightarrow -\langle \sum_{i=1}^n \phi_{x_i},\, \omega\rangle. \tag{2.19}$$

Moreover, by (2.15) and (2.16)

$$\langle \phi,\, \sum_{i=1}^n \psi_\delta(u)u^\delta_{x_i}\rangle \longrightarrow \langle \phi,\, \sum_{i=1}^n u u_{x_i}\rangle. \tag{2.20}$$

Combining (2.12), (2.17), (2.18) and (2.19) shows that $(u,\, \omega)$ the limit of $(u^\delta,\, \omega^\delta)$ is a weak solution to the Boussinesq system (2.1).

Since the bounds in (2.4) are independent of δ for each T and δ sufficiently small, the weak limit (u, ω) of the sequence $(u^\delta, \omega^\delta)$ will satisfy the bounds

$$\begin{aligned} \frac{d}{dt}\left[\int_{I\!R^n} |u|^2 dx + \int_{I\!R^n} |\sigma_0(\omega,\, \bar\omega) dx + \int_{I\!R^n} |D^p u|^2 dx\right] + \int_{I\!R^n} |\nabla u|^2 dx \\ + \int_{I\!R^n} \sigma_0''|\nabla\omega|^2 dx + \int_{I\!R^n} |\nabla D^p u|^2 dx dt \le 0. \end{aligned} \tag{2.21}$$

Note that to obtain (2.21) we have used the fact that C_0^∞ is dense in L^2 and hence if $(\omega^\delta,\, u^\delta)$ converges weakly in the sense of distributions to $(\omega,\, u)$ it also converges weakly in L^2 and the bound (2.21) follows by Fatou's Lemma. Moreover, by standard Sobolev inequalities and (2.4) we also have

$$|u|_\infty + |\omega|_\infty \le C_0.$$

□

3. Decay Results

In this section we establish the decay of the L_2-norm of the solutions to the Boussinesq system,

$$
\left\{
\begin{aligned}
\omega_t + \sum_{i=1}^{n} (\omega u)_{x_i} \; &= \; \Delta \omega, \\
u_t + \sum_{i=1}^{n} (\omega + \frac{u^2}{2})_{x_i} \; &= \; \sum_{|\alpha|=2p} \Delta^{\alpha} u_t + \Delta u + \Delta^{p+1} u
\end{aligned}
\right.
\tag{3.1}
$$

The decay results are first obtained for the approximate solutions constructed in the previous section, and then passing to the limit. For this we will use the so called Fourier splitting method [5], [6]. To implement this method an inequality which combines the time-derivative of the L_2-norm with the L_2-norm of the gradient is necessary. Such an inequality is given by (2.4). It is also necessary to have some knowledge of the Fourier transform near the origin. We use the Fourier splitting method and formally apply it to the approximating solutions and then pass to the limit.

We recall that for the Fourier splitting method the Fourier space is subdivided into two time-dependent sets one of which is a ball of radius $g(t)$. The starting point of the Fourier splitting method is a differential inequality of the form

$$\frac{d}{dt} \int |U|^2 dx \; \leq \; - \int |\nabla U|^2 dx.$$

¿From here Plancherel's theorem yields

$$\frac{d}{dt} \int |\hat{U}|^2 d\xi \; \leq \; - \int |\xi|^2 |\hat{U}|^2 d\xi.$$

Using the Fourier splitting method we have

$$\frac{d}{dt} \int |\hat{U}|^2 d\xi \; \leq \; -g^2(t) \int_{I\!R^n} |\hat{U}|^2 d\xi + g^2(t) \int_{S(t)} |\hat{U}|^2 d\xi. \tag{3.2}$$

The following conditions are imposed on g to make the Fourier splitting technique work

1) g^2 is an integrable function,

2) $g(t)$ decreases monotonically to zero and

3) $\exp(\int_0^t g^2(s)ds) \longrightarrow \infty$ as $t \longrightarrow \infty$.

In the next proposition we show that the above conditions 1, 2, 3 on g yield functions of the type

$$
\begin{aligned}
g_1(t) \; &= \; \left(\frac{\alpha}{(t+1)} \right)^{\frac{1}{2}}, \\
g_2(t) \; &= \; \left[\frac{1}{\ln{(t+e)}(t+e)} \right]^{\frac{1}{2}} \quad \text{or} \\
g_n(t) \; &= \; \left[\frac{1}{\ln \ln \cdots \ln{(t+e)}(t+e)} \right]^{\frac{1}{2}}.
\end{aligned}
$$

Since we will work with u such that $|\hat{U}(\xi,t)| \leq C_0$ for $\xi \in S(t)$ from (3.2) we obtain a differential inequality of the form

$$\frac{d}{dt}\left[y(t)h(t)\right] \leq Cg^{\frac{n}{2}+2}h(t) \tag{3.3}$$

where $h(t) = exp\left(\int_0^t g^2(s)ds\right)$ and $y(t) = |U(t)|_2^2$. The following auxiliary proposition will give the range of g which will yield the best decay rate for $y(t)$ the solution of (3.3).

Proposition 3.1

Let $g(t) \in L^2(I\!R^n)$ which satisfies

$$\int_0^t g^2(s)ds \to \infty \text{ as } t \to \infty. \tag{3.4}$$

Then,

(i) For all $\epsilon > 0$ there exists a sequence $\{t_n\}$ and a constant t (depending on ϵ) such that

$$g^2(t_n) \geq \frac{C}{(1+t_n)^{1+\epsilon}}.$$

(ii) If in addition to (3.4) $g(t)$ decreases to zero as $t \to \infty$, then for any $y(t)$ which satisfies (3.3) above we have

$$y(t) \leq C_0 max\left\{ h(t)^{-1}, g^{\frac{n}{2}} \right\}.$$

Proof. We prove the theorem by contradiction. Thus we suppose that the conclusion of the theorem does not hold. Then there exists a t_0 such that for all $t \geq t_0$

$$g(t) \leq \frac{C}{(1+t)^{1+\epsilon}}.$$

It then follows that

$$\int_0^t g^2(s)ds \leq \int_0^t \frac{C}{(1+s)^{1+\epsilon}}ds = C - \frac{C}{(1+t)^{\epsilon}},$$

contradicting (3.4).

For the second part of the theorem we integrate the differential inequality (3.3) to get

$$y(t)h(t) \leq h(0)y(0) + C\int_0^t g^{\frac{n}{2}+2}h(s)ds.$$

Note that $h^{'}(s) = h(s)g^2(s)$. An integration by parts now yields

$$y(t)h(t) \le h(0)y(0) + Ch(t)g^{\frac{n}{2}}(t) - Ch(0)g^{\frac{n}{2}}(0) - C\int_0^t h(s)g^{\frac{n}{2}-1}g^{'}(s)ds.$$

Since $g^{'} < 0$ with $g > 0$ and $h > 0$ and increasing we have

$$\begin{aligned} y(t)h(t) &\le h(0)y(0) + Ch(t)g^{\frac{n}{2}}(t) + C\int_0^t h(s)g^{\frac{n}{2}-1}(-g^{'}(s))ds \\ &\le h(0)y(0) + Ch(t)g^{\frac{n}{2}}(t) + Ch(t)\int_0^t g^{\frac{n}{2}-1}(-g^{'}(s))ds \\ &\le h(0)y(0) + Ch(t)g^{\frac{n}{2}}(t) + Ch(t)g^{\frac{n}{2}}(t). \end{aligned}$$

Hence,

$$y(t) = \frac{h(0)y(0)}{h(t)} + Cg^{\frac{n}{2}}(t)$$

which completes the proof of the proposition. □

Let

$$\mathcal{A} = \{g \in L_1 : g(0) = 1,\ g\ satisfying\ (i)\ and\ (ii)\ of\ Proposition\ 3.1\}.$$

Corollary 3.2.
Let y_g be the solution of the equation (3.3). Then the $g \in \mathcal{A}$ which makes $\|y_g\|_2$ decay the fastest satisfies

$$if\ g^{\frac{n}{2}}(t) \ge h(t)^{-1}\ then\ \|y\|_2 \le \left(\frac{C}{1+t}\right)^{\frac{n}{2}} \tag{3.5}$$

and

$$if\ g^{\frac{n}{2}}(t) \le h(t)^{-1}\ then\ \|y_g\|_2 \le \frac{1}{\ln(t+e)}. \tag{3.6}$$

Proof.
Conditions (3.5) and (3.6) follow since

$$y_g = y(t) \le C_0 max\left\{h(t)^{-1}, g^{\frac{n}{2}}(t)\right\}.$$

Let

$$f(t) = exp\left(\int_0^t g^2\right) g^{\frac{n}{2}}(t).$$

Then

$$f'(t) = g^{\frac{n}{2}} g^2 exp\left(\int_0^t g^2\right) + g^{\frac{n}{2}-1} g' exp\left(\int_0^t g^2\right). \tag{3.7}$$

This implies that if $f'(t) > 0$ then $f(t) > f(0)$ which is equivalent to the statement that $g^{\frac{n}{2}} > h(t)^{-1}$. Hence we see that if $f' > 0$ then $-g' < g^3$ and (3.7) yields

$$g(t) \leq \frac{C_\star}{(1+t)^{\frac{1}{2}}}.$$

Hence the largest value of g^2 is $\frac{C}{(1+t)}$ and (3.5) follows from Proposition 3.1.

On the other hand if $f' < 0$ then $f(t) < f(0)$ which implies that $h^{-1}(t) > g^{\frac{n}{2}}$. In this case an analysis as above yields

$$g^2(t) \leq \frac{C}{(e+t)} \leq \frac{C}{(e+t)\ln(e+t)} \leq \cdots .$$

etc. In particular if $g^2 = \frac{C}{(e+t)}$ or if $g^2(t) = \frac{C}{(e+t)\ln(e+t)}$ etc, then we have

$$\int g^2(s) = C\ln(t+e) \text{ or } \int g^2(s) = \ln\ln(t+e)$$

etc, which yields

$$h^{-1}(t) \leq \frac{1}{\ln(t+e)} \text{ or } h^{-1}(t) \leq \frac{1}{\ln\ln(t+e)}$$

etc. We remark here that if we choose $g^2(s) = \frac{C}{(e+t)^{1+\epsilon}}$, then $\int g^2(s)ds = \frac{C}{(e+t)^{\epsilon}}$. This implies that in particular that $h^{-1}(t) \to 1$ as $t \to \infty$. Thus this form of g will not give us any decay. This completes the proof of the corollary using Proposition (3.1). □

Corollary 3.3

From Corollary 3.2 it follows that the g that gives the best decay must have the form

$$g_1(t) = \left(\frac{\alpha}{(t+1)}\right)^{\frac{1}{2}},$$

$$g_2(t) = \left[\frac{1}{\ln(t+e)(t+e)}\right]^{\frac{1}{2}} \quad or$$

$$g_n(t) = \left[\frac{1}{\ln\ln\cdots\ln(t+e)(t+e)}\right]^{\frac{1}{2}}.$$

□

Recall that

$$\mathcal{X}_\beta = \left\{(u_0,\omega_0) : |u_0(t)|^2_{L_2} \leq C(t+1)^{-\beta},\ |\omega_0(t)-\bar{\omega}|^2_{L_2} \leq C(t+1)^{-\beta}\right\}.$$

We now establish the main theorem of the paper.

Theorem 3.4.

Let $(\omega_0 - \bar{\omega},\, u_0) \,\in\, \left(C_0^2 \cap H^{2p} \cap L^1 \cap \mathcal{X}_{\alpha_0}\right) \times \left(C_0^2 \cap H^{2p} \cap L^1 \cap \mathcal{X}_{\alpha_0}\right)(I\!\!R^n)$ *with* $n \,\geq\, 2$. *Assume that* $(\omega_0(t), u_0(t)) \in \mathcal{X}_{\alpha_0}$. *Here* $0 \,=\, \lim_{|x_i|\to\infty} u_0(x)$ *and* $\bar{\omega} \,=\, \lim_{|x_i|\to\infty} \omega_0(x)$. *If* $n \,>\, 2$, *then the weak solution constructed in Section 2 satisfies*

$$|u(\cdot,\, t)|_2^2 \,+\, |\omega(\cdot,\, t) - \bar{\omega}| \,+\, |D^p u(\cdot,\, t)|_2^2 \,\leq\, C(t+1)^{-\frac{n}{2}}. \tag{3.8}$$

If $n \,=\, 2$ *the solution satisfies the estimate*

$$|u(\cdot,\, t)|_2^2 \,+\, |\omega(\cdot,\, t) - \bar{\omega}| \,+\, |D^p u(\cdot,\, t)|_2^2 \,\leq\, C[\ln(t+e)]^m \tag{3.9}$$

for all $m \geq 3$. *Here* C *dependsonly on the norms of the data.*

Proof.

The proof will first be given for the approximating sequences. Let (ω, u) be the approximate solution (the solution of (2.2).) Making the change of variables $z \,=\, \omega - \bar{\omega}$ and $y \,=\, D^p u$ we find that (z, u) satisfy the following system of equations

$$\begin{aligned} z_t + \sum(\psi_\delta(u)z)_{x_i} + \sum_{i=1}^{n} \bar{\omega} u_{x_i} \,&=\, \Delta z \\ u_t + \sum_{i=1}^{n}(z + \frac{1}{2}\psi_\delta(u)u)_{x_i} \,&=\, (-1)^{p+1} \sum_{|\alpha|=2p} D^\alpha u_t + \Delta u + \Delta^{p+1} u \end{aligned} \tag{3.9}$$

We wish to obtain the decay rate of the solution. We note that by Theorem 2.6 we have the inequality,

$$\frac{d}{dt}\left[\|u(t)\|_2^2 + \|y(t)\|_2^2 + \int_{I\!\!R^n} \sigma_0(\omega, \bar{\omega})dx\right] \,\leq\, -\|\nabla u(t)\|_2^2 - \|\nabla y(t)\|_2^2 - \int_{I\!\!R^n} \sigma_0''|\nabla\omega|^2 dx. \tag{3.10}$$

We wish to use the Fourier splitting method. For this, we need the integrand on the right-hand side of (3.10) to be the gradient of the terms on the left-hand side. Thus in particular we need to obtain a bound of the type

$$-\int_{I\!\!R^n} \sigma_0''|\nabla\omega|^2 dx \,\leq\, -C\int_{I\!\!R^n} |\nabla\sigma_0^{\frac{1}{2}}|^2 dx \tag{3.11}$$

where C is a fixed constant independent of ω and u. Rearranging terms this is equivalent to showing that

$$4\sigma_0''\sigma_0 \,\geq\, C{\sigma_0'}^2.$$

Explicitly writing out the expression for σ_0, σ_0' and σ_0'' this implies that we must show that

$$\frac{\omega(\ln\omega - \ln\bar{\omega}) + \bar{\omega} - \omega}{\omega|\ln\omega - \ln\bar{\omega}|^2} \geq C \tag{3.12}$$

for all ω. For this, we calculate the minimum value of the expression on the left-hand side of (3.12). A straightforward calculation then shows that this condition is always satisfied for a suitable constant C. Hence (3.11) always holds and (3.10) then implies that

(3.13)

$$\frac{d}{dt}\left[\|u(t)\|_2^2 + \|y(t)\|_2^2 + \int_{I\!R^n} \sigma_0(\omega,\bar{\omega})dx\right] \leq -\|\nabla u(t)\|_2^2 - \|\nabla y(t)\|_2^2 - \int_{I\!R^n} |\nabla {\sigma_0'}^{\frac{1}{2}}|^2 dx.$$

We can now apply the Fourier splitting method to (3.13) to obtain the decay rate for the solution. Rewriting (3.13) using Plancherel's theorem, we obtain

$$\begin{aligned}\frac{d}{dt}&\left[\int_{I\!R^n} |\mathcal{F}u(t)|^2 d\xi + \int_{I\!R^n} |\mathcal{F}y(t)|^2 d\xi + \int_{I\!R^n} |\mathcal{F}\sigma_0^{\frac{1}{2}}(\omega,\bar{\omega})|^2 dx\right] \\ &\leq -\int_{I\!R^n} |\xi|^2|\mathcal{F}u|^2 d\xi - \int_{I\!R^n} |\xi|^2|\mathcal{F}y(t)|^2 d\xi - \int_{I\!R^n} |\xi|^2|\mathcal{F}\sigma_0^{\frac{1}{2}}|^2 d\xi\end{aligned} \tag{3.14}$$

If $n > 2$, we define the set $S(t)$ by

$$S(t) = \left\{\xi : |\xi| \leq \left(\frac{4n}{(t+1)}\right)^{\frac{1}{2}}\right\}.$$

Writing the integrals over space on the right-hand side of (3.14) as the sum of integrals over the set $S(t)$ and it compliment $S(t)^c$, we get

$$\begin{aligned}\frac{d}{dt}&\left[\int_{I\!R^n} |\mathcal{F}u(t)|^2 d\xi + \int_{I\!R^n} |\mathcal{F}y(t)|^2 d\xi + \int_{I\!R^n} |\mathcal{F}\sigma_0^{\frac{1}{2}}(\omega,\bar{\omega})|^2 dx\right] \\ &\leq -\int_{S(t)} |\xi|^2|\mathcal{F}u(t)|^2 d\xi - \int_{S(t)} |\xi|^2|\mathcal{F}y(t)|^2 d\xi - \int_{S(t)} |\xi|^2|\mathcal{F}\sigma_0^{\frac{1}{2}}(t)|^2 d\xi \\ &\quad -\int_{S(t)^c} |\xi|^2|\mathcal{F}u(t)|^2 d\xi - \int_{S(t)^c} |\xi|^2|\mathcal{F}y(t)|^2 d\xi - \int_{S(t)^c} |\xi|^2|\mathcal{F}\sigma_0^{\frac{1}{2}}(t)|^2 d\xi\end{aligned} \tag{3.15}$$

Using the definition of the set $S(t)$ we get

$$\begin{aligned}\frac{d}{dt}&\left[\int_{I\!R^n} |\mathcal{F}u(t)|^2 d\xi + \int_{I\!R^n} |\mathcal{F}y(t)|^2 d\xi + \int_{I\!R^n} |\mathcal{F}\sigma_0^{\frac{1}{2}}(\omega,\bar{\omega})(t)|^2 dx\right] \\ &\leq -\frac{4n}{(t+1)}\int_{S(t)^c} |\mathcal{F}u|^2 d\xi - \frac{4n}{(t+1)}\int_{S(t)^c} |\mathcal{F}y(t)|^2 d\xi - \frac{4n}{(t+1)}\int_{S(t)^c} |\mathcal{F}\sigma_0^{\frac{1}{2}}(t)|^2 d\xi.\end{aligned}$$

Hence we can write the inequality as follows

$$\begin{aligned}
&\frac{d}{dt}\left[(t+1)^{4n}\int_{I\!R^n}\left(|\mathcal{F}u(t)|^2+|\mathcal{F}y(t)|^2+|\mathcal{F}\sigma_0^{\frac{1}{2}}(t)|^2\right)d\xi\right]\\
&\qquad \le 4n(t+1)^{4n-1}\int_{S(t)}|\mathcal{F}u(t)|^2d\xi+4n(t+1)^{4n-1}\int_{S(t)}|\mathcal{F}y(t)|^2d\xi\\
&\qquad\qquad +n(t+1)^{4n-1}\int_{S(t)}|\mathcal{F}\sigma_0^{\frac{1}{2}}(t)|^2d\xi.
\end{aligned}\tag{3.16}$$

Thus it remains to estimate the integrals on the right-hand side of (3.16). In what follows we consider (3.9) as a system of equations rather than treating each equation separately. This enables us to simplify the ensuing calculations. Taking the Fourier transform of (3.9) and using matrix notation we obtain,

$$\begin{aligned}
\frac{d}{dt}\begin{pmatrix}\mathcal{F}(z(t))\\ \mathcal{F}(u(t))\end{pmatrix}+\left[\begin{pmatrix}|\xi|^2 & 0\\ 0 & \frac{|\xi|^2+|\xi|^{2(p+1)}}{1+|\xi|^{2p}}\end{pmatrix}+i\begin{pmatrix}0 & \bar{\omega}\sum_1^n\xi_j\\ \frac{\sum_1^n\xi_j}{1+|\xi|^{2p}} & 0\end{pmatrix}\right]\cdot\begin{pmatrix}\mathcal{F}(\omega(t))\\ \mathcal{F}(u(t))\end{pmatrix}\\
+i\begin{pmatrix}\sum_1^n\xi_j\mathcal{F}(\psi_\delta(u)\omega)\\ \frac{1}{2}\frac{\sum_1^n\xi_j}{1+|\xi|^{2p}}\mathcal{F}(\psi_\delta(u)u)\end{pmatrix}.
\end{aligned}$$

Solving this matrix equation we get,

(3.17)

$$\begin{pmatrix}\mathcal{F}(z(t))\\ \mathcal{F}(u(t))\end{pmatrix}=\begin{pmatrix}\mathcal{F}(z_0)\\ \mathcal{F}(u_0)\end{pmatrix}exp(-At)+\int_0^t exp(-A(t-s))\begin{pmatrix}\sum_1^n\xi_j\mathcal{F}(\psi_\delta(u)z)\\ \frac{1}{2}\sum_1^n\frac{\xi_j}{1+|\xi|^{2p}}\mathcal{F}(\psi_\delta(u)u)\end{pmatrix}ds.$$

Here z_0 and u_0 denote the initial data for $z(t)$ and $u(t)$ respectively and the matrix A is given by

$$A=\begin{pmatrix}|\xi|^2 & \bar{\omega}\sum_1^n\xi_j\\ \sum_1^n\frac{\xi_j}{1+|\xi|^{2p}} & \frac{|\xi|^2+|\xi|^{2(p+1)}}{1+|\xi|^{2p}}\end{pmatrix}.$$

A simple but tedious calculation (and as such will be omitted) shows that for $|\xi|$ sufficiently small

1. The eigenvalues $\lambda_1,\ \lambda_2$ have positive real part and nonzero imaginary part.

2. We write $exp(-At)=Pexp(-\Lambda t)P^{-1}$ where Λ is the diagonal matrix with entries λ_1 and λ_2. and P is the eigenvector matrix. Hence using the Hahn-Banach Theorem there exists ϕ such that for $q\ge 1$

$$\|exp(-At)\|=\|exp(-At)\phi\|_{L^q}=\|Pexp(-\Lambda t)P^{-1}\phi\|_{L^q}$$

¿From the explicit calculations of $\Lambda,\ P$ and P^{-1} it follows that

$$\|exp(-At)\|\le M_0$$

is finite and hence we have

$$\|exp(-At)H\| \leq M_0\|H\|_{L^q}$$

for $q = 2, \infty$. Here we use

$$|\mathcal{F}D^p u|^2 = |\xi|^2|\mathcal{F}(y)|^2 \leq |\mathcal{F}(y)|^2$$

if $|\xi| \leq 1$. Now let

$$\gamma_0(t) = \|exp(-At)\begin{pmatrix}\mathcal{F}(z_0)\\ \mathcal{F}(u_0)\end{pmatrix}\|_2^2$$

That is, $\gamma_0(t)$ is the L_2 norm of the underlying linear equation.

¿From (3.17) it then follows that

$$\left|\begin{pmatrix}\mathcal{F}(z(t))\\ \mathcal{F}(u(t))\end{pmatrix}\right|_\infty \leq C + Ct^{\frac{1}{2}} \tag{3.18}$$

It remains to estimate $|\mathcal{F}\sigma_0^{\frac{1}{2}}|$ over the set $S(t)$. By definition, we can write

$$\sigma_0(\omega,\bar{\omega}) = \frac{1}{\bar{\omega}}(\omega-\bar{\omega})^2 + \frac{1}{\bar{\omega}}(\omega-\bar{\omega})^3 + \frac{1}{\omega^\star}(\omega-\bar{\omega})^4.$$

Hence it follows that

$$\sigma_0^{\frac{1}{2}}(\omega,\bar{\omega}) \leq \sqrt{\frac{1}{\bar{\omega}}}|\omega-\bar{\omega}| + \sqrt{\frac{1}{\bar{\omega}}}|\omega-\bar{\omega}|^{\frac{3}{2}} + \sqrt{\frac{1}{\omega^\star}}|\omega-\bar{\omega}|^2$$

which implies that

$$\mathcal{F}(\sigma_0^{\frac{1}{2}}) \leq \sqrt{\frac{1}{\bar{\omega}}}\mathcal{F}(|\omega-\bar{\omega}|) + \frac{1}{\omega^\star}\mathcal{F}(|\omega-\bar{\omega}|^{\frac{3}{2}}) + \mathcal{F}(\frac{1}{\sqrt{\omega^\star}}|\omega-\bar{\omega}|^2)$$

¿From (3.18) it follows that

$$\sqrt{\frac{1}{\bar{\omega}}}|\mathcal{F}(\omega-\bar{\omega})|_\infty \leq Ct^{\frac{1}{2}} \tag{3.19}$$

and that

$$\begin{aligned}|\mathcal{F}(\frac{1}{\sqrt{\omega^\star}}|\omega-\bar{\omega}|^2)|_\infty &= \left|\int_{-\infty}^{\infty}\frac{1}{\sqrt{\omega^\star}}|\omega-\bar{\omega}|^2 e^{ix\xi}d\xi\right| \\ &\leq C|\omega-\bar{\omega}|_2^2 \\ &\leq C\end{aligned} \tag{3.20}$$

Hence it remains to estimate the L_∞-norm of the term $\frac{1}{\omega^\star}\mathcal{F}(|\omega-\bar{\omega}|^{\frac{3}{2}})$. Since

$$|\mathcal{F}(|\omega-\bar{\omega}|^{\frac{3}{2}})|_\infty \le \int_{I\!R^n} |\omega-\bar{\omega}|^{\frac{3}{2}}dx$$

and

$$\left(\int_{I\!R^n} |\omega-\bar{\omega}|^{\frac{3}{2}}dx\right)^{\frac{2}{3}} \le \left(\int_{I\!R^n} |\mathcal{F}(\omega-\bar{\omega})|^3\,dx\right)^{\frac{1}{3}}$$

we have

$$\begin{aligned}|\mathcal{F}(|\omega-\bar{\omega}|^{\frac{3}{2}})|_\infty &\le \left(\int_{I\!R^n} |\mathcal{F}(\omega-\bar{\omega})|^3\right)^{\frac{1}{2}}\\ &\le C|\mathcal{F}(\omega-\bar{\omega})|\left(\int_{I\!R^n} |\mathcal{F}(\omega-\bar{\omega})|dx\right)^{\frac{1}{2}}\end{aligned}$$

Moreover, we have

$$|\mathcal{F}(\omega-\bar{\omega})| = \mathcal{F}(\omega_0)exp(-At) + \int_0^t exp(-A(t-s))\mathcal{F}((\omega u)_x).$$

Hence,

(3.21)

$$\begin{aligned}\int_{I\!R^n} |\mathcal{F}(\omega-\bar{\omega})|dx &\le \int_{I\!R^n} |\mathcal{F}(\omega_0)|dx + \int_0^t\int_{I\!R^n} exp(-A(t-s))\frac{(1+|\xi|^{2p})}{(1+|\xi|^{2p})}\mathcal{F}((\omega u)_x)d\xi d\tau\\ &\le \int_{I\!R^n} |\mathcal{F}(\omega_0)|dx\\ &+ \int_0^t\left(\int_{I\!R^n}\frac{exp(-2A(t-s))}{(1+|\xi|^{2p})^2}d\xi\right)^{\frac{1}{2}}\left(\int_{I\!R^n}(1+|\xi|^{2p})^2|\mathcal{F}((\omega u)_x)|^2d\xi\right)^{\frac{1}{2}}d\tau\end{aligned}$$

The last integral on the right-hand side of (3.18) can be can be bounded by a constant, since the term

$$\left(\int_{I\!R^n}(1+|\xi|^{2p})^2|\mathcal{F}((\omega u)_x)|^2d\xi\right)$$

is bounded if the derivatives of ω and u are bounded. This is true provided that the initial data is sufficiently smooth. The term

$$\int_{I\!R^n}\frac{exp(-2A(t-s))}{(1+|\xi|^{2p})^2}d\xi$$

is bounded since $2p > n$.

Hence we find that

$$|\mathcal{F}(|\omega-\bar{\omega}|^{\frac{3}{2}})| \le C. \tag{3.22}$$

Using the estimates obtained in (3.17) and (3.18) in (3.15) we obtain

$$\frac{d}{dt}\left[(t+1)^{4n}\int_{\mathbb{R}^n}\left(|\mathcal{F}u(t)|^2+|\mathcal{F}y(t)|^2+|\mathcal{F}\sigma_0^{\frac{1}{2}}(t)|^2\right)d\xi\right]$$
$$\leq\ 4n(t+1)^{4n-1}(t+1)(t+1)^{-\frac{n}{2}}+4n(t+1)^{4n-1}(t+1)(t+1)^{-\frac{n}{2}}$$
$$+\,C4n(t+1)^{4n-1}(t+1)^{-\frac{n}{2}}$$

Thus integrating this last inequality with respect to time, and simplifying the result, we obtain

$$\|u(t)\|_2^2+\|y(t)\|_2^2+\int_{\mathbb{R}^n}\sigma_0 dx\ \leq\ C(t+1)^{-(\frac{n}{2}-1)}$$

In particular this implies that

$$\int_{\mathbb{R}^n}\sigma_0 dx\ \leq\ C(t+1)^{-(\frac{n}{2}-1)}$$

By Corollary 2.7 we know that there is a constant C_0 such that

$$\|\omega(t)\|_2^2\ \leq\ C_0\int_{\mathbb{R}^n}\sigma_0 dx, \tag{3.23}$$

which implies that

$$\|u(t)\|_2^2+\|y(t)\|_2^2+\|\omega(t)\|_2^2\ \leq\ C(t+1)^{-(\frac{n}{2}-1)}. \tag{3.20}$$

Using this estimate in (3.17) we get

$$\left|\begin{pmatrix}\mathcal{F}(z(t))\\ \mathcal{F}(u(t))\end{pmatrix}\right|_\infty\ =\ C \tag{3.25}$$

Using (3.25) in (3.16) and simplifying the result we obtain on integration the estimate

$$\|u(t)\|_2^2+\|y(t)\|_2^2+\int_{\mathbb{R}^n}\sigma_0 dx\ \leq\ C(t+1)^{-\frac{n}{2}}.$$

Moreover, from (3.23) it then follows that

$$\|u(t)\|_2^2+\|y(t)\|_2^2+\|\omega(t)\|_2^2\ \leq\ C(t+1)^{-\frac{n}{2}}.$$

Thus this proves the theorem in the case when $n>2$. We treat the case $n=2$ separately. In this case we choose

$$S(t)\ =\ \{\xi\ :\ |\xi|\ \leq\ g(t)\} \tag{3.26}$$

where $g(t) = \left(\frac{3}{(t+e)\ln(t+e)}\right)^{\frac{1}{2}}$. Using the definition of the set $S(t)$ as given in (3.26) in (3.15) and simplifying the resulting inequality we get (after some manipulations),

(3.27)

$$\frac{d}{dt}\left[[\ln(t+e)]^3 \int_{\mathbb{R}^n} \left(|u(t)|^2 + |y(t)|^2 + \sigma_0(\omega,\bar{\omega})9t)\right) dx\right] \leq C\frac{1}{\ln(t+e)(t+e)}[\ln(t+e)]^3$$
$$\int_{S(t)} \left(|\mathcal{F}(u(t))|^2 + |\mathcal{F}(y(t))|^2 + |\mathcal{F}(\sigma_0^{\frac{1}{2}})(t)|^2\right) d\xi$$

We thus need to estimate $\mathcal{F}(u)$ and $\mathcal{F}(\sigma_0^{\frac{1}{2}})$ on the set $S(t)$. Note that the term $\mathcal{F}(u)$ is evaluated as follows

$$\left|\begin{pmatrix}\mathcal{F}(\omega)\\ \mathcal{F}(u)\end{pmatrix}\right| \leq e^{-At}\left|\begin{pmatrix}\mathcal{F}(\omega_0)\\ \mathcal{F}(u_0)\end{pmatrix}\right| + \int_0^t |\xi| \left|\begin{pmatrix}\mathcal{F}(\omega u)\\ \mathcal{F}(u^2)\end{pmatrix}\right| e^{-A(t-s)}ds.$$

Hence it follows that

$$\begin{aligned}\left|\begin{pmatrix}\mathcal{F}(\omega)\\ \mathcal{F}(u)\end{pmatrix}\right|_\infty &\leq C + C|g(t)|t \\ &\leq C\left(\frac{(t+e)}{\ln(t+e)}\right)^{\frac{1}{2}}.\end{aligned} \tag{3.28}$$

Using (3.28) in (3.27) we obtain,

$$\begin{aligned}\frac{d}{dt}\left[[\ln(t+e)]^3 \int_{\mathbb{R}^n} \left(|u(t)|^2 + |y(t)|^2 + \sigma_0(\omega,\bar{\omega})(t)\right) dx\right] &\leq C\frac{1}{\ln(t+e)(t+e)}[\ln(t+e)]^3 \\ &\quad \cdot \frac{1}{\ln(t+e)(t+e)} \\ &= \frac{C}{(t+e)}\end{aligned}$$

Integrating this inequality in time yields

$$\begin{aligned}\int_{\mathbb{R}^n}\left(|u(t)|^2 + |y(t)|^2 + \sigma_0(\omega,\bar{\omega})(t)\right) dx \leq &\frac{1}{[\ln(t+e)]^3}\int_{\mathbb{R}^n}\left(|u_0|^2 + |D^p u_0|^2 + \sigma_0(\omega_0,\bar{\omega})\right) dx \\ &+ \frac{C}{[\ln(t+e)]^2}\end{aligned}$$

We now use induction to show that if

$$\int_{\mathbb{R}^n}\left(|u|^2 + |D^p u|^2 + \sigma_0(\omega,\bar{\omega})\right) dx \leq [\ln(t+e)]^{-k}$$

for $k < m$, then the decay can be improved to $k = m$. The method is standard and analogous to that given in [6] and is thus omitted. This completes the proof of the theorem. □

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