On Decay of Solutions to the Navier-Stokes Equations

Maria Elena Schonbek
Department of Mathematics,
University of California,
Santa Cruz

October 10, 2001

Abstract

We first recall results on space-time decay of solutions to the Navier-Stokes equation in the whole space $I\!R^n$ which were developed in [9] and [1]. Next we give an example of a solution with radial vorticity to the Navier-Stokes equations in 2D, where the space-time decay rate can be computed explicitly.

1 Introduction

In this note we discuss the pointwise space time decay of solutions to the Navier-Stokes equations in the whole space $I\!R^n$, with $2 \leq n \leq 5$. We present some results that show the interplay between the space and time decay of the solutions and give an example of an explicit solution were this relation is clear. This kind of interplay is already present at the level of the solutions to the Heat equation. In particular for the Heat kernel sharp rates can be established.
The space time decay for solutions to the Navier-Stokes equations is algebraic and seems not to be as fast as for the heat kernel itself. This raises

questions of what causes the loss of decay. The proofs developed in [1] and [9]for solutions to the Navier-Stokes equations will naturally also work for solutions to the Heat equation. The question stands if for solutions to the Heat equations and solutions to the Navier-Stokes equations depending on the data one could refine our results to obtain faster decay. The example we give at the end seems to indicate that this rates could be optimal.

We present only the results. For details on the proofs we refer the reader to our joint papers with T. P. Schonbek [9] and with Amrouche, Girault and T.P. Schonbek[1]. Questions of time decay of solutions to the Navier-Stokes equations in different norms have been studied, among many others, by R. Kajikiya and T. Miyakawa [2], T. Kato [3], H. Kozono [4], H. Kozono and T. Ogawa [5], M.E. Schonbek [7], [8], M. Wiegner [13], and Zhang-Linghai [14]. In the direction of space-time decay of particular interest in the are the results by Takahashi [12]. In this reference, Takahashi studies the pointwise decay in space and time of the solutions, and their first derivatives, to the Navier-Stokes equations with zero initial data and an external force which decays at an algebraic rate in both space and time. In our case the data is nonzero and the external force vanishes. Our results follow by moment estimates combined with a Gagliardo-Nirenberg estimate. Specifically in [1] we show that strong solutions to the Navier-Stokes equations with data in appropriate spaces for $0 \leq k \leq n/2$:

$$|D^\alpha u(x,t)| \leq C_{k,m} \frac{1}{(t+1)^{\rho_0}(1+|x|^2)^{k/2}}$$

where $\rho_O = (1-2k/n)(m/2+\mu+n/4)$, $|\alpha| = m$, $\mu > \frac{n}{4}$ and where μ is the L^2 time rate of decay of the solution. We recall that this decay depends only on norms of the data [6],[7], [13].

In this paper we first recall the results obtained in the papers we mention above, the we discuss questions of optimality related to the rates we obtained. Finally we analyze an explicit example. This example is a solution to the Navier-Stokes in 2 dimensions with radial vorticity, which turns out to be simultaneously a solution to the Heat equation, [10] with very special data which depends on the initial vorticity. Extensions of these types of solutions can be constructed in all even space dimensions [10]

We use the notation

Let $\alpha = (\alpha_1, \ldots, \alpha_n)$ be a multi-index with $\alpha_i \geq 0$.

$$(1.\ 1) \qquad D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}} \,,$$

where

$$(1.\ 2) \qquad |\alpha| = \alpha_1 + \ldots + \alpha_n \,,$$

and

$$(1.\ 3) \qquad D_i = \frac{\partial}{\partial x_i} \,.$$

For any integer $m \geq 0$, we set

$$D^m f(x) = \left(\sum_{|\alpha|=m} |D^\alpha f(x)|^2 \right)^{1/2} \,,$$

where $x = (x_1, \ldots, x_n)$. The L^2 norm (or energy norm) will be denoted by

$$(1.\ 4) \qquad \|u\| = \|u(.,t)\|_2 = [\int_{I\!R^n} |u(x,t)|^2 dx]^{1/2},$$

where $dx = dx_1 \ldots dx_n$. More generally we denote the L^p norm for $1 \leq p < \infty$ by

$$(1.\ 5) \qquad \|u(.,t)\|_p = [\int_{I\!R^n} |u(x,t)|^p dx]^{1/p},$$

and the L^∞ norm by

$$(1.\ 6) \qquad \|u(.,t)\|_\infty = \text{ess sup}_x |u(x,t)|.$$

The H^m norm is defined by

$$(1.\ 7) \qquad \|u(.,t)\|_{H^m} = [\int_{I\!R^n} \sum_{|\alpha| \leq m} |D^\alpha u(x,t)|^2 dx]^{1/2} \,.$$

For $s = 0, 1, 2, \ldots$, we define the (s, α) moments

$$M_{s,\alpha}(t) = \int_{I\!R^n} |x|^s |D^\alpha u(x,t)|^2 \, dx,$$

and in particular for $s \geq 0$, $t \geq 0$, we define the moment of order s of u by

$$M_s((u)(t)) = M_{s,0}(t) = \int_{I\!R^n} |x|^s |u(x,t)|^2 \, dx = \left(\|u(t)\|_{L^2_{s/2}} \right)^2 .$$

2 Pointwise Decay

The main difficulty in establishing spatial-time decay is to obtain a time independent estimate for the moments of the solution and their derivatives. In the presence of such a bound the time decay of the moments is straightforward. Once the estimates on the moments are established we use a Gagliardo-Nirenberg's estimate to obtain an L^∞ algebraic time decay for $v(x,t) = (1+|x|^2)^{k/2} D^\alpha u(x,t)$. From where the space time decay will follow. Specifically we use Gagliardo-Nirenberg inequality to show

$$(2.\ 8)\quad |(1+|x|^2)^{k/2} D^\alpha u(x,t)| \leq \|v(\cdot,t)\|_\infty \leq \|v(\cdot,t)\|_2^{1-a} \|D^s v(\cdot,t)\|_2^a \,.$$

We note that the L^2 norms on the right are nothing else than energy norms of the moments of the solution and the moments of their derivatives. Thus the decay is a consequence of the following theorem. For its statement, we need to introduce the real numbers ν, q, r and r_1 which satisfy the relations

$$(2.\ 9)\qquad 0 \leq s < n, \quad 2 \leq r_1 \leq r, \quad 1 \leq q \leq \infty, \quad r > n\,,$$

Theorem 2.1 *Let $u_0 \in W^{m,r} \cap W^{m,r_1} \cap H^1(I\!\!R^n)^n$ with r, s, r_1 as above. Let u be a strong solution of the Navier-Stokes equations with data u_0, satisfying*

$$(2.\ 10)\qquad \|u(t)\|_2 \leq C(t+1)^{-\mu} \quad \text{where } \mu > n/4\,.$$

Then

$$\tilde{M}_{s,m}(t) \leq C(t+1)^{-(2\mu+m)(1-\frac{s}{n})}\,,$$

for $m = 0, 1, 2, \ldots$, $s = 0, 1, \ldots, n$.

Proof: For a proof we refer the reader to [1]. As remarked above that the main step in this proof is a uniform bound on the moments of the solution and derivatives. The decay will follow by an appropriately chosen Hölder inequality.

This theorem combined with (2. 8) yields

Theorem 2.2 *Let $2 \leq n \leq 5$. We retain the assumptions of the last theorem and we consider a strong solution u of the Navier-Stokes equations with data u_0. Let $k \leq n/2$. Then*

$$|D^\alpha u(x,t)| \le C_{k,m} \frac{1}{(t+1)^{\rho_0}(1+|x|^2)^{k/2}}\,, \tag{2. 11}$$

where $\rho_0 = (\mu + m/2 + n/4)(1 - 2k/n)$ *and* $|\alpha| = m$.

Proof: See [1] and [9]. The restriction to dimensions $n = 2, 3, 4, 5$ is due to the fact that we are using decay results for the derivatives of the solutions to Navier-Stokes which were established under those conditions, [11].

3 Questions on optimality of the decay

In order to understand the interplay between the time and space decay of our solutions we compare the situation with solutions to the Heat equation. In particular we first recall the behavior of the Heat Kernel. It is easy to show that the fundamental solution of the Heat equation,

$$E(x,t) = (4\pi t)^{-n/2} e^{-|x|^2/4t}$$

has the following asymptotic behavior:

$$|D^\alpha E(x,t)| \le c_0 |x|^{-a} t^{-b},$$

where $a + 2b = n + m$, with $m = |\alpha|$. The proof of the last fact follows by a simple induction argument on the order of derivation. On the other hand depending on the data, solutions to the Heat equation will decay at different algebraic order. If the data is compact it is easy to show that the solution decays at the same rates as the Heat Kernel. On the other extreme if the data is constant there is no decay. A simplified version of the proofs used for solutions to the Navier-Stokes equations can also be used for solutions to the Heat equation. The question that remains is if given appropriate data this decay rate for the solutions to the Navier-Stokes equations can be improved. Even if the data is compact our method will only show the decay obtained in theorem (2.2).

For solutions to the Navier-Stokes equations, the interplay between the time and space decay can be described as follows. Let $\mu = n/4 + \gamma$ be such that

$$\|u(t)\|_{L^2} \le C(t+1)^{\mu}.$$

Note that such type of decay in the L^2 norm can be obtained easily when the appropriate data is given. See [6],[7], [13].
The relation that holds between the space and time decay which follows by theorem (2.2) is

$$2\rho_0 + 2k = m + n + 2\gamma - \frac{2km}{n} - \frac{4\gamma m}{n} \ . \tag{3. 12}$$

For $k = 0$, we recover the same decay of the Heat equation, but this only gives decay in time, [11]. If $m = 0$ we have the relation $2\rho_0 + 2k = n + 2\alpha(1 - 2k/n)$, For $m \geq 0$ the decay improves since $k \leq n/2$. To have the same interplay between the space and time decay as for the Heat Kernel,

$$2\rho_0 + k = m + n$$

we would need $\gamma = k/2(\frac{n+2m}{n-2k})$ and this would imply that $\gamma \to \infty$ as $k \to n/2$. Since γ determines the order of the time decay of the L^2 norm of the solution, this would be equivalent to require that there is exponential L^2 time decay for the solutions . The above comments leave open the question of optimality of the decay rates.

4 A Special Example

We will first give an example in two spatial dimensions and then mention how to extend it to all even spatial dimensions. The main purpose of this 2D example is to show explicitly the interplay between the time and space decay. In this case we compute directly the space-time decay of the solution and show that it agrees with the one obtained in the general theorem. We are going to require less conditions on the data and thus our resulting decay will only be for the solution and not for the derivatives.

Let u(x,t) be a solution to the 2-D Navier-Stokes equation with radial vorticity. Suppose that $u(x,0) = u_o \in L^2 \cap L^1$ and $\nabla u_o \in L^2$. Let $\omega_o = curl u_o$ and $\omega_o \in L^1$. It is well known that a solution can be expressed as

$$u(x,t) = 1/r^2 \int_o^r s\omega(s) ds Ax,$$

where $x = (x_1, x_2)$, $r = {x_1}^2 + {x_2}^2$, ω is the vorticity and

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

One can show that in this case the nonlinear term of the solution $u\nabla u$ is a gradient of some function p. (See [10]). Thus u is a solution to both the Navier-Stokes equation and the Heat equation with data

$$u(x,0) = 1/r^2 \int_o^r s\omega_o(s)dsAx,$$

We can bound the solution pointwise in the following manner.
Let $u = (u_1, u_2)$, then

$$|u_1(x,t)|^2 + |u_2(x,t)|^2 \leq \frac{1}{r^2}\left(\int_o^r s\omega(s)ds\right)^2 = G(r) \tag{4. 13}$$

Now let $\alpha + \beta = 2$, then the right hand side of equation (4. 13) can be bounded as follows

$$\begin{aligned} G(r) &\leq \\ \frac{1}{r^2}\left[\frac{1}{2\pi}\left(\int_o^{2\pi}\int_o^r s|\omega(s)|ds\right)^\alpha \left(\int_o^{2\pi}\int_o^r s|\omega(s)|ds\right)^\beta\right] &\leq \\ \frac{1}{r^2}\frac{1}{2\pi}\left(\int_{\mathbb{R}^2}|w|dxdy\right)^\alpha \left(\int_o^{2\pi}\int_o^r sds\right)^{(\beta)/2}\left(\int_o^{2\pi}\int_o^r s|\omega(s)|^2ds\right)^{(\beta)/2} & \end{aligned} \tag{4. 14}$$

Combining equations (4. 13) and (4. 14) yields

$$\begin{aligned} |u_1(x,t)|^2 + |u_2(x,t)|^2 &\leq \\ \frac{1}{r^2}\frac{1}{2\pi}\left(\int_{\mathbb{R}^2}|w|dxdy\right)^\alpha r^\beta \left(\int_{\mathbb{R}^2}|w|^2dxdy\right)^{(\beta)/2} & \end{aligned} \tag{4. 15}$$

Since ω is the radial vorticity of a solution to the 2D Navier-Stokes equation it satisfies the Heat equation, which in our case has data in L^1. Thus we know that the vorticity is bounded in L^1, since $\nabla u_o \in L^1$. Moreover since the data $u_o \in L^2 \cap L^1$ and $\nabla u_o \in L^2$ it follows that ω decays in L^2 at a rate of $(1+t)^{-n/4-1/2}$, with $n = 2$, [11]. Thus estimate (4. 15) yields

$$|u(x,t)| \leq \left(|u_1(x,t)|^2 + |u_2(x,t)|^2\right)^{1/2} \leq$$
$$Cr^{-\alpha/2}(1+t)^{-\beta/2(n/4+1/2)}$$

From the last equation we have a clear interplay be-teen the time and space decay. Moreover we can check the relation we had obtained in (3. 12). In our case we have $m = 0, \gamma = 0, \mu = n/4$ and (3. 12) thus reduces to

$$2\rho_0 + 2k = n$$

This relation holds in our example, since we have $\rho_0 = \beta/2(n/4 + 1/2)$ and $k = \alpha/2$ thus the last relation translates for $n = 2$ into

$$\alpha + \beta = 2$$

which follows by the definition of α and β.

To extend this example to all even dimensions we use the solution constructed in [10]. We quote the theorem that gives the extension.

Theorem 4.1 *Suppose n is even and let 1. $\omega : [0,\infty) \times I\!R \to I\!R$ be such that the function $v(x,t) = \omega(|x|,t)$ is a solution of the Heat equation $v_t = \triangle v$; so*

$$\omega_t = \omega_{rr} + \frac{n-1}{r}\omega_r;$$
$$2.g(r,t) = r^{-n}\int_o^r s^{n-1}w(s,t)ds;$$
$$3.A = (a_{ij}) \quad \textit{is an} \quad n \times n \quad \textit{matrix with real entries such that}$$
$$A^2 = \lambda I \quad \textit{for some } \lambda \in I\!R, x^tAx = O \quad \textit{for all} \quad x \in I\!R^n$$

Then the function $u(x,t) = g(|x|,t)Ax$ satisfies
a) $u_t = \triangle u$.
b) There exists a function p such that (u,p) is a solution to the incompressible Navier-Stokes equations.

Proof : See [10].
We note that the matrix A will be an $n \times n$ and will have block in the diagonal of order 2×2 which will coincide with the block for the 2D case. The rest of the matrix will have zeroes.

Since the solutions constructed in Theorem (4.1) is of similar structure as the 2D solutions with radial vorticity one can compute the space-time decay as in the example above.

References

[1] C. Amrouche, V. Girault, M.E. Schonbek, and T.P. Schonbek, *Pointwise decay of solutions and higher derivatives to Navier-Stokes equations,* Preprint.

[2] R. Kajikiya and T. Miyakawa, *On the L^2 decay of weak solutions of the Navier-Stokes equations in $\mathbb{R}^n$,* Math. Z. **192** (1986), 135-148.

[3] T. Kato, *Strong L^p solutions of the Navier-Stokes equations with applications to weak solutions,* Math. Zeit., Vol = 187, 1982, 471-480

[4] H. Kozono, *Weak and classical solutions of the two-dimensional magneto-hydrodynamic equations,* Tohoku Math. J. , **41** (1989), 471–488.

[5] H. Kozono and T. Ogawa, *Two dimensional Navier-Stokes equations in unbounded domains,* Math. Ann., Vol 297, 1993, 1-31.

[6] M. E. Schonbek, *L^2 decay of weak solutions of the Navier-Stokes equations,* Arch. Rational Mech. Anal. **88** (1985), 209-222.

[7] M. E. Schonbek, *Large time behavior of solutions to the Navier-Stokes equations,* Comm. P.D.E. **11** (1986), 733-763.

[8] M. E. Schonbek, *Lower bounds of rates of decay for solutions to the Navier-Stokes equations,* J.A.M.S. **4**, (1991), 423-449.

[9] M.E.Schonbek and T.P. Schonbek, *On the boundedness and decay of moments of solutions of the Navier-Stokes equations,* Preprint.

[10] SCHONBEK, M.E. AND SCHONBEK, T.P. AND SÜLI, E. *Large-time behavior of solutions to the Magneto-Hydrodynamic equations* Mathematische Annalen, 304 (1996) no.4 717-756..

[11] M.E. SCHONBEK AND M. WIEGNER, *On the decay of higher order norms of the solutions of Navier-Stokes equations,* Proceedings of the Royal Society of Edinburgh, section A -Mathematics , Vol= 126 , 1996 , 677-685.

[12] S. TAKAHASHI, *A weighted equation approach to decay rate estimates for the Navier-Stokes equations,* Preprint.

[13] M. WIEGNER, *Decay results for weak solutions to the Navier-Stokes equations in* $\mathbb{R}^n$, J. London Math. Soc. (2) **35** (1987), 303-313.

[14] ZHANG LINGHAI, *Sharp rates of decay of global solutions to 2-dimensional Navier-Stokes equations,* Preprint.