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Energy Decay of Solutions to the Boussinesq, Primitive, and Planetary Geostrophic Equations

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In this paper we study the long time behavior of the energy of solutions to the Boussinesq, planetary geostrophic, and primitive equations. The equations are considered in the whole space R^3. The asymptotic behavior will depend on the type of data and how many damping constants are nonzero in the equations. In several cases we are able to establish an algebraic rate of decay of the same order as the solutions of the underlying linear equations. In the case with less damping our results establish that either the energy of the solutions decays with no rate to an equilibrium or it will be oscillating. © 1999 Academic Press

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1. INTRODUCTION

The Navier–Stokes equations are usually considered to be the basic equations for modeling atmospheric and oceanic flow phenomena. However, in three dimensions the equations are very complex, possibly ill behaved, and contain phenomena, such as sound waves, that are not normally considered important in geophysical flows. For these reasons the equations are normally simplified using rational physical approximation and asymptotic methods before analytic or numerical solutions are sought. Among the various simpler sets of equations are the Boussinesq equations, the so-called primitive equations, and the planetary geostrophic equations.

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In this paper we study the energy decay properties of these sets of equations.

In the Boussinesq equations (e.g., Tritton [10]) variations in density are considered only when coupled to gravity; the mass conservation equation is simple conservation of volume. These equations are commonly used for modeling convection in liquids. The primitive equations additionally impose hydrostatic balance in the vertical direction, and are commonly used to model large-scale atmospheric and oceanic flow. (In this paper we consider the primitive equations as a simplification of the Boussinesq equations, and not the slightly differing primitive equations appropriate for gases.) The planetary geostrophic equations make an additional simplification: the inertial terms are ignored in the horizontal momentum equation, and geostrophic balance is assumed, possibly with a small frictional correction. These equations are useful as a model of very large scale flow in the ocean and atmosphere by Phillips in [4].

The Boussinesq equations in a frame of reference rotating about the vertical axis, and with no thermodynamic source term, may be written

$$\frac{DU}{Dt} + f\mathbf{k} \times U = -\nabla p + \nu \Delta U - \mathbf{k} g\theta \tag{1.1}$$

$$\frac{D\theta}{Dt} = \kappa \Delta \theta \tag{1.2}$$

$$\nabla \cdot U = 0. \tag{1.3}$$

In these equations, the mean density is taken as unity, g is a constant (henceforth also set to unity). U is the three dimensional velocity field, θ is proportional to temperature, and p is equivalent to the pressure. (We use the notation $U = (u, v, w)$ and $V = (u, v)$.) f is the Coriolis parameter, and ν and κ are constant coefficients of viscosity and diffusivity.

It is common when considering large-scale geophysical flows to suppose that the vertical accelerations are small compared to gravitational or buoyancy forces, and that "hydrostatic balance" holds. This leads to the "primitive equations" as described by Holton in [2], commonly used for weather forecasting and other large numerical simulations. To represent this, the equations of motion are written

$$\frac{DV}{Dt} + f\mathbf{k} \times U = -\nabla p + \nu \Delta V \tag{1.4}$$

$$\alpha \frac{Dw}{Dt} + g\theta = -\frac{\partial p}{\partial z} + \gamma \nu \Delta w \tag{1.5}$$

$$\frac{D\theta}{Dt} = \kappa\Delta\theta \tag{1.6}$$

$$\nabla \cdot U = 0 \tag{1.7}$$

where $\alpha = 1$ and $\gamma = 1$ in the Boussinesq equations, and $\alpha = 0$ and $\gamma = 0$ for the primitive equations.

For large-scale flow further simplification is possible. Appropriate scaling and asymptotic analysis leads to the "planetary geostrophic (PG) equations," namely,

$$-fv = -\frac{\partial p}{\partial x} + \nu\Delta u - \epsilon u \tag{1.8}$$

$$fu = -\frac{\partial p}{\partial y} + \nu\Delta v - \epsilon v \tag{1.9}$$

$$g\theta = -\frac{\partial p}{\partial z} - rw \tag{1.10}$$

$$\frac{D\theta}{Dt} = \kappa\Delta\theta - k\theta \tag{1.11}$$

$$\nabla \cdot U = 0 \tag{1.12}$$

These equations are, respectively, approximations to the momentum equations in the x, y, and z directions, a thermodynamic equation and volume conservation. Additional dissipative ("Rayleigh damping") terms have been added to the momentum equation and thermodynamic equations: In all such large-scale equations for large-scale flow, the scales at which molecular dissipation is important are hopelessly unresolved, and rather *ad hoc* frictional and diffusive terms are often added. The PG equations, or variations around them, have been used extensively in theoretical and analytic studies of the large-scale ocean circulation (e.g., Samelson and Vallis [5]).

Since a number of simplifications have plainly been made in deriving all these equations, and the frictional terms are *ad hoc*, it is important to understand their properties. One wishes to know whether the system one is dealing with is well behaved, or has pathological characteristics which might lead to singularities (as, for example, in Burgers equation). It is for this reason that we are interested in the general properties of solutions.

Interest will be focused on several cases. First is the case where the diffusion of the temperature is reduced by setting $k = 0$ and we add diffusion in the third variable of the velocity, i.e., in w. This case is analyzed in order to explain the technique of Fourier splitting which will give decay of the energies. Second we suppose that there is no diffusion in $w, k > 0$ and the energy of w goes to a finite limit. The case where the diffusion in w is zero but $k > 0$ is also considered. We also study the case where there is no diffusion in the third variable and $k = 0$ but here we need additional hypotheses on the behavior of the energy of w. Finally we consider the quasi-stationary case, where $k = 0$ and $\nu = 0$ and the dependence on time of the velocity is through the temperature. It is interesting to note that in most cases we get the same algebraic rate of decay, indicating that the temperature is probably driving the energy of the velocity to a zero equilibrium as time increases.

Ideally, one would like to prove global regularity. Such a proof is beyond the scope of this paper; here we restrict ourselves to proving energy decay under various circumstances.

2. SPECIFIC EQUATIONS

We analyze the above equation sets in whole space. For specificity, we first consider a single set of equations representing both the Boussinesq and hydrostatic primitive equations, using parameters that are either zero or unity to differentiate between them. We write the equations as

$$\frac{\partial U_\alpha}{\partial t} + U \cdot \nabla U_\alpha + fAU = -\nabla p - BU + \bar{\theta} + \nu \tilde{\Delta} U \tag{2.1}$$

$$\frac{\partial \theta}{\partial t} + U \cdot \nabla \theta = \kappa_h \left[\theta_{xx} + \theta_{yy} \right] + \kappa_\nu \theta_{zz} - k\theta \tag{2.2}$$

$$\nabla \cdot U = 0 \tag{2.3}$$

where $U_\alpha = (u, v, \alpha w)$ is the velocity vector, p is the pressure, $\bar{\theta} = (0, 0, \theta)^t$, and f is the Coriolis parameter which may be a function of y. The notation $\tilde{\Delta}$ is used to indicate that the Laplacian in the coordinate w is multiplied by a constant γ which in some cases will be zero. Specifically $\tilde{\Delta} = (\Delta, \Delta, \gamma\Delta)$. The matrices A and B are given by

$$A = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} \epsilon & 0 & 0 \\ 0 & \epsilon & 0 \\ 0 & 0 & r \end{pmatrix} \tag{2.4}$$

The constants ϵ, r, κ_h, κ_ν and k are frictional, damping, and diffusive coefficients of various types. These constants are greater or equal to zero. In what follows we will always specify which of the constants are zero. Making any of the constants zero is equivalent to removing some of the damping and hence the decay will be slower. The "ideal" equations have all these parameters set to zero. However, this case is not the most physically realistic, nor will the method in this paper carry over to this case since there would not be enough diffusion to implement it.

In what follows we will use the notation

$$\alpha = (\alpha_1, \alpha_2, \alpha_3), \qquad |\alpha| = \alpha_1 + \alpha_2 + \alpha_3, \qquad \alpha_i \geq 0, \tag{2.5}$$

and

$$D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \partial x_3^{\alpha_3}}. \tag{2.6}$$

The L^2 norm (or energy norm) will be denoted by

$$\|u\| = \|u(\cdot, t)\|_2 = \left[\int_{R^3} |u(X, t)|^2 \, dX \right]^{1/2}, \tag{2.7}$$

where $X = (x, y, z)$, $dX = dx\,dy\,dz$. More generally we denote the L^p norm for $1 \leq p < \infty$ by

$$\|u(\cdot, t)\|_p = \left[\int_{R^3} |u(X, t)|^p \, dX \right]^{1/p}, \tag{2.8}$$

and L^∞ by

$$\|u(\cdot)\|_\infty = \operatorname{ess\,sup} |u(x)|. \tag{2.9}$$

The H^m norm is defined by

$$\|u(\cdot, t)\|_{H^m} = \left[\int_{R^3} \sum_{|\alpha| \leq m} |D^\alpha u(X, t)|^2 \, dX \right]^{1/2}, \tag{2.10}$$

$$\mathscr{A}_\alpha = \{U : D^\alpha U \to 0, \qquad |\alpha| \geq 1 \quad |X| \to \infty\} \tag{2.11}$$

The following notation will be used for the Fourier transform

$$\hat{u}(\xi) = \mathscr{F}u(\xi) = \int_{R^3} u(X) e^{-iX\cdot\xi} \, dX \tag{2.12}$$

where $\xi = (\xi_1, \xi_2, \xi_3)$.

We show that the solutions of (2.1)–(2.3) decay, in the case where α and γ are nonzero, at the same rate as the solutions of the underlying heat equation. This is the best decay we can expect since the temperature will not decay faster than its linear counterpart. The decay rate of the velocity will be driven by the decay of the temperature; hence we do not expect better decay for the velocity. If we start with data in a more restricted space we can obtain better decay for the temperature and hence for the velocity. One such space would be

$$D_\alpha = \Big\{V_0\text{: where } V(x,0) = V_0, \text{ and } V(x,t) \text{ satisfies } \partial V/\partial t = \Delta V, \\ \text{and } \|V(t)\|_2 \le C(t+1)^{-\alpha}\Big\}. \quad (2.13)$$

In particular we note that $L^1 \cap L^2 \subset D_{3/4}$. In such situation our method will yield a decay of order $(t+1)^{-3/4}$. We first show that both the L^2 norms of the temperature and velocity are bounded uniformly in time by constant depending on norms of the data. These bounds are then used to obtain the decay of the temperature, which is shown to decay at the same rate as the solution to the underlying heat equation. We note that if we would consider the equations in a bounded domain with zero boundary condition, the problem is considerably simpler, since then exponential decay would be an immediate consequence of Poincare's inequality. In the unbounded case we will use a technique used for solutions to the Naver–Stokes equations (Schonbek [7]) and for parabolic conservations laws (Schonbek [8]). This technique is the Fourier splitting method. In what follows we suppose that our solutions decay to zero as $|X| \to \infty$, $X = (x, y, z)$. Such solutions can be constructed easily if the data satisfy such a condition.

3. SOME REMARKS ON EXISTENCE

In this section we make a few general remarks on existence of solutions. We expect that it is easy to establish existence of local in time solutions in good spaces. In particular we will always suppose that our solutions and derivative tend to zero as $|x|$ tends to ∞. More precisely if $\alpha \neq 0$ using fixed point techniques it is easy to show

THEOREM 1. *Let $(U_0, \Theta_0) \in H^1$. Then there exist t_0 depending only on the H^1 norm of the data such that there exists a solution (U, θ) to* (2.1) *with data (U_0, Θ_0) which belongs to H^m for all $m \geq 0$.*

As for Navier–Stokes the question of regularity reduces to show that the solutions are in H^1. We remark that there are several proofs of regularity for solutions to three-dimensional Navier–Stokes equations with small data in H^1. We expect that these proofs with minor modifications will yield regularity of the geostrophic equations. More precisely the following should hold.

THEOREM 2. *Let* $(U_0, \theta_0) \in H^1$. *Suppose that* $\|U_0\|_{H^1} + \|\theta_0\|_{H^1} \le \epsilon$ *with* ϵ *as small as needed. Then* (2.1) *has a smooth global solution with data* (U_0, θ_0).

If $\alpha = 0$ and $\gamma = 0$, then the term w_z is the only part of w that plays a role in the equation. This term can be replaced by $-(u_x + v_y)$ and local existence will again follow by fixed point techniques.

If the data are large as for Navier–Stokes we expect that weak solutions can be constructed. To obtain decay the idea would be to use approximating solutions which are obtained via linearizations. The linearizations can be obtained with minor modifications of the ones constructed (with minor modifications) for solutions to the Navier–Stokes equations. These linearizations we are referring to were constructed by Leray [3], by a retarded mollification such as the ones used by Caffarelli *et al*. [1] or by Sohr *et al*. [9]. The existence of weak solutions will follow passing to the limit.

In what follows it is supposed (in addition to the hypotheses given in the theorems) that the data are small in H^1 and we will look at the proof as being formal; that is, it can be applied to approximations. Then using Fatou's lemma one can pass to the limit and obtain the decay for the limiting equations, i.e., the μ-geostrophic equations.

4. UNIFORM BOUNDS FOR THE TEMPERATURE AND VELOCITY

In this section we obtain uniform bounds for the temperature and velocity. These estimates will be the basis for the decay estimates which will be obtained in Section 4.

Recall that in what follows our data are either supposed small in H^1 or we are using a formal argument which can be applied to approximating equations and we have to pass to the limit to obtain the decay for (2.1). We note that we suppose that we are working first with a solution for which $(U, \theta) \in L^2$. This follows by easy energy estimates if γ and α in (2.1) are nonzero and the data are in L^2. In the case these constants are zero we will need additional hypothesis.

THEOREM 3. *Let* $(U_0(x, y, z), \theta_0) \in L^2 \cap L^1 \cap \mathscr{A}_1$. *Let* $(U, \theta) \in L^2$ *be a solution to* (GE) *with data* (U_0, θ_0). *We suppose* $k = 0$ *and the other constants in* (2.1) *are nonzero. Then the energy of the solutions to* (2.1) *will be bounded uniformly in* L^2.

Moreover we have

$$\|\theta(t)\|^2 + k_0 \int_0^t \int_{R^3} |\nabla\theta|^2 \, dX \le C_0, \tag{4.1}$$

$$\|U(t)\| \le C_1 \tag{4.2}$$

where $C_0 = \int_{R^3} |\theta_0|^2 \, dX$ and $C_1 = \max\{2C_0/\gamma, \|U(0)\|^2\}$, $\delta = \min(r, \epsilon)$ and $k_0 = \min(k_h, k_\nu)$.

Proof.

Bounds for the L^2 norm of the temperature: We suppose we are working with smooth solutions. This solution exists for small enough data in $L^1 \cap H^1$. For a nonsmooth solution the process is to obtain the bounds for approximations and then pass to the limit. Multiply the equation of the temperature by θ and integrate in space

$$\frac{d}{dt} \int_{R^3} |\theta|^2 \, dX = -\int_{R^3} \theta U \cdot \nabla\theta \, dX + k_h \int_{R^3} \theta \Delta\theta \, dX + k_v \int_{R^3} \theta\theta_{zz} \, dX. \tag{4.3}$$

Let $k_0 = \min(k_h, k_v)$. Then after some integration by parts, since the boundary terms vanish and the convective term integrates to zero, due to the fact that $\nabla \cdot U = 0$. Thus we have

$$\frac{1}{2} \frac{d}{dt} \int_{R^3} |\theta|^2 \, dX \le -k_0 \int_{R^3} |\nabla\theta|^2 \, dX. \tag{4.4}$$

Hence integrating in time the last inequality yields

$$\int_{R^3} |\theta|^2 \, dX + k_0 \int_0^t \int_{R^3} |\nabla\theta|^2 \, dX \le \int_{R^3} |\theta_0|^2 \, dX = C_0. \tag{4.5}$$

L^2 bounds for the velocity: Now multiply the velocity equations by U and integrate in space to obtain

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} \int_{R^3} |u|^2 + |v|^2 + |w|^2 \, dX \, dX + k_0 \int_{R^3} f U^t A U \, dX \\ &\quad = \int_{R^3} U \cdot \nabla U \, dX - \int_{R^3} U \cdot \nabla p \, dX - \int_{R^3} U^t B U \, dX + \int_{R^3} w\theta \, dX \\ &\qquad - \nu \int_{R^3} \left(|\nabla u|^2 + |\nabla v|^2 + \gamma |\nabla w|^2 \right) dx \end{aligned} \tag{4.6}$$

where we integrated the integral with the Laplacian by parts and use that the boundary terms are zero. By the definition of the matrix A it follows that the $U^2AU = 0$; hence the second integral on the left hand side vanishes. Since $\nabla \cdot U = 0$ the pressure integral and the convective term also vanish. Let $\delta = \min(\epsilon, r)$. Then (4.7) yields

$$\frac{1}{2}\frac{d}{dt}\int_{R^3}|U|^2\,dX \le -\delta\int_{R^3}|U|^2\,dX + \int_{R^3} w\theta\,dX. \tag{4.7}$$

Here the negative term $-\nu\int_{R^3}|\nabla U|^2$ on the right hand side was dropped. Thus by Hölder's inequality and the last inequality we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_{R^3}|u|^2 + |v|^2 + |w|^2\,dX\\ &\qquad \le -\delta\int_{R^3}|U|^2\,dX + \left(\int_{R^3}|w|^2\,dX \times \int_{R^3}|\theta|^2\,dX\right)^{1/2}.\end{aligned} \tag{4.8}$$

Therefore the L^2 bound of the temperature yields

$$\frac{1}{2}\frac{d}{dt}\int_{R^3}|U|^2\,dX \le -\delta\int_{R^3}|U|^2\,dX + \left(C_0\int_{R^3}|U|^2\,dX\right)^{1/2}. \tag{4.9}$$

The last equation will give

$$\frac{1}{2}\frac{(d/dt)\int_{R^3}|U|^2\,dX}{\left[\int_{R^3}|U|^2\,dX\right]^{1/2}} \le -\delta\int_{R^3}|U|^2\,dX^{1/2} + C_0. \tag{4.10}$$

This can be rewritten as

$$\frac{d}{dt}\left[\int_{R^3}|U|^2\,dX\right]^{1/2} \le -\delta\int_{R^3}|U|^2\,dX^{1/2} + C_0. \tag{4.11}$$

Thus

$$\frac{d}{dt}e^{\delta t}\left[\int_{R^3}|U|^2\,dX\right]^{1/2} \le C_0 e^{\delta t}. \tag{4.12}$$

Now integrating over $[0, T]$ yields

$$\begin{aligned}e^{\delta t}\left[\int_{R^3}|U|^2\,dX\right]^{1/2} &\le \left[\int_{R^3}|U_0|^2\,dX\right]^{1/2} + C_0\int_{R^3} e^{\delta s}\,ds\\ &= \left[\int_{R^3}|U_0|^2\,dx\right]^{1/2} + C_0\frac{e^{\delta t}-1}{\gamma}.\end{aligned} \tag{4.13}$$

Hence

$$\left[\int_{R^3} |U|^2\, dX\right]^{1/2} \le e^{-\delta t}\left[\int_{R^3} |U_0|^2\, dX\right]^{1/2} + \frac{C_0}{\delta}. \tag{4.14}$$

The last inequality completes the proof of the Theorem 2.1.

In the case that $r = 0$ the proof shows that the L^2 norm of temperature is bounded and is still valid. To insure that the energy of the velocity is bounded in the case $r = 0$ we will need more decay in the temperature. Specifically

THEOREM 4. *Let* $(U_0(x, y, z), \theta_0) \in L^2 \cap L^1 \cap \mathscr{A}_1$. *Let* $(U, \theta) \in L^2$ *be a solution to* (GE) *with data* (U_0, θ_0). *We suppose* $r = 0$, $k \neq 0$, *and the other constants in* (2.1) *are nonzero. Then the energy of the solutions to* (2.1) *will be bounded uniformly in* L^2.

Moreover we have

$$\|\theta(t)\|^2 + k_0 \int_0^t \int_{R^3} |\nabla\theta|^2 dX \le C_0, \tag{4.15}$$

and

$$\|\theta(t)\|^2 \le C_0 \exp^{-kt}, \tag{4.16}$$

$$\|U(t)\| \le C_1, \tag{4.17}$$

where the constants depend only on norms of the data.

Proof. By the same steps as last theorem we have

$$\frac{1}{2}\frac{d}{dt}\int_{R^3} |\theta|^2\, dX \le -k\int_{R^3} |\theta|^2\, dX. \tag{4.18}$$

Thus the exponential decay of the temperature follows. For the bounds of the velocity we note that the energy methods of the proof of the last theorem (specifically see (4.8)) combined with the exponential decay of the temperature yield

$$\frac{d}{dt}\left[\int_{R^3} |U|^2\, dX\right]^{1/2} \le +C_0\, e^{-kt/2}. \tag{4.19}$$

And thus the bound on the energy of the velocity follows after an integration in time.

5. L^2 DECAY OF THE TEMPERATURE AND VELOCITY

In this section we first analyze the case where we have both α and γ or r nonzero. For simplicity we put these constants equal to one, keeping in mind that if we leave the constants α and γ, our estimates will depend on these constants and will not be valid for $\alpha = 0$ and $\gamma = 0$. We present this case only to introduce the ideas of the Fourier splitting technique ([7, 8]) since the ideas are clearer in this context. The bounds of the theorems of last section will now be used to establish the desired algebraic rates of decay.

THEOREM 5. *Let $U_0(x, y, z) \in L^2 \cap L^1(\mathscr{R}^3) \cap \mathscr{A}_1$, $\theta_0 \in L^2 \cap L^1(\mathscr{R}^3) \cap \mathscr{A}_1$. Let (U, θ) be a solution to* (2.1) *with data (U_0, θ_0). We suppose $k = 0$ and that all the other constants in* (2.1) *are nonzero. Then the energy of the equations will decay at the following algebraic rate*

$$\|U(\cdot, t))\|_2^2 + \|\theta(\cdot, t)\|_2^2 \leq C_*(t+1)^{-3/2} \tag{5.1}$$

where the constant C_* depends on the L^2 norms of the data.

Remark. This proof is valid also if $\nu = 0$.

Proof. The proof we give is formal. We recall again that a rigorous proof would follow by the method presented here and applied to a sequence of approximating solutions which are smooth and then pass to the limit. Such an approximating sequence, as was mentioned above, can be obtained in a similar fashion to the approximations to the Navier–Stokes equations [1]. Once the theorem is established for approximations the result for weak solutions will follow by Fatou's lemma. Hence from now on we work as if we have a smooth solution.

We next show that the temperature decays at the expected rate. For this multiply the temperature equation by θ and integrate in space. We obtain as before,

$$\frac{1}{2}\frac{d}{dt}\int_{R^3} |\theta|^2\, dX \leq -k_0 \int_{R^3} |\nabla\theta|^2\, dX. \tag{5.2}$$

This energy inequality is the starting point of the Fourier splitting method. The idea is to obtain an ordinary differential inequality for the energy norm of the temperature. This is obtained by working in the Fourier domain and splitting the space into two appropriately chosen time dependent subspaces. In a bounded domain case we would use Poincare's inequality and exponential decay would follow immediately.

By Plancherel's theorem inequality (5.2) reads as follows in frequency space

$$\frac{1}{2}\frac{d}{dt}\int_{R^3}|\hat{\theta}|^2\,d\xi \leq -k_0\int_{R^3}|\xi|^2|\hat{\theta}|^2\,d\xi, \tag{5.3}$$

where $\xi = (\xi_1, \xi_2, \xi_3)$. We subdivide the frequency space into two time dependent subspaces: $S(t)$ and its complement $S^c(t)$ where $S(t)$ is defined by

$$S(t) = \left\{\xi : |\xi| \leq \left(\frac{3}{2k_0(t+1)}\right)^{1/2}\right\}. \tag{5.4}$$

Note that as $k_0 \to 0$ the volume of $S(t)$ will tend to R^3, and so our estimates will not be valid if $k_0 = 0$. That is, the decay is only obtained if there is diffusion present in all three coordinates of the temperature. Hence from (5.3) it follows that

$$\frac{d}{dt}\int_{R^3}|\tilde{\theta}|^2\,d\xi \leq -2k_0\int_{S(t)}|\xi|^2|\hat{\theta}|^2\,d\xi - 2k_0\int_{S(t)^c}|\xi||\hat{\theta}|^2\,d\xi. \tag{5.5}$$

As $t \to 0$ the term coming from the integral over $S(t)$ will tend to zero since the volume of $S(t)$ tends to 0. Hence this term will not be useful in this proof, and thus will be dropped. The inequality still holds. Thus the integral of the frequency squared over

$$S(t)^c = R^3 \setminus S(t)$$

can be bounded by the least value of the frequency in the exterior of the ball $S(t)$. Thus

$$\frac{d}{dt}\int_{R^3}|\hat{\theta}|^2\,d\xi \leq -\frac{3}{(t+1)}\int_{S(t)^c}|\hat{\theta}|^2\,d\xi. \tag{5.6}$$

This last inequality can be rewritten as follows:

$$\frac{d}{dt}\int_{R^3}|\hat{\theta}|^2\,d\xi \leq -\frac{3}{(t+1)}\int_{R^3}|\hat{\theta}|^2\,d\xi + \frac{3}{(t+1)}\int_{S(t)}|\hat{\theta}|^2\,d\xi. \tag{5.7}$$

Thus we have a linear ordinary differential inequality for the L^2 norm of the temperature. Recall that the L^2 norm of the energy is equal to the L^2 norm of the Fourier transform. To obtain the decay of the L^2 norm of the temperature we need an intermediate estimate for the Fourier transform of the temperature for frequency values in $S(t)$. For this we note that the

Fourier transform of the temperature satisfies the following differential equation:

$$(\mathscr{F}\theta)_t + \left[k_n\left(|\xi_1|^2 + |\xi_1|^2\right) + k\nu|\xi_3|^2\right]\mathscr{F}\theta = -\mathscr{F}(U \cdot \nabla\theta). \quad (5.8)$$

Hence if we let $|\beta| = [k_n(|\xi_1|^2 + |\xi_1|^2) + k_\nu(|\xi_3|^2)]$, then

$$\mathscr{F}\theta(\xi) = \mathscr{F}(\theta_0)e^{-|\beta|t} + \int_0^t \sum_{j=1}^{3} \xi_j \mathscr{F}(u_j\theta)e^{-|\beta|(t-s)}\, ds. \quad (5.9)$$

[We changed for convenience the notation $U = (u, v, w) = (u_1, u_2, u_3)$.]

Hence

$$\mathscr{F}(\theta(\xi)| \le |\mathscr{F}(\theta_0)e^{-k_0|\xi|^2 t} + \sum_{j=1}^{3} \int_0^t |\xi|\,|\mathscr{F}(u_2\theta)|e^{-k_0|\xi|^2(t-s)}\, ds. \quad (5.10)$$

Since the data were in L^1 it follows that $\mathscr{F}(\theta_0)| \le C_*$. And since by Theorem 3.1 the L^2 norms of the temperature and velocity are bounded, the last inequality yields

$$\mathscr{F}(\theta(\xi)| \le C_0 + \sum_{j=1}^{3} \int_0^t |\xi|\,|\mathscr{F}(\theta u_j)|e^{-k_0|\xi|^2(t-s)}\, ds. \quad (5.11)$$

Thus

$$\mathscr{F}(\theta(\xi)| \le C_0 + \sum_{j=1}^{3} \int_0^t \xi \int_{R^3} |\hat{\theta} u_j| e^{-k_0|\xi|^2(t-s)}\, ds. \quad (5.12)$$

Thus by the Hölder inequality

$$\mathscr{F}(\theta(\xi)| \le C_0 + \sum_{j=1}^{3} \int_0^t \xi \int_{R^3} \|\theta\|^{1/2}\|u_j\|^{1/} e^{-k_0|\xi|^2(t-s)}\, ds. \quad (5.13)$$

And finally the right hand side can be bounded by

$$\begin{aligned}\mathscr{F}(\theta(\xi)| &\le C_0 + C_0C_1\int_0^t \xi \int_{R^3} e^{-k_0|\xi|^2(t-s)}\, ds \\ &= \frac{C_0 + C_1}{\kappa_0|\xi|} e^{-k_0|\xi|^2(t-s)}\Big|_0^t \le C_*\left[1 + \frac{1}{k_0|\xi|}\right]. \quad (5.14)\end{aligned}$$

With this bound in hand we return to inequality (4.4) to obtain

$$\frac{d}{dt}\int_{R^3}|\hat{\theta}|^2\,d\xi \leq -\frac{3}{(t+1)}\int_{R^3}|\hat{\theta}|^2\,d\xi + \frac{3}{(t+1)}\int_{S(t)} 2C_*{}^2 + \left[\frac{2}{k_0|\xi|}C_0C_1\right]^2 d\xi. \tag{5.15}$$

Hence integrating the last integral on the right hand side yields

$$\begin{aligned}\frac{d}{dt}\int_{R^3}|\hat{\theta}|^2\,d\xi + \frac{3}{(t+1)}\int_{R^3}|\hat{\theta}|^2\,d\xi &\leq \frac{C_2}{(t+1)}\int_{S(t)}\left[1+\frac{1}{|\xi|^2}\right]d\xi\\ &\leq \frac{C_3}{(t+1)^{5/2}} + \frac{C_4}{(t+1)^{3/2}}\\ &\leq \frac{C_*}{(t+1)^{3/2}}\end{aligned} \tag{5.16}$$

Using $(t+1)^3$ as a multiplier the last inequality can be expressed as follows

$$\frac{d}{dt}\left[(t+1)^3\int_{R^3}|\hat{\theta}|^2\,d\xi\right] \leq \frac{C}{(t+1)^{3/2}}(t+1)^3 \tag{5.17}$$

Integrating in time yields

$$(t+1)^3\int_{R^3}|\hat{\theta}|^2\,d\xi \leq \int_{R^3}|\hat{\theta}_0|^2\,d\xi + C(t+1)^{5/2}. \tag{5.18}$$

Thus

$$\int_{R^3}|\hat{\theta}|^2\,d\xi \leq (t+1)^{-3}C_0 + c(t+1)^{-1/2} \leq C(t+1)^{-1/2} \tag{5.19}$$

This last inequality establishes an intermediate rate of decay. For the temperature from where an intermediate decay for $\|U\|_2$. These two estimates are used to get a better bound for $\mathscr{F}\theta$. This bound will be used to yield the optimal decay for $\|\theta\|_2$ and repeating the process yields the decay for the energy of the velocity. More precisely it is shown that the solutions decay at the same rate as their underlying linear counterpart.

Now we use the decay of $\|\theta(t)\|_2$ to obtain an intermediate decay for the velocity. Here we first suppose α and r are nonzero.

5.1. *Auxiliary L^2 Decay of the Velocity*

The ideas here are the same as for the decay of the temperature. That is, the main tool will be the Fourier splitting. Multiplying the velocity equations by U, we obtain as before in the first part of Theorem 3

$$\frac{1}{2}\frac{d}{dt}\int_{R^3}|U|^2\,dX \le -\delta\int_{R^3}|U|^2\,dX + \left[\int_{R^3}|U|^2\,dX\int_{R^3}|\theta|^2\,dX\right]^{1/2} - \nu\int_{R^3}|\nabla U|^2\,dx. \tag{5.20}$$

By Schwartz' inequality and dropping the last term on the right hand side we obtain

$$\int_{R^3}|U|^2\,dX \le -\delta\int_{R^3}|U|^2\,dX + \frac{\delta}{2}\int_{R^3}|U|^2\,dX + \frac{1}{2\delta}\cdot\int_{R^3}|\theta|^2\,dX. \tag{5.21}$$

We note that the last term on the right hand side can be used to show that the decay rate has perhaps a smaller constant, but it will not improve the rate of decay and thus we omit it. Hence

$$\frac{1}{2}\frac{d}{dt}\int_{R^3}|U|^2\,dX \le -\frac{\delta}{2}\int_{R^3}|U|^2\,dX + \frac{1}{2\delta}\int_{R^3}|\theta|^2\,dX. \tag{5.22}$$

The decay of the temperature now yields

$$\frac{d}{dt}\int_{R^3}|U|^2\,dX + \delta\int_{R^3}|U|^2\,dX \le \frac{C}{\delta}(t+1)^{-1/2}. \tag{5.23}$$

Now using $\exp(\delta t)$ as a multiplier, the last equation yields

$$\frac{d}{dt}[\exp(\delta t)]\int_{R^3}|U|^2\,dX \le \exp(\delta t)\frac{C}{\gamma}(t+1)^{-1/2}. \tag{5.24}$$

Integrating in time over $[0, t]$ it follows that

$$\exp(\delta t)\int_{R^3}|U(X,t)|^2\,dX \le \int_{R^3}|U_0|^2\,dX + \int_0^t \exp(\delta s)\frac{C}{\delta}(s+1)^{-1/2}\,ds = I_1 + I_2. \tag{5.25}$$

We need to analyze integral I_2

$$I_2 = \int_0^{t/2}\exp(\delta s)\frac{C}{\delta}(t+1)^{-1/2}\,ds + \int_{t/2}^{t}\exp(\delta s)\frac{C}{\delta}(t+1)^{-1/2}\,ds \le C\exp\left(\frac{\delta}{2}t\right) + C(t/2+1)^{-1/2}\exp(\delta t). \tag{5.26}$$

Thus combining (4.40) and (4.41) it follows that

$$\int_{R^3} |U(X,t)|^2\, d\tilde{X} \leq \exp(-\delta t) \int_{R^3} |U_0|^2\, dX + C(t+1)^{-1/2}. \quad (5.27)$$

which gives an intermediate decay for the velocity.

The estimates above on the L^2 norms of the temperature and velocity yield a new estimate for $\mathscr{F}(\theta)$. More precisely, if $\xi \in S(t)$ then $\xi \leq (1+t)^{-1/2}$. Thus

$$\begin{aligned} |\mathscr{F}(\theta(\xi))| &\leq C_0 + C_0 C_1 \int_0^t \xi \int_{R^3} \|\theta\|_2 \|U\|_2 e^{-k_0|\xi|^2(t-s)}\, ds \\ &\leq C_0 + (1+t)^{-1/2} \int_0^t (1+s)^{-1/2} \leq C_*. \end{aligned} \quad (5.28)$$

Now repeating the argument which gave the intermediate decay of $\|\theta\|_2$, but replacing the bound of $\mathscr{F}(\theta)$ by the one we just obtained will yield the expected decay of the temperature. More precisely, it gives

$$\|\theta(t)\|_2^2 \leq C_0(1+t)^{-3/2}.$$

To obtain the right rate of decay for the velocity, proceed as follows. Repeat the argument described by inequalities (4.35)–(4.42), replacing in inequality (4.38) the old bound on $\|\theta\|_2$ by the new bound if $\|\theta\|_2^2$. This yields the new and optimal rate of

$$\|U(t)\|_2^2 \leq C_0(t+1)^{-3/2}.$$

We note that we did not make use of the term with the parameter ν. Thus the result includes the case $\nu = 0$. The last inequality completes the proof of Theorem 3.1.

Next we analyze the case where $r = 0$ and $\gamma \neq 0$.

THEOREM 6. *Let $U_0(x,y,z) \in L^2 \cap L^1(\mathscr{R}^3) \cap \mathscr{A}$, $\theta_0 \in L^2 \cap L^1(\mathscr{R}^2) \cap \mathscr{A}$. Let (U, θ) be a solution to* (2.1) *with data* (U_0, θ_0). *We suppose* $r = 0$ *and that all the other constants in* (2.1) *are nonzero. Then the energy of the equations will decay at the following algebraic rate*

$$\|U(\cdot,t)\|_2^2 + \|\theta(\cdot,t)\|_2^2 \leq C_*(t+1)^{-3/2} \quad (5.29)$$

where the constant C_* depends on the L^2 norms of the data.

Proof. Since $k \neq 0$ it follows that the temperature decays exponentially. Thus

$$\frac{1}{2}\frac{d}{dt}\int_{R^3}|U|^2\,dX \leq -\delta\int_{R^3}|U|^2\,dX + \left[\int_{R^2}|U|^2\,dX\int_{R^2}|\theta|^2\,dX\right]^{1/2} - \nu\int_{R^3}|\nabla U|^2\,dx \tag{5.30}$$

yields

$$\frac{1}{2}\frac{d}{dt}\int_{R^3}|U|^2\,dX \leq -\delta\int_{R^3}|U|^2\,dX + C_0\exp^{-(1/2)kt} - \nu\int_{R^2}|\nabla U|^2\,dx. \tag{5.31}$$

Now applying the Fourier splitting method will again yield the algebraic rate for the L^2 norm of the velocity. Since this proof is very similar to the steps used in the first case to obtain algebraic rate of decay for the temperature, it is omitted. For details on how to handle this case we refer the reader to Fourier splitting method (M. E. Schonbek [7, 8]).

6. THE PRIMITIVE EQUATIONS AND EXTENSIONS

In this section we are going to study the equations when there is no diffusion in the variable w, and no vertical acceleration. In fact it is the lack of diffusion that causes most difficulty in obtaining an estimate, omitting vertical acceleration while keeping vertical diffusion is easier. Thus, we set $\gamma = 0$ in the (2.1)–(2.3). The first problem we find here is that it is not obvious at all if there are solutions for which the L^2 norm is bounded independently of time. That a time dependent bound exists is clear by the former section. We note that in the case that the constant k is not zero, then as shown before there is such a bound and the Fourier splitting method can be used to obtain decay for the energy of the first two variables. This is an easy consequence of the last section.

Attention will be focused now on the case where $\gamma = 0$ and $k = 0$. There are two possibilities for solutions for which the L^2 norm remains finite. Either there are oscillations where the L^2 norm of u, v increases while the L^2 norm of w decreases and then the process is reversed, i.e., the L^2 norm of u, v decreases and the l^2 norm of w increases and so on or the L^2 norm of w tends to a finite limit L, in which case one can show that the L^2 norms of u and v tend to zero. In what follows we suppose the existence of good solutions.

THEOREM 7. *Let (U, θ) be a solution of the primitive equations. Suppose that the L^2 norm of w is bounded, then either we have oscillations as described above or there exists L such that*

$$\|w(t)\| \to L$$

and

$$\|u\| + \|v\| \to 0.$$

Proof. Since we are supposing that the L^2 norm is bounded, there has to be a limit or the norm has to oscillate. Hence we suppose that such a limit L exists. Multiplying the first two equations by $V = (u, v)$, the last equation by w, and summing yields

$$\frac{1}{2}\frac{d}{dt}\int |u|^2 + |v|^2 + |w|^2\,dx \le -\nu\frac{1}{2}\frac{d}{dt}\int |\nabla u|^2 + |\nabla v|^2\,dx - \int \theta w\,dx. \tag{6.1}$$

Let $G(s) = \|V(s)\|^2 + \|w(s)\|^2$ and $V = (u, v)$. Then the last equation yields

$$G(t) - G(s) \le -v\int_s^t\int |\nabla V|^2\,dx\,ds - \int_s^t\int \theta w\,dx\,ds.$$

By the boundedness of w in L^2 and the decay of the temperature obtained in previous section we have

$$G(t) - G(s) \le -v\int_s^t\int |\nabla V|^2\,dx\,ds + C_0\int_s^t\frac{1}{(s+1)^{1/4}}.$$

Note that we only were able to use the auxiliary decay of temperature which is not optimal. We will now apply an extension to the Fourier splitting method due to Wiegner [11]. Let $S(t) = \{\xi : |\xi| \le [g(t)/2]^{1/2}\}$; g will be specified below. Combined with the last equation, the Fourier splitting method yields

$$\begin{aligned} G(t) - G(s) &+ \int_s^t g(r)^2\|V(r)\|^2\,dr \\ &\le \int_s^t g(r)^2\int_{S(t)} |u(\hat{\xi}, r)|^2\,d\xi\,dr + C_0\int_s^t\frac{1}{(r+1)^{1/4}}\,dr \\ &\le C\int_s^t g(r)^2\left[g(r)^{3/2} + (t+1)^{3/4} + g(r)\right]dr \\ &\quad + C_0\int_s^t\left[\frac{1}{r+1}\right]^{1/4}\,dr \end{aligned} \tag{6.2}$$

Note that here we are integrating on a sphere of radius $[g(t)/2]^{1/2}$ and that the Fourier transform of V can be easily bounded by

$$|V(\hat{\xi}, t)| \leq C + C_0 \int_0^t \xi C \|V\|^2 e^{-|\xi|^2(t-s)}\, ds + \int_0^t \|\theta\|\, \|w\| e^{-|\xi|^2(t-s)}\, ds$$
$$\leq C + \frac{C_0}{\|\xi|} + \int_s^t \left[\frac{1}{r+1}\right]^{1/4} dr.$$

Now following Wiegner [11] define

$$e(t) = \exp\left[\int_0^t g(r)^2\, dr\right].$$

Hence

$$e(t) - e(t-h) = e(t-h)\int_{t-h}^t g(r)^2\, dr + h\epsilon(h) \tag{6.3}$$

where $\epsilon(h) \to 0$ as $h \to 0$. Now write

$$\begin{aligned} e(t)(G(t) - L) &- e(t-h)(G(t-h) - L) \\ &= (e(t) - e(t-h))(G(t) - L) \\ &\quad + e(t-h)((G(t) - L) - (G(t-h) - L)) \end{aligned}$$

so that by (6.3)

$$\begin{aligned} &e(t)(G(t) - L) - e(t-h)(G(t-h) - L) \\ &\quad = e(t-h)\int_{t-h}^t g(r)^2\, dr (G(t) - L) \\ &\qquad + e(t-h)[(G(t) - G(t-h)) + h\epsilon(h)(G(t) - L)] \\ &\quad = e(t-h)\left[\int_{t-h}^t g(r)^2 [G(t) - G(r)]\, dr + G(t) - G(t-h) \right. \\ &\qquad\qquad \left. + \int_{t-h}^t g(r)^2 (G(r) - L)\, dr\right] + h\epsilon(h)(G(t) - L). \end{aligned}$$

Recalling that $G(r) - L = \|U(r)\|_2^2 - L$, we get

$$
\begin{aligned}
e(t)(G(t) - L) &- e(t-h)(G(t-h) - L) \\
&= e(t-h)\left[\int_{t-h}^{t} g(r)^2 [G(t) - G(r)]\, dr + G(t) - G(t-h) \right. \\
&\qquad \left. + \int_{t-h}^{t} g(r)^2 \|v(r)\|_2^2\, dr \right] \\
&\quad + e(t-h)\int_{t-h}^{t} g(r)^2 \left(\|w(r)\|_2^2 - L\right) dr + h\epsilon(h)(G(t) - L).
\end{aligned}
\tag{6.4}
$$

Let $g(t)^2 = \alpha(t+1)^{-1}$, with α sufficiently large. Then

$$
e(t) = e^{\alpha \int_0^t dr/(r+1)} = (t+1)^{\alpha}.
$$

Let T_0 be such that for $t \geq T_0$ we have $|\,\|w(t)\|_2^2 - L| < \epsilon$. Using (6.4) it follows that, for $t \geq T_0$,

$$
\begin{aligned}
e(t)[G(t) - L] &- e(t-h)[G(t-h) - L] \\
&\leq e(t-h)\int_{t-h}^{t} g(r)^2 [G(t) - G(r)]\, dr + C\int_{t-h}^{t} e(r)(r+1)^{-3/2}\, dr \\
&\quad + \epsilon e(t-h)\int_{t-h}^{t} g(r)^2\, dr + h\epsilon(h)(G(t) - L) \\
&\quad + C\int_{t-h}^{t} e(r) g(r)^2 \left[\frac{1}{r+1}\right]^{1/4} dr,
\end{aligned}
\tag{6.5}
$$

$$
C\int_{t-h}^{t} e(r) g(r)^2 \left[\frac{1}{r+1}\right]^{1/4} dr = C\int_{t-h}^{t} e(r)(r+1)^{-5/4}\, dr.
$$

Hence the last integral bounds the second integral on the right.

It follows easily after integrating (6.1) in time that if $|t - r| < h$

$$
G(t) - G(r) \leq -\int_r^t \int |\nabla V|^2\, dx\, ds + \int_r^t \int \theta w\, dx\, ds
$$

where we used (6.2).

From this

$$|G(t) - G(r)| \leq \int_t^r \int \theta w \, dx \, ds \leq C_0 \int_r^t (r+1)^{-1/4} \, dr = O(h^{3/4}).$$

Hence summing (6.5) over intervals of length h it follows that for $t \geq T$

$$\begin{aligned} e(t)[G(t) - L] &- e(T)[G(T) - L] \\ &\leq O(h^{3/4}) \int_T^t e(r) g(r)^2 \, dr + \int_T^t e(r)(r+1)^{-5/4} \, dr \\ &\quad + \epsilon \int_T^t g(r)^2 e(r) \, dr + \epsilon(h)[G(0) - L]. \end{aligned}$$

Let $h \to 0$; then

$$e(t)[G(t) - L] \leq e(T)[G(T) - L] + C(t+1)^{\alpha - 1/4} + \epsilon C_0 e(t). \tag{6.6}$$

Here we used that

$$\int_T^t g(r)^2 e(r) \, dr = C_0 \int_T^t (r+1)^{\alpha - 1} \, dr \leq C(t+1)^{\alpha}.$$

Dividing by $e(t)$ yields

$$\begin{aligned} \int |u(t)|^2 \, dx + \int |w(t)|^2 \, dx - L &\leq \frac{e(T)}{e(t)} (G(0) + L) \\ &\quad + C(t+1)^{-1/4} + \epsilon C_0. \end{aligned}$$

Since ϵ tends to zero the proof is now complete.

7. QUASI-STATIONARY CASE: PLANETARY GEOSTROPHIC EQUATIONS

Let $\nu = 0$ in (2.1) and we suppose that U depends on time only through the temperature. We suppose that we have solutions in L^2. Hence we have equations of the form

$$fAU = -\nabla p - BU + \bar{\theta} \tag{7.1}$$

$$\frac{\partial \theta}{\partial t} + U \cdot \nabla \theta = \kappa_h [\theta_{xx} + \theta_{yy}] + \kappa_\nu \theta_{zz} \tag{7.2}$$

$$\nabla \cdot U = 0 \tag{7.3}$$

where the constants ϵ, r, k_h, k_ν are strictly positive and $\mu \geq 0$. We will need to impose some conditions on the size of ϵ and r for existence in this case. We note that local existence of solutions to the quasistationary equations will follow by a fixed point argument. For this first linearize the equations

$$\phi_n = P\left[B^{-1}(-fAU_n - \mu V \cdot \nabla U_n) + \bar{\theta}_n\right] \tag{7.4}$$

$$\psi_n = e^{-\tilde{A}t}(\theta_0) - \int_0^t e^{-\tilde{A}(t-s)}(V \cdot \nabla\theta_n)\, ds. \tag{7.5}$$

For local existence we note that one can apply a fixed point argument to the temperature variable and use the first equation to describe the velocity in function of temperature. Since we are interested in decay we are going to suppose that such solutions in L^2 exist provided that the data are small.

THEOREM 8. *Let $\theta(x,0)$ and $\mathbf{U}_0(\mathbf{x})$ lie in $L^2(\mathscr{R}^3)$. Let $(\mathbf{U}, \theta)$ be a solution of* (2.1)–(2.3) *with data $(\mathbf{U}_0, \theta_0)$ such that $(\mathbf{U}_0, \theta) \to \mathbf{0}$ as $|x| \to \infty$. Then*

(i) $$\int_{R^3} |\theta|^2\, dx + k_* \int_0^t \int_{R^3} |\nabla\theta|^2\, dx\, ds \leq C_0 \tag{7.6}$$

where $k_* = \min(k_h, k_\nu)$ and

(ii) $$\epsilon \int_{R^3}\int |u|^2\, dx + \epsilon \int_{R^3} |v|^2\, dx + \frac{r}{2}\int_{R^3} |w|^2\, dx \leq \frac{1}{2r}\int_{R^3} |\theta_0|^2\, dx \tag{7.7}$$

(where $r > 0$). Here, $C_0 = \int_{R^3} |\theta_0|^2\, dx$.

Proof. The proof is formal. To make it rigorous as before it is necessary to apply the proof to approximating solutions and then pass to the limit, using Fatou's lemma.

Part i: Follows the same way as Theorem 4, that is, multiply Eq. (2.2) by θ and integrate in space and time. Notice that the convective term vanishes since the velocity is divergence free and there are no boundary terms since the solution tends to zero as $|x| \to \infty$. Notice also that the diffusive term is integrated by parts to yield the gradient square. And again there are no boundary terms for the same reason as before.

Part ii: We suppose that the data are small and hence we have a smooth solution or otherwise we work with approximations and pass to the limit. Thus our computations will be formal. Multiply the velocity equation by U

and integrate in space to obtain

$$-\epsilon\int_{R^3}|u|^2\,d\mathbf{x}+\epsilon\int_{R^3}|v|^2\,d\mathbf{x}+r\int_{R^3}|w|^2\,d\mathbf{x}+\int_{R^3}\nabla p\cdot\mathbf{u}\,dx=\int_{R^3}w\theta\,d\mathbf{x}. \tag{7.8}$$

Since the pressure term vanishes (by incompressibility) the last equation yields

$$\begin{aligned}\epsilon\int_{R^3}|u|^2\,d\mathbf{x}+\epsilon\int_{R^3}|v|^2\,d\mathbf{x}+r\int_{R^3}|w|^2\,d\mathbf{x}\\ \le\left(\int_{R^3}w\theta\,d\mathbf{x}\right)^{1/2}\times\left(\int_{R^3}|\theta|^2\,d\mathbf{x}\right)^{1/2}.\end{aligned} \tag{7.9}$$

Since by part (i) of this theorem we have $\theta\in L^2$; it follows that

$$\epsilon\int_{R^3}|u|^2\,d\mathbf{x}+\epsilon\int_{R^3}|v|^2\,d\mathbf{x}+r\int_{R^3}|w|^2\,d\mathbf{x}\le\frac{r}{2}\int_{R^3}|w|^2\,d\mathbf{x}+\frac{1}{2r}\int_{R^3}|\theta|^2\,dx. \tag{7.10}$$

Thus,

$$\epsilon\int_{R^3}|u|^2\,d\mathbf{x}+\epsilon\int_{R^3}|v|^2\,d\mathbf{x}+\frac{r}{2}\int|w|^2\,d\mathbf{x}\le\frac{1}{2r}\int_{R^3}|\theta|^2\,d\mathbf{x}. \tag{7.11}$$

This completes the proof.

The next theorem addresses the decay of the energy of the solutions to the quasistationary equations. The decay of the L^2 norm of the temperature will follow as in Theorem 4, that is, by the Fourier splitting method as described in Schonbek ([7, 8]). The energy decay of the velocity is a consequence of the decay of temperature.

THEOREM 9. *Let* $(\mathbf{U}_0(x),\theta_0(x))\in(L^2(\mathscr{R}^3))^2$. *Let* $(\mathbf{U},\theta)$ *be as in Theorem* (4.1), *and then*

(i) $$\int_{R^3}|\theta|^2\,dx\le C_0(t+1)^{-3/2} \tag{7.12}$$

(ii) $$\int_{R^3}|\mathbf{U}|^2\,dx\le C_1(t+1)^{-3/2} \tag{7.13}$$

where C_1 *depends on* ϵ, r, *and norms of the data. Moreover,* $C_1\to\infty$ *if* ϵ *or* $r\to0$. C_0 *depends on norms of the data.*

Proof. To obtain the decay described above one has to first obtain an auxiliary decay of $(t+1)^{-1/2}$ for the L^2 norm of the temperature. This follows from the bounds obtained in Theorem (5.1) and then proceeds exactly as Theorem (3.1). This decay of the energy of the temperature yields easily using (5.49) a decay of $(t+1)^{-1/2}$ for the energy of the velocity. More precisely by this theorem and (5.49) we have

$$\epsilon\int_{R^3}|u|^2\,d\mathbf{x} + \epsilon\int_{R^3}|v|^2\,d\mathbf{x} + \frac{r}{2}\int|w|^2\,d\mathbf{x} \le \frac{1}{2r}\int_{R^3}|\theta|^2\,d\mathbf{x} \le \frac{C_0}{2r}(t+1)^{-1/2}. \tag{7.14}$$

Now with this decay in hand we proceed to use a bootstrap argument to refine the decay order of the temperature. That is, we note that now we have as in Theorem 4

$$\|\mathscr{F}(\theta u_j)\|_{L^\infty} \le \|\theta(t)\|_2\|U(t)\|_2 \le C_0(1+t)^{-1/2}.$$

Hence we repeat the estimate which yields the decay of the L^2 norm of the temperature using this new estimate of the Fourier transform of the product of the temperature and velocity. This time just as in Theorem 4 we obtain the optimal rate of $(t+1)^{-3/2}$ for the square of the L^2 norm of the temperature. Finally to obtain the decay of the L^2 norm of the velocity we repeat the former argument but replace the old decay of the temperature with the new one. That is,

$$\epsilon\int_{R^3}|u|^2\,d\mathbf{x} + \epsilon\int_{R^3}|v|^2\,d\mathbf{x} + \frac{r}{2}\int|w|^2\,d\mathbf{x} \le \frac{1}{2r}\int_{R^3}|\theta|^2\,d\mathbf{x} \le \frac{C_0}{2r}(t+1)^{-3/2}. \tag{7.15}$$

This completes the proof of the theorem.

8. SUMMARY COMMENTS

We have extended and applied the Fourier splitting method, formerly used for the incompressible Navier–Stokes equations, to a slightly compressible (Boussinesq) fluid in which the expansion of a fluid by a change in temperature feeds into the momentum equation via the buoyancy term. Similar methods were also applied to various simplifications of these equations that are commonly used in geophysical settings. We have shown that in several cases the solutions decay algebraically, at a rate of the same order as solutions of the underlying equations. It is important that energy decay can be proved these cases, since without such reassuring mathematical properties one should be hesitant about applying the equations—which are really just models—to study real phenomena. Finally, we note that Samelson *et al*. [6] have recently obtained some existence results for the planetary geostrophic equations with certain types of dissipative terms.

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