STRONG SOLUTIONS TO THE INCOMPRESSIBLE NAVIER-STOKES EQUATIONS IN THE HALF-SPACE

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Abstract

We derive an exact formula for solutions to the Stokes equations in the half-space with an external forcing term. This formula is used to establish local and global existence and uniqueness in a suitable Besov space for solutions to the Navier-Stokes equations. In particular, well-posedness is proved for initial data in $L^3(\mathbb{R}^3_+)$.

Introduction and definitions

The Cauchy problem for the Navier-Stokes equations governing the time evolution of the velocity $u(x,t) = (u_1(x,t), u_2(x,t), u_3(x,t))$ and the pressure $p(x,t)$ of an incompressible fluid filling all of $\mathbb{R}^3_+$ is described by the system

$$\left\{ \begin{array}{rcl} \dfrac{\partial u}{\partial t} & = & \Delta u - \nabla \cdot (u \otimes u) - \nabla p, \\ \nabla \cdot u & = & 0, \\ u(x,0) & = & u_0(x), \\ u(x',0,t) & = & 0,\ x = (x', x_3) \in \mathbb{R}^3_+,\ t \geq 0, \end{array} \right. \tag{1}$$

where $x' = (x_1, x_2)$. Strong solutions of this system are traditionally studied via semi-group techniques. Mild solutions, that is, strong solutions to the integral equation derived from (1), which are continuous in time with value in some Banach space, have been constructed in the half-space in [21], in Lebesgue spaces. Ukai gave in [19] an exact formula for solutions to the Stokes problem in the half-space, and remarked that this allows to construct solutions to the Navier-Stokes equations in the same fashion as for [9] in the whole space. Recently attention has been focused on the whole space problem, as a variety of tools from harmonic analysis allowed several authors to gain valuable insight on the system [11, 2, 4, 16, 6]. These results relied heavily on the use of the Fourier transform, and on the systematic use of various scales of Besov spaces, which, unlike Lebesgue ones, do not have local versions. Thus, a priori it is unclear if such results can be easily extended over other domains, such as bounded or exterior domains. However, the half-space turns out to be a particular case of a domain, where, as originally remarked by Ukai, it is possible to obtain an exact representation formula. In this case the corresponding Besov spaces are well-defined, and related in an easy way to their whole space counterpart. Therefore it is possible to extend the theory developed in [2, 16], and to obtain various existence and uniqueness results, with very rough initial data. We should note that in [8] a similar direct approach is used to obtain estimates for the Stokes flow in Hardy spaces. Let us also remark here that Ukai's formula was also successfully used in some previous papers by H. Kozono [12, 13].

This paper is organized as follows : in the first section the definition of Besov spaces is recalled, on both the whole space and the half-space, and the few results that will be used later are summarized. In the second section, following Ukai's celebrated paper [19], an exact representation formula for the Stokes system in the half-space with an external force is derived. This differs from the semi-group approach as we don't need to introduce the projection operator on divergence-free vectors, and only use the heat kernel in the half-space. Our result differs from [19] since we have an external force, and thus we have to adapt Ukai's estimates to handle the new term. The third section is devoted to the Navier-Stokes equations, where solutions are constructed with initial data $u_0 \in L^3(\mathbb{R}^3_+)$. More precisely we obtain global solutions for small initial data in a Besov space, and local in time solutions for arbitrary large data. Moreover, we have uniqueness of such solutions in $C([0,T), L^3(\mathbb{R}^3_+))$.

1 Besov spaces in $\mathbb{R}^3_+$.

We recall the definition of Besov spaces in the whole space using the characterization with a continuous parameter instead of the more usual dyadic one. This form of the characterization will be helpful in the following sections to relate the special structure of the bilinear term in the Navier-Stokes equations, with bounds in Besov spaces. For other definitions of Besov spaces see [15, 18, 1].

Definition 1
Let $\psi(x) \in C^\infty$ be such that

$$\hat{\psi}(\xi) = |\xi|^2 e^{-|\xi|^2}.$$

Let $p,q \in (1, +\infty]$, $s \in \mathbb{R}$, $s < 1$. Then, $f \in \dot{B}^{s,q}_p$ if and only if

$$\left(\int_0^\infty \|t^{-s}\psi_t * f\|^q_{L^p} \frac{dt}{t} \right)^{\frac{1}{q}} < +\infty, \tag{2}$$

where ψ_t is the rescaled function $\frac{1}{t^3}\psi(\frac{\cdot}{t})$, and this norm is equivalent to the usual dyadic norm. If $q = \infty$, we replace the L^q norm by sup_t.

The reader familiar with both scales of spaces will note that we can replace $\hat{\psi}(\xi)$ with any $\hat{\phi}(\xi)$ in the Schwartz class, whose support in disjoint from 0. The usual characterization involves such a function, with a compact support in a ring. Following Triebel [18], we can define the Besov spaces on the half-space as restrictions (in the distributional sense) of the Besov spaces in the whole space.

Definition 2

Let $p,q \in (1,+\infty]$, $s \in \mathbb{R}$. Then $\dot{B}_p^{s,q}(\mathbb{R}^3_+)$ is the collection of all restrictions of elements of $\dot{B}_p^{s,q}(\mathbb{R}^3)$. If f is the restriction of g on $\mathbb{R}^3_+$, its norm is defined by

$$\|f\|_{\dot{B}_p^{s,q}(\mathbb{R}^3_+)} = \inf \|g\|_{\dot{B}_p^{s,q}(\mathbb{R}^3)} \tag{3}$$

where the infimum is to be taken over all g whose restriction coincides with f.

This definition is not well-suited to any practical purpose. For positive regularity indices, it turns out that direct definitions can be given, but no such definitions exist for negative regularity indices. Since our main interest lies in the Besov spaces $\dot{B}_q^{-(1-\frac{3}{q}),\infty}(\mathbb{R}^3_+)$ with $q > 3$, this could present a serious problem. However, for this range of indices, the usual extension operator e, i.e. the extension by zero, is continuous from the Besov space on the half-space to its counterpart on the whole space. Specifically, if f is a function defined on $\mathbb{R}^3_+$, we set

$$e(f) = \begin{cases} f(x) & \text{for} \quad x_3 \geq 0 \\ 0 & \text{for} \quad x_3 < 0. \end{cases} \tag{4}$$

Therefore,

$$\|ef\|_{\dot{B}_p^{-(1-3/p),\infty}(\mathbb{R}^3)} \leq C\|f\|_{\dot{B}_p^{-(1-3/p),\infty}(\mathbb{R}^3_+)}. \tag{5}$$

This is a consequence of the characterization of Fourier multipliers on Besov spaces, and we refer the reader to Triebel ([18] p 167,168) for a complete explanation.

Let $S(t)$ denote the heat semi-group in the whole space. We recall the following equivalent characterization of Besov spaces with a negative regularity index.

Proposition 1

Take $\alpha > 0$, $\gamma \geq 1$, $f \in \mathcal{S}'(\mathbb{R}^3)$ a tempered distribution, then

$$\|f\| = \sup_{t>0} t^{\frac{\alpha}{2}} \|S(t)f\|_{L^\gamma} \tag{6}$$

is a norm in $\dot{B}_\gamma^{-\alpha,\infty}(\mathbb{R}^3)$ equivalent to the usual dyadic one.

We remark first that in the whole space it is very useful to use this definition together with estimates of the heat kernel ([2, 16]). In what follows, we want to adapt these estimates to the half-space. The following proposition will be needed. Here $E(t)$ denotes the heat operator in the half-space

Proposition 2
Take $0 < \alpha < 1$, $\gamma \geq 1$, $f \in \mathcal{S}'(\mathbb{R}^3_+)$, *then*

$$\sup_{t>0} t^{\frac{\alpha}{2}} \|E(t)f\|_{L^\gamma(\mathbb{R}^3_+)} \leq C\|f\|_{\dot{B}_\gamma^{-\alpha,\infty}(\mathbb{R}^3_+)}. \tag{7}$$

We recall that $E(t)$ can be easily represented using $S(t)$. Let f be defined on the half-space, let $\tilde{e}f$ be its extension to the whole space in the following sense:

$$\tilde{e}(f) = \left\{ \begin{array}{rcl} f(x) & \text{for} & x_3 \geq 0 \\ -f(x', -x_3) & \text{for} & x_3 < 0. \end{array} \right. \tag{8}$$

That is, $\tilde{e}$ completes f by the opposite of its mirror image with respect to the hyperplan $x_n = 0$. Then, it is well known that

$$E(t)f = rS(t)\tilde{e}f. \tag{9}$$

Where r is the restriction from the whole space to the half-space.

$$rf = f_{|\mathbb{R}^3_+}. \tag{10}$$

We may consider the case $\alpha = 1 - 3/p$, $\gamma = p$ since this is the one we will need. Thus

$$\begin{aligned} \sup_t t^{-\alpha/2} \|E(t)f\|_{L^p(\mathbb{R}^3_+)} &= \sup_t t^{-\alpha/2} \|rS(t)\tilde{e}f\|_{L^p(\mathbb{R}^3_+)} \\ &\leq \sup_t t^{-\alpha/2} \|S(t)\tilde{e}f\|_{L^p(\mathbb{R}^3)} \\ &\leq 2\sup_t t^{-\alpha/2} \|S(t)ef\|_{L^p(\mathbb{R}^3)} \\ &\leq C\|ef\|_{\dot{B}_p^{-(1-3/p),\infty}(\mathbb{R}^3)} \\ &\leq C\|f\|_{\dot{B}_p^{-(1-3/p),\infty}(\mathbb{R}^3_+)}. \end{aligned}$$

Thus, the initial data is in such a Besov space $\dot{B}_{\gamma}^{-\alpha,\infty}(\mathbb{R}^3_+)$. Expressions of the type (6) for the half-space will be bounded. To conclude the section we recall some useful relations between Besov and Sobolev spaces. Let $p \leq q$:

$$\dot{B}_p^{\frac{3}{p}-1,\infty}(\mathbb{R}^3) \hookrightarrow \dot{B}_q^{\frac{3}{q}-1,\infty}(\mathbb{R}^3)\ ,$$

and, for $p > 3$

$$\dot{H}^{1/2}(\mathbb{R}^3) \hookrightarrow L^3(\mathbb{R}^3) \hookrightarrow \dot{B}_p^{\frac{3}{p}-1,\infty}(\mathbb{R}^3).$$

Here $\dot{H}^s(\mathbb{R}^3)$ denotes the usual homogeneous Sobolev space. These inclusions are in turn true for all spaces over $\mathbb{R}^3_+$. From now, we will drop the space reference for spaces over $\mathbb{R}^3$ like L^p and write L^p_+ for spaces over $\mathbb{R}^3_+$.

2 The Stokes system with an external force

In this section we intend to obtain an exact formula for the solution of the Stokes system in the half-space,

$$\left\{ \begin{array}{rcl} \dfrac{\partial u}{\partial t} & = & \Delta u + f - \nabla p, \\ \nabla \cdot u & = & 0, \\ u(x,0) & = & u_0(x), \\ u(x',0,t) & = & 0, \\ f(x',0,t) & = & 0, \;\; x = (x', x_3) \in \mathbb{R}^3_+,\ t \geq 0. \end{array} \right. \tag{11}$$

Existence of exact formulas, without the use of semi-group techniques, was first obtained in [17]. In [19] Ukai gave a complete formulation of the problem under different boundary conditions. For our purposes we need to obtain a formula when an external force is present. We will assume that the boundary value of this external force is zero. If not additional terms would appear. To obtain the exact expression of the solution we proceed following the steps in Ukai's paper. For details we refer the reader to [19]. A careful use of a combination of Ukai's results would lead to the same formula but rederiving it directly provides a better understanding of the underlying difficulties due to the presence of the boundary. For convenience the notations are kept the same as in Ukai's paper. Therefore, denote by $u = (u^{(1)}, u^{(2)}, u^{(3)})$, $u' = (u^{(1)}, u^{(2)})$, R_j

the Riesz transforms defined by $R_j = \partial_j(-\Delta)^{-\frac{1}{2}}$ and S_j the Riesz transforms in $\mathbb{R}^2$, which can also be extended in a natural way to $\mathbb{R}^3$. Define

$$\begin{aligned} V_1 u_0 &= -S \cdot u_0' + u_0^3 \\ V_2 u_0 &= u_0' + S u_0^3 \end{aligned}$$

Here V_1 is acting on vectors to give scalars, and V_2 is vectorial. Note that Sf stands for $(S_1 f, S_2 f)$. Let U be defined by

$$Uf = rR' \cdot S(R' \cdot S + R_3)ef, \tag{12}$$

where r and e are respectively the restriction and the extension to the half-space defined earlier. We first establish the following formula,

Theorem 1

The solution to the Stokes system (11) is given by

$$\begin{aligned} u' &= E(t)V_2 u_0 - SUE(t)V_1 u_0 \\ &\quad - \int_0^t E(t-s)S\tilde{M}f ds - SU \int_0^t E(t-s)\tilde{N}f ds, \end{aligned} \tag{13}$$

and

$$u^{(3)} = UE(t)V_1 u_0 + U \int_0^t E(t-s)\tilde{N}f ds. \tag{14}$$

$\tilde{M}$ and $\tilde{N}$ are two pseudo-differential operators of order 0 *defined below.*

Proof: take the divergence of the first equation in (11), to get

$$\begin{aligned} \Delta p &= \nabla \cdot f \\ p(x', 0) &= b(x') \end{aligned} \tag{15}$$

where b is the pressure on the boundary. Recall that $\tilde{e}f$ is the extension of f over the whole space with the opposite of its mirror image, thus the solution of (15) is

$$p(x) = \frac{1}{|x|} \star \tilde{e}(\nabla \cdot f) + Db, \tag{16}$$

where D is the single layer potential solution of the Laplace equation when there is no source:

$$Db = \int_{\mathbb{R}^2} \partial_3 \frac{C}{\sqrt{|x' - y'|^2 + x_3^2}} b(y')dy'. \tag{17}$$

At this stage, it is useful to recall the following important lemma ([19]), which results from simple manipulations on the symbols of the operators:

LEMMA 1

- *Both operators V_1 and V_2 commute with any partial derivatives in the space variable.*
- *The operator U commutes with ∂_1 and ∂_2, as does $E(t)$.*
- *For ∂_3 we have*

$$\partial_3 U = (I - U)|\nabla'|. \tag{18}$$

Note $U\partial_3 = \partial_3 U$ in the very particular case where it is applied to a function which has a null boundary value ([19]). Next apply the pseudo-differential operator $\partial_3 + |\nabla'|$ to p, to get

$$(\partial_3 + |\nabla'|)p = (\partial_3 + |\nabla'|)\frac{1}{|x|} \star \tilde{e}(\nabla \cdot f) = M\tilde{e}(\nabla \cdot f). \tag{19}$$

Note that the second term disappeared, specifically the operator was applied to annihilate such a term. This follows since

$$\mathcal{F}_{x'}(Db)(\xi') = e^{-|\xi'|x_3}\mathcal{F}_{x'}(b)(\xi'). \tag{20}$$

Hence we can apply our pseudo-differential operator to the kernel. Thus taking the Fourier transform with respect to x' yields

$$(\partial_3 + |\xi'|)(e^{-|\xi'|x_3}) = 0.$$

Using the same pseudo-differential operator we define

$$z(x,t) = (\partial_3 + |\nabla'|)u^{(3)}(x,t). \tag{21}$$

Thus z is a solution of

$$\begin{aligned}
\partial_t z - \Delta z &= (\partial_3 + |\nabla'|)f^{(3)} - (\partial_3 + |\nabla'|)\partial_3 p \\
&= (\partial_3 + |\nabla'|)f^{(3)} - \partial_3 M\tilde{e}(\nabla \cdot f) \\
&= Nf \\
z(x',0,t) &= 0 \\
z(x,0) &= |\nabla'|V_1 u_0.
\end{aligned}$$

Note that due to the boundary condition the projection onto the hyperplane commutes with ∂_2 and ∂_3 for the second term. The divergence-free property is used to commute the projection and the first term. The initial condition is a consequence of the divergence free property of the velocity field. As before let $E(t)$ be the heat operator for the half-space, thus

$$z = |\nabla'|E(t)V_1 u_0 + \int_0^t E(t-s)Nf(s)ds. \tag{22}$$

The term $u^{(3)}$ can be recovered from (21), (as in [19])

$$u^{(3)} = \int_0^{x_3}\int_{\mathbb{R}^2} \frac{1}{\sqrt{|x'-y'|^2 + |x_3 - y_3|^2}} z(y', y_3)dy'dy_3, \tag{23}$$

to yield

$$u^{(3)} = UE(t)V_1 u_0 + U\frac{1}{|\nabla'|}\int_0^t E(t-s)Nf(s)ds. \tag{24}$$

As in Ukai, define U from (23) and note that this definition of U coincides with the previous one. In our case, unlike in [19], there is an additional term due to the external force. Here the operator $\frac{U}{|\nabla'|}$ cannot be seen as a composition of U and $|\nabla'|^{-1}$, since this last operator makes no a priori sense in $\mathbb{R}^3$. Therefore it is necessary to rewrite the operator N in a more convenient way. For this we rewrite the second term in N:

$$\begin{aligned}\partial_3 M &= \partial_3 \frac{\partial_3 + |\nabla'|}{|\nabla|^2} \\ &= 1 + |\nabla'| \frac{\partial_3 - |\nabla'|}{|\nabla|^2}\end{aligned}$$

then, indeed

$$\begin{aligned}\partial_3 M \tilde{e}(\nabla \cdot f) &= \frac{\partial_3^2 + \partial_3 |\nabla'|}{|\nabla|^2} (\partial_1 \tilde{e} f^{(1)} + \partial_2 \tilde{e} f^{(2)}) \\ &\quad + \frac{\partial_3^2 + \partial_3 |\nabla'|}{|\nabla|^2} \partial_3 (\tilde{e}(f^{(3)})) \\ \tilde{M} f &= Q(f^{(1)}, f^{(2)}) + (1 + \frac{|\nabla'|(\partial_3 - |\nabla'|)}{|\nabla|^2} \partial_3 \tilde{e}(f^{(3)}).\end{aligned}$$

This new expression "isolates" the normal coordinate, and allows us to express N in a more suitable way (recall N is actually defined on the half-space, which allows to cancel the $\partial_3 f^{(3)}$ with $\partial_3 \tilde{e} f^{(3)}$):

$$\begin{aligned}Nf &= (\partial_3 + |\nabla'|) f^{(3)} - \partial_3 M \tilde{e}(\nabla \cdot f) \\ &= |\nabla'| f^{(3)} - \frac{\partial_3^2 + \partial_3 |\nabla'|}{|\nabla|^2} (\partial_1 \tilde{e} f^{(1)} + \partial_2 \tilde{e} f^{(2)}) \\ &\quad - |\nabla'| \frac{\partial_3 - |\nabla'|}{|\nabla|^2} \tilde{e}(\partial_3 f^{(3)}).\end{aligned}$$

Given this last formula, commute $E(t)$ and $|\nabla'|$ to obtain

$$u^3 = U E(t) V_1 u_0 + U \int_0^t E(t-s) \tilde{N} f(s) ds, \tag{25}$$

where $\tilde{N}$ is defined on vectors as

$$\tilde{N} f = -(R_3^2 + R_3 \frac{|\nabla'|}{|\nabla|})(S_1 \tilde{e}(f^{(1)}) + S_2 \tilde{e}(f^{(2)})) + (1 - R_3^2 + R_3 \frac{|\nabla'|}{|\nabla|}) \breve{e}(f^{(3)}), \tag{26}$$

where $\check{e}(f)$ is the antisymmetrical extension of f:

$$\check{e}(f) = \begin{cases} f(x) & \text{for} \quad x_3 \geq 0 \\ -f(x', -x_3) & \text{for} \quad x_3 < 0. \end{cases} \tag{27}$$

Here we used the simple fact that $\tilde{e}\partial_3 f = \partial_3 \check{e} f$, which allows to rewrite the last term of (26) in a simple way.

The next step is to recover the rest of the velocity field, i.e. the non-tangential part. For this, introduce $w = V_2 u = u' + S u^{(3)}$, and solve another heat equation (again with a null boundary condition)

$$\partial_t w - \Delta w = -S(|\nabla'| + \partial_3)p = -S\tilde{M}f. \tag{28}$$

The solution can be expressed as

$$w = E(t)V_2 u_0 + \int_0^t E(t-s)[-S\tilde{M}f]ds,$$

which yields the following expression for u'

$$\begin{aligned} u' &= E(t)V_2 u_0 - SUE(t)V_1 u_0 \\ &\quad - \int_0^t E(t-s)S\tilde{M}f ds - SU\int_0^t E(t-s)\tilde{N}f ds. \end{aligned} \tag{29}$$

Proceeding in the same fashion one can obtain an explicit formula for the pressure. Such a formula will not be needed, and thus omitted. Let us summarize what we have obtained so far: we now have a nice representation formula for the solution to the Stokes system, in terms of the heat operator on the half-space, and a few pseudo-differential operators A_1, A_2, B_1, B_2 of order zero, that is, we have

$$u = A_1 E(t) A_2 u_0 + \int_0^t B_1 E(t-s) B_2 f ds. \tag{30}$$

3 Navier-Stokes equations in the half-space

We intend to transform the system (1) into an integral equation. We therefore use (30), letting $f = \nabla \cdot (u \otimes u)$, to obtain

$$u(x,t) = A_1 E(t) A_2 u_0 + \int_0^t B_1 E(t-s) B_2 \nabla \cdot (u \otimes u)(s) ds. \tag{31}$$

This equation will be solved by a classical fixed point method, (see [2],[9],[10]). Following [2], we note that the study of the bilinear term in the previous equation can be reduced to the study of the corresponding simplified scalar operator. More precisely, by lemma 1 it follows that, for $i = 1, 2$ and $j, k \in 1, 2, 3$

$$B_1 E(t-s) B_2 \partial_i (u_j u_k) = B_1 \partial_i E(t-s) B_2 (u_j u_k). \tag{32}$$

For the last partial derivation, ∂_3, the situation is slightly more complicated to write down, but not more difficult to handle. In fact, if $f = 0$ on $x_3 = 0$, then

$$E(t)\partial_3 f = rS(t)\tilde{e}\partial_3 f = r\partial_3 S(t)\check{e} f, \tag{33}$$

Thus even if $E(t)$ and ∂_3 do not commute, we have $E(t)\partial_3 = \partial_3 \check{E}$ where $\check{E} = rS(t)\check{e}$. Any remaining B_i operators between $E(t)$ and ∂_3 can be handled by lemma 1. Since B_1 and B_2 are bounded on all the spaces under consideration, the second term in (31) can be expressed as

$$B(f,g) = \int_0^t \mathcal{G}(t-s)(fg)\, ds, \tag{34}$$

where $\mathcal{G}$ is a scalar version of the operator $E(t-s)\nabla\cdot$. Extending to the whole space via $e, \tilde{e}, \check{e}$, the last expression becomes simply a convolution operator by a function $\frac{1}{(t-s)^2} G(\frac{x}{\sqrt{t-s}})$, where

$$|G(x)| \leq \frac{C}{1+|x|^4} \tag{35}$$

$$|\nabla G(x)| \leq \frac{C}{1+|x|^4}. \tag{36}$$

This is an easy consequence of the structure of the derivative of the heat kernel $\nabla E(t-s)$. However, in order to solve (31) by iteration, it is necessary that the bilinear term $\nabla\cdot(u\otimes u)$ has null boundary value. Writing this term in the usual way, $\Sigma_i u^{(i)}\partial_i u$, it is clear that we only need to verify that ∇u is bounded also on the boundary. As we intend to solve the equation by a fixed-point method, we will have to ensure that this is the case on all iterates. In what follows we will concentrate on the bilinear operator defined by (34). Now our problem has been reduced to a generic setting, and thus everything will be treated by keeping in mind the case of the whole space. Therefore in what follows we will in general only sketch the proofs and refer the reader to [3, 6] for details.

A well suited functional space to study (1) is L^3_+ ([21, 9]), since $\|u_\lambda\|_{L^3_+}=\|u\|_{L^3_+}$. Note that the dilation $u_\lambda(x,t)=\lambda u(\lambda x,\lambda^2 t)$ is well defined on the half-space, for $\lambda>0$. However, it has been shown in [2, 16] that a convenient framework is provided by larger classes of functions, such as the homogeneous Besov spaces $\dot{B}^{-(1-\frac{3}{p}),\infty}_{p,+}$. We will see later how they arise.

The following theorem extends the results of [20], for Dirichlet boundary conditions. We restrict the initial data to L^3_+ for convenience, more details will appear in [5].

Theorem 2
Let $u_0\in L^3_+$, $\nabla\cdot u_0=0$, and suppose there exists $q>3$ such that

$$\|u_0\|_{B^{-(1-\frac{3}{q}),\infty}_{q,+}}<\eta(q) \tag{37}$$

where $\eta(q)$ is a constant only depending on q, then there exists a unique solution of (1) such that

$$u\in L^\infty([0+\infty),L^3_+), \tag{38}$$

and

$$t^{\frac{1}{2}-\frac{3}{2q}}u\in L^\infty([0+\infty),L^q_+)). \tag{39}$$

Theorem 2 will be a consequence of a fixed point argument via the following abstract lemma (Picard's theorem in a Banach space).

Lemma 2
Let $\mathcal{E}$ be a Banach space, B a continuous bilinear application, $x, y \in \mathcal{E}$

$$\|B(x,y)\|_{\mathcal{E}} \leq \gamma \|x\|_{\mathcal{E}} \|y\|_{\mathcal{E}} \,. \tag{40}$$

Then, if $4\gamma\|x_0\|_{\mathcal{E}} < 1$, the sequence defined by

$$x_{n+1} = x_0 + B(x_n, x_n)$$

converges to $x \in \mathcal{E}$ and

$$x = x_0 + B(x,x) \quad , \quad \|x\|_{\mathcal{E}} < 2\|x_0\|_{\mathcal{E}} \,. \tag{41}$$

Define for $q > 3$ the space

$$F_q = \{f(x,t) \mid \sup_{t>0}(t^{\frac{1}{2}-\frac{3}{2q}}\|f(x,t)\|_{L^q_+}) < +\infty\}. \tag{42}$$

Using the Sobolev inclusion,

$$L^3_+ \hookrightarrow \dot{B}^{\frac{3}{q}-1,\infty}_{q,+} \ ,$$

for $3 \leq q$, it follows that if $u_0 \in L^3_+$ then $u_0 \in \dot{B}^{\frac{3}{q}-1,\infty}_{q,+}$, and thus by proposition 2

$$[E(t)u_0](x) \in F_q,$$

for all $q > 3$. To apply lemma 2 to F_q, we are left to prove that $B(\cdot,\cdot)$ is bicontinuous on F_q. Take f and g in F_q. Denote $h = fg \in F_{q/2}$ and recall that the bilinear operator can be written as follows

$$B(f,g) = \int_0^t \mathcal{G}(t-s)h(x,s)ds \ .$$

Thus, Hölder and Young inequalities (remember that up to extensions and restrictions, $\mathcal{G}$ is a convolution operator) will yield

$$\|B(f,g)\|_{F_q} \leq \int_0^1 \frac{Cd\lambda}{(1-\lambda)^{\frac{1}{2}+\frac{3}{2q}}\lambda^{1-\frac{3}{q}}}\|f\|_{F_q}\|g\|_{F_q} \tag{43}$$

which gives us the existence of $\eta(q)$. Proceeding the same way we find

$$\|B(f,g)\|_{F_3} \leq \int_0^1 \frac{Cd\lambda}{(1-\lambda)^{\frac{3}{p}}\lambda^{1-\frac{3}{p}}}\|f\|_{F_p}\|g\|_{F_p}\ , \tag{44}$$

provided $p < 6$ which establishes (38).

Hence our solution belongs to F_q. We note that such a solution u actually satisfies

$$u(x,t) \in F_p, \quad \text{for all} \quad p \geq 3\ , \tag{45}$$

(45) is of course true for the linear part and the following estimates gives the rest of the needed estimate:

$$\|B(f,g)\|_{F_p} \leq \int_0^1 \frac{Cd\lambda}{(1-\lambda)^{\frac{1}{2}+\frac{3}{2q}}\lambda^{1-\frac{3}{2q}-\frac{3}{2p}}}\|f\|_{F_p}\|g\|_{F_q}\,. \tag{46}$$

Thus if $p < q$ it follows that our solution is in F_p. Otherwise, the procedure is done in several steps to reach a value $q' > q$, by a bootstrap argument. The great amount of flexibility provided by inequalities of type (43), (46) allows us to obtain the result in many different ways. Using (44) it follows that the bilinear term is bounded in L^3_+.

We now address the question of whether or not the representation formula (31) can be used as a substitute for our initial problem. We note that it suffices to obtain an L^3 bound of the gradient of each iterate in order to insure that the "external force" vanishes on the boundary. This is an easy consequence of formula (31). We illustrate this with the first two iterates (the general case is exactly the same). The first iterate $u_{(1)}$ is the solution to the heat equation in the half-space, and thus verifies the estimate

$$\|\nabla u_{(1)}\|_{L^3} \leq \frac{C}{\sqrt{t}}.$$

Therefore, $\partial_i(u_j)u_k$ has null trace on the boundary, since it is a product of a function which is bounded and one which has a null trace. The next iterate $u_{(2)}$ will be the sum of $u_{(1)}$ and of $B(u_{(1)}, u_{(1)})$, for which we easily control the gradient:

$$\begin{aligned}\|\nabla B(f,g)\|_{F_3} \leq & \int_0^1 \frac{C d\lambda}{(1-\lambda)^{\frac{1}{2}+\frac{3}{2q}}\lambda^{\frac{1}{2}-\frac{3}{2p}}}(\|\nabla f\|_{F_3}\|g\|_{F_q} \\ & +\|\nabla g\|_{F_3}\|f\|_{F_q}).\end{aligned}$$

Note that our functional setting, if not for the restriction $u_0 \in L^3_+$, would allow initial data which are homogeneous of degree -1, thus providing self-similar solutions. Details will appear in [5]. The case $u_0 \in L^3_+$ is of special interest for several reasons. Besides being the most studied case, it provides a good example of what we can do if we restrict u_0 in some nice subclass of the Besov spaces used in the previous section. Since the Schwartz class is dense in L^3, we are able to obtain strong continuity at zero as well as local existence. Specifically,

Theorem 3
If $u_0 \in L^3_+$ is a divergence free vector field, then there exists a unique local in time solution to equation (31), such that

$$u \in C([0,T), L^3_+), \tag{47}$$

where T depends on the initial data.

Note that uniqueness is obtained for solutions of the integral equation (31) where the bilinear term is replaced by a sum of terms of the form (34). This derivation can be done only if it is known a priori that the non-linear term has null boundary value. This isn't actually a real concern, since we can avoid the commutation between $E(t)$ and ∂_3 in all our computation. More precisely we go back to (31) and forget about the B_i which are irrelevant since they are bounded operators. We have to estimate the operator $E(t)\partial_i$ on L^p_+. As noticed in [7], the adjoint is $\partial_i E(t)$, for which the usual $L^{q'} - L^{p'}$ estimate holds, for all $1 < p' < q' < \infty$. Taking the adjoint estimate gives us the needed $L^p - L^q$ estimate. This is the main fact used. That the operator under the integral might be a convolution of some sort is not essential here.

As in the whole space [3], the following result can be obtained

THEOREM 4
Let $u(x,t)$ be a solution of (31) in $C([0,T),L^3_+)$, with initial data $u_0 \in L^3_+$ and denote by w the function $w = u - S(t)u_0$, then

$$\nabla w \in C([0,T),L^{3/2}_+). \tag{48}$$

We begin with the proof of theorem 3. We apply theorem 2, with a slight modification: define the space F_p for $0 < t < T$. Thus, we have to verify first that such a finite T exists for any u_0 in L^3, and the smallness assumption is satisfied in the modified F_p. This is, in turn, a consequence of the density of smooth functions in L^3_+: one can find a sequence $f_n \in C_0^\infty$ converging to $f \in L^3$, and

$$\begin{aligned}
\lim_{T\to 0}\sup_{[0,T]} t^{\frac{1}{2}-\frac{3}{2p}}\|E(t)f(x)\|_{L^p_+} &= \lim_{T\to 0}\sup_{[0,T]} t^{\frac{1}{2}-\frac{3}{2p}}\|E(t)(f-f_n)\|_{L^p_+} \\
&\quad + \lim_{T\to 0}\sup_{[0,T]} t^{\frac{1}{2}-\frac{3}{2p}}\|E(t)f_n\|_{L^p_+} \\
&\leq C\|f_n - f\|_{L^3_+} + \lim_{T\to 0}\sup_{[0,T]} t^{\frac{1}{2}-\frac{3}{2p}}\|E(t)f_n\|_{L^p_+},
\end{aligned}$$

and both terms tend to zero with T. Thus by the fixed point lemma it follows that $t^{1/2-3/2p}u \in C((0,T),L^p_+)$ for all $p > 3$ and u satisfies also

$$\lim_{T\to 0}\sup_{[0,T]} t^{\frac{1}{2}-\frac{3}{2p}}\|u(x,t)\|_{L^p_+} = 0. \tag{49}$$

The last equality ensures the continuity at zero. It's worth noting that as t goes to infinity, the L^3_+ norm of u also tends to zero, as this is true for the linear part, and thus true for all the iterates of the fixed point scheme, and therefore true for the limit. As in the whole space, the proof of the above theorems will follow by a serie of lemmata ([14] or [3] for a proof similar to the following)

LEMMA 3
Let $3/2 < q < 3$. The bilinear operator $B(f,g)$ is bicontinuous from $L^\infty_t(L^{3,\infty}_{+,x}) \times L^\infty_t(L^{3,\infty}_{+,x})$ into $L^\infty_t(\dot{B}^{\frac{3}{q}-1,\infty}_{q,+})$.

We will prove the estimate by duality, and fix $q = 2$, which gives $\frac{1}{2}$ as the regularity index, merely as a convenience. For an arbitrary test function $\varphi(x) \in C_0^\infty$

consider the functional

$$I_t = \langle B(f,g), \varphi \rangle. \tag{50}$$

That is

$$\begin{aligned} I_t &= \int_0^t \Big\langle \mathcal{G}(t-s)(fg), \varphi \Big\rangle \, ds \\ &= 2\int_0^{\sqrt{t}} \Big\langle \tilde{G}_\eta fg(t-\eta^2),\ H_\eta \varphi \Big\rangle \, d\eta \end{aligned} \tag{51}$$

where, using $\mathcal{G}(\alpha) = E(\alpha/2)E(\alpha/2)\nabla\cdot$, we define $\tilde{G}_\eta$ as $\eta E((\eta^2)/2)\nabla\cdot$ after a change of variables $\eta^2 = t-s$, and similarly for H (which is therefore nothing else than the heat kernel up to rescaling). Then,

$$|I_t| \leq \int_0^{\sqrt{t}} \|\tilde{G}_\eta fg(t-\eta^2)\|_{L^2_+} \|H_\eta \varphi\|_{L^2_+} \, d\eta. \tag{52}$$

Using the decay properties of $\tilde{G}$ (which is nothing but $\mathcal{G}$ rescaled)

$$\|\tilde{G}_\eta fg(t-\eta^2)\|_{L^2_+} \leq C\eta^{-\frac{1}{2}} \|fg\|_{L^{\frac{3}{2}}_+}, \tag{53}$$

we find

$$|I_t| \leq C \sup_t (\|g\|_{L^{3,\infty}_+} \|f\|_{L^{3,\infty}_+}) \int_0^\infty \sqrt{\eta} \|H_\eta \varphi\|_{L^2_+} \frac{d\eta}{\eta}. \tag{54}$$

The last integral is then less than $\|\varphi\|_{\dot{B}^{-\frac{1}{2},1}_{2,+}}$, which, by duality, yields the proof. In fact from

$$|\langle B(f,g), \varphi \rangle| \leq C \|\varphi\|_{\dot{B}^{-\frac{1}{2},1}_{2,+}}$$

we have that $B(f,g)$ belongs (uniformly in time) to the dual of $\dot{B}^{-\frac{1}{2},1}_{2,+}$, which is $\dot{B}^{\frac{1}{2},\infty}_{2,+}$. This only gives the uniqueness of strong solutions in $C_t(L^3_+)$ for small initial data. To prove time-local uniqueness for arbitrary initial data a third lemma is necessary

LEMMA 4
Let $f(x,t) \in L^\infty_t(L^{3,\infty}_+)$ *and* $g(x,t) = E(t)G$*, where* $G \in L^3_+$*. Then the bilinear operator* $B(f,g)$ *belongs to* $L^\infty_t(\dot{H}^{\frac{1}{2}}_+)$*.*

For uniqueness, we don't actually need $\dot{H}^{\frac{1}{2}}_+$ (which is included in $\dot{B}^{\frac{1}{2},\infty}_{2,+}$), but this result shows how an exact knowledge of the time dependence of g can be exploited.
Proof of the lemma: as $G \in L^3_+$, g verifies all the usual estimates for a solution to the heat equation. For example, if we denote

$$\|g|_{4,T} = \sup_{t<T} t^{\frac{1}{8}} \|g(x,t)\|_{L^4_+},$$

then by Young inequality

$$\sup_{t<T} t^{\frac{1}{8}} \|g(x,t)\|_{L^4_+} \leq C\|G\|_{L^3_+}$$

and the left hand side goes to zero as T goes to zero (remember that $\sup_t = \sup_{t\in[0,T]}$ where T is to be chosen). Thus

$$\|g(x,t-s^2)\|_{L^4_+} \leq \frac{\|g|_{4,T}}{(t-s^2)^{\frac{1}{8}}}.$$

Returning to the estimates of the previous lemma, we obtain, starting from (52)

$$|I_t| \leq \int_0^{\sqrt{t}} \|\tilde{G}_\eta f g(t-\eta^2)\|_{L^2_+} \|H_\eta \varphi\|_{L^2_+} \, d\eta$$

Now we chose different exponents, to get

$$\|\tilde{G}_\eta f g(t-\eta^2)\|_{L^2_+} \leq C\eta^{-\frac{1}{4}} \|fg\|_{L^{\frac{12}{7}}_+}, \tag{55}$$

to obtain,

$$|I_t| \leq C \sup_t (\|g|_{4,T} \|f\|_{L^{3,\infty}_+}) \int_0^{\sqrt{t}} \frac{\|H_\eta \varphi\|_{L^2_+}}{(t-\eta^2)^{\frac{1}{8}} \eta^{\frac{1}{4}}} d\eta. \tag{56}$$

Applying Cauchy-Schwarz to this last integral, we bound it from above by

$$\left(\int_0^1 \frac{d\theta}{(1-\theta^2)^{1/4}\theta^{1/2}}\right)^{\frac{1}{2}} \left(\int_0^{\sqrt{t}} \|H_\eta \varphi\|^2_{L^2_+} d\eta\right)^{\frac{1}{2}} \leq C\|\varphi\|_{\dot{H}_+^{-\frac{1}{2}}}$$

which concludes the proof of Lemma 4.

From these three lemmata, we can deduce the uniqueness result, in the same way as this was done in the whole space [3]. Consider two solutions $u(x,t)$ and $v(x,t)$ with the same initial data u_0 and for which u is actually the solution constructed via the fixed point method. We denote $w = u - E(t)u_0$ and $\tilde{w} = v - E(t)u_0$. Then, if we temporarily forget that the bilinear operator appearing in (1) is vectorial and non-commutative, we may abuse the notation and write (as in the scalar case)

$$w - \tilde{w} = 2B(E(t)u_0, w - \tilde{w}) + B(w + \tilde{w}, w - \tilde{w}).$$

We know from Lemma 3 that both w and $\tilde{w}$ belong to $L_+^{3,\infty}$. By applying Lemma 4 to the first term, and Lemma 3 to the second, we obtain

$$\sup_t \|w - \tilde{w}\|_{L_+^{3,\infty}} \leq C(\|E(t)u_0|_{4,T} + \sup_t \|w + \tilde{w}\|_{L_+^3}) \sup_t \|w - \tilde{w}\|_{L_+^{3,\infty}}$$

so that $w = \tilde{w}$ at least on a small interval in time, since both quantities $\|E(t)u_0|_{4,T}$ and $\sup_t \|w + \tilde{w}\|_{L_+^3}$ tend to zero as T tends to zero (the first by density of smooth functions and the second by the strong continuity in L_+^3 of the solutions). We conclude by a standard continuation argument, which guarantees that both solutions are the same on the interval where they are defined. This ends the proof of Theorem 3.

We now proceed with the proof of Theorem 4. This would be essentially a rewriting of the same theorem for the whole space. Therefore we just state the lemma and refer the reader to [3] for details of the proof. By a change of variables, we have that

$$B(f,g) = 2\int_0^{\sqrt{t}} \mathcal{G}_s fg(x, t - s^2) ds \tag{57}$$

where, as usual, $\mathcal{G}_s$ is like a convolution by $G_s(x) = \frac{1}{s^3}G(\frac{x}{s})$. In addition, it is useful to work with the following operator, $A(f,g) = \Lambda B(f,g)$, where Λ is the

Calderón operator, with symbol $|\xi|$. Then

$$A(f,g) = 2\int_0^{\sqrt{t}} \tilde{\mathcal{G}}_s fg(x, t-s^2)\frac{ds}{s}, \tag{58}$$

and $\tilde{\mathcal{G}} = \nabla E(1)\nabla\cdot$, which is in our case, like a convolution by $\mathcal{F}(\tilde{G})(\xi) = |\xi|^2 e^{-|\xi|^2}$. We will proceed in two steps. First, we will deal with the bilinear operator applied to the linear part. Afterwards we will estimate the remaining part using a result similar to Lemma 3.

Lemma 5
Let F and G be in L^3_+. If $f(x,t) = E(t)F(x)$ and $g(x,t) = E(t)G(x)$, then $A(f,g) \in C_t(L^{\frac{3}{2}}_+)$.

We simply remark that if we were to replace the dependence in s of f and g in the definition of B, by a dependence on t, then we could integrate with respect to s, to obtain the operator $I - E(t)$. The actual proof uses this idea by carefully splitting the operator in several terms we can control using different properties.

We now state another lemma, which will allow us to conclude the proof.

Lemma 6
Let $p > 3$. If $f(x,t) \in C_t(\dot{F}^{1,2}_{\frac{3}{2},+})$ and $g(x,t)$ satisfies the estimate,

$$|g\|_p = \sup_t (t^{\frac{1}{2}-\frac{3}{2p}} \|g(x,t)\|_{L^p_+}) < \infty \tag{59}$$

then

$$A(f,g) \in C_t(L^{\frac{3}{2}}_+).$$

In order to give an idea of the proof, let's write, with $\mathcal{G}^* = \Lambda E(1)$ (and replacing the $\nabla\cdot$ by ∂_i, for any $i = 1, 2, 3$)

$$A(f,g) = 2\int_0^{\sqrt{t}} \mathcal{G}^*_s \partial_i (fg)(x, t-s^2) ds$$

As we control $\partial_i(fg)$ via $\partial_i(f)g + f\partial_i(g)$, we have to study $B(\partial_i(f), g)$. This is done essentially in the same way we carried the study of B. Then we are done with the lemma, and also with the theorem as the previous estimates allow us to preserve the property verified by $B(E(t)u_0, E(t)u_0)$ on all iterates of the fixed point scheme used to solve (31). Note that this fixed point scheme needs to be applied to the difference $w = u - E(t)u_0$, for the linear part doesn't in general verify the estimates.

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