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POINTWISE DECAY OF SOLUTIONS AND OF HIGHER DERIVATIVES TO NAVIER–STOKES EQUATIONS*

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Abstract. In this paper we study the space-time asymptotic behavior of the solutions, and their derivatives, to the incompressible Navier–Stokes equations in dimension $2 \le n \le 5$. Using moment estimates we obtain that strong solutions to the Navier–Stokes equations which decay in L^2 at the rate of $\|u(t)\|_2 \le C(t+1)^{-\mu}$ will have the following pointwise space-time decay, for $0 \le k \le n/2$:

$$|D^\alpha u(x,t)| \le C_{k,m} \frac{1}{(t+1)^{\rho_0}(1+|x|^2)^{k/2}},$$

where $\rho_O = (1-2k/n)(m/2+\mu+n/4)$, $|\alpha| = m$ and $\mu > \frac{n}{4}$.

Key words. Navier–Stokes equations, derivatives, pointwise algebraic decay

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1. Introduction.

In this paper, we study the space-time decay of solutions to the incompressible Navier–Stokes equations in $\mathbb{R}^n$

$$\begin{aligned} u_t + u \cdot \nabla u + \nabla p &= \triangle u\,, \\ \operatorname{div} u &= 0\,, \\ u(x,0) = u_0(x) &\in \mathbf{X}\,, \end{aligned} \tag{1.1}$$

and of their derivatives. We assume $2 \le n \le 5$ and the space $\mathbf{X}$ will be specified below. Using moment techniques, we show that strong solutions and their derivatives of all orders decay pointwise at an algebraic rate as $|x| \to \infty$ and $t \to \infty$.

Questions of decay of solutions to the Navier–Stokes equations in different norms have been studied, among others, by Knightly [6], Kajikiya and Miyakawa [4], Kato [5], Kozono [7], Kozono and Ogawa [8], Schonbek [13], [10], Wiegner [18], and Zhang [20]. Of particular interest in the direction of the present paper are the results by S. Takahashi [17]. In this reference, Takahashi studies the pointwise decay of solutions with zero initial data to the Navier–Stokes equations with an external force, as well as the decay of the first derivatives of these solutions. Using a weighted-equation approach, he obtains pointwise decay rates both in time and space. The external force is assumed to decay at an algebraic rate in both space and time and the solutions are assumed bounded in some weighted $L^{q,s}$ norms, with $n/q + 2/s = 1$ and $q, s \in [2, \infty]$,

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(the limiting Serrin class), where $L^{q,s}$ denotes the space of all $u : \mathbb{R}^n \times (0,\infty) \to \mathbb{R}^n$ such that

$$\left\{\int_0^\infty \left(\int_{\mathbf{R}^n} |u(x,t)|^q dx\right)^{s/q} dt\right\}^{1/s} < \infty.$$

Our results complement and extend Takahashi's results in the sense that in our case we have nonzero initial data but zero external force. Moreover, we are able to establish decay for derivatives of all orders. We note that since we are obtaining decay results for derivatives, we will work directly with strong solutions. These results can be derived for weak solutions provided we start at a sufficiently large time. Since in this case we are already in the regime where the solutions are smooth, we prefer to simplify notation and work directly with smooth solutions. The reader can also refer to [17], which presents a very detailed outline of what other authors in the field have done with related questions.

It is already clear at the level of the heat equation that there is a relation between the time decay and the space decay. This kind of balance will also be found for solutions to the Navier–Stokes equations. In particular the balance relation we obtain between the decay in space and in time coincides with the relation for the heat equation when we consider the solutions themselves.

The plan of the paper is the following. We begin with a section of notation (section 2). In section 3, we construct a solution of the Navier–Stokes equations as the limit of a sequence of solutions of a linearized approximation of the Navier–Stokes equations. By standard uniqueness results, this solution coincides with the one constructed by Kato in [5]. We recall some essential estimates on the moments of this sequence of approximate solutions and of their derivatives and then we show that these bounds are also valid for the limit solution and its derivatives. The first bounds we obtain are not sufficient for yielding a uniform time decay; they are valid for all time but depend on time. However, owing to the results of [11], we already have uniform bounds for the moments, though not for the moments of the derivatives; for this reason we dedicate section 4 to showing that these moments are also bounded independently of time. The last section deals with the space-time pointwise decay of the solution, which follows from the uniform bound of the moments and an appropriate form of the Gagliardo–Nirenberg inequality.

2. Notation and assumptions. Let $\alpha = (\alpha_1, \ldots, \alpha_n)$ be a multi-index with $\alpha_i \geq 0$. We will use the notation

$$D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}\,, \tag{2.1}$$

where

$$|\alpha| = \alpha_1 + \cdots + \alpha_n\,, \tag{2.2}$$

and

$$D_i = \frac{\partial}{\partial x_i}\,. \tag{2.3}$$

For any integer $m \geq 0$, we set

$$D^m f(x) = \left(\sum_{|\alpha|=m} |D^\alpha f(x)|^2\right)^{1/2},$$

where $x = (x_1, \dots, x_n)$. The L^2 norm (or energy norm) will be denoted by

$$\|u\| = \|u(.,t)\|_2 = \left[\int_{\mathbf{R}^n} |u(x,t)|^2 dx\right]^{1/2}, \tag{2.4}$$

where $dx = dx_1 \cdots dx_n$. More generally we denote the L^p norm for $1 \le p < \infty$ by

$$\|u(.,t)\|_p = \left[\int_{\mathbf{R}^n} |u(x,t)|^p dx\right]^{1/p} \tag{2.5}$$

and the L^∞ norm by

$$\|u(.,t)\|_\infty = \text{ess sup}_x |u(x,t)|. \tag{2.6}$$

The H^m norm is defined by

$$\|u(.,t)\|_{H^m} = \left[\int_{\mathbf{R}^n} \sum_{|\alpha| \le m} |D^\alpha u(x,t)|^2 dx\right]^{1/2}. \tag{2.7}$$

In what follows, we assume that $u = u(x,t) = (u_1(x,t), \dots, u_n(x,t))$ is a global solution of the Navier–Stokes equations with the following decay: there exist constants $C, \mu > n/4$ such that

$$\|u(t)\|_2 \le C(t+1)^{-\mu} \quad \text{for } t \ge 0\,. \tag{2.8}$$

Under these conditions, assuming as always $2 \le n \le 5$, it is proved in [12] that the decay given by (2.8) generalizes to

$$\|D^j u(t)\|_2 \le C(t+1)^{-\mu - j/2} \quad \text{for } t \ge 0\,, j = 0, 1, 2, \dots\,. \tag{2.9}$$

We recall the Gagliardo–Nirenberg inequality; if $f \in H^m$, then

$$\|D^j f\|_\infty \le C \|f\|_2^{1-a} \|D^m f\|_2^a\,,$$

with $a = a_{jm} = \frac{j + \frac{n}{2}}{m}$, as long as $j + \frac{n}{2} < m$. Taking m large enough (assuming we can do this) we get from (2.8) and (2.9)

$$\|D^j u(t)\|_\infty \le C(t+1)^{-\mu - j/2 - n/4} \quad \text{for } j = 0, 1, \dots\,. \tag{2.10}$$

Combining (2.9) and (2.10) we get as in [12]

$$\|D^j u(t)\|_p \le C(t+1)^{-\mu - j/2 - n/4(1-2/p)} \quad \text{for } j = 0, 1, \dots \tag{2.11}$$

for $p \in [2, \infty]$, $t > 0$.

Since we are interested in decay of derivatives and hence in smooth solutions, we are going to work with solutions that start with small data, or the results we establish will only be valid for large t.

The main idea in order to obtain pointwise decay is to prove decay of the moments and then combine this with an appropriate Gagliardo–Nirenberg inequality to yield decay in L^∞, whence the pointwise decay. With this in mind, we introduce the following weighted spaces:

$$f \in L^{r_1}_\nu \quad \text{iff} \quad \left(\int_{\mathbf{R}^n} |x|^{\nu r_1} |f|^{r_1}\, dx\right)^{1/r_1} < \infty. \tag{2.12}$$

For $s = 0, 1, 2, \ldots$, we define the (s, α) moments

$$M_{s,\alpha}(t) = \int_{\mathbf{R}^n} |x|^s |D^\alpha u(x,t)|^2 \, dx,$$

and in particular for $s \geq 0$, $t \geq 0$, we define the moment of order s of u by

$$M_s((u)(t)) = M_{s,0}(t) = \int_{\mathbf{R}^n} |x|^s |u(x,t)|^2 \, dx = \left(\|u(t)\|_{L^2_{s/2}} \right)^2.$$

Finally, define for $s, m = 0, 1, 2, \ldots$,

$$\tilde{M}_{s,m}(t) = \sum_{|\alpha|=m} M_{s,\alpha}(t) = \int_{\mathbf{R}^n} |x|^s |D^m u(x,t)|^2 \, dx.$$

3. Preliminaries. To start our calculations we need to recall some weighted-norms estimates satisfied by approximate solutions to the Navier–Stokes equations [11]. These solutions satisfy a "linearized Navier–Stokes equation," in which both the convective and the pressure terms are linearized in "explicit form." To this purpose, the pressure is expressed as a product of Riesz transforms. Specifically, we construct the sequence $\{u^\ell\}$ of approximate solutions as follows: $v = u^{\ell+1}$ is the solution of

$$\begin{aligned} v_t - \Delta v + u^\ell \cdot \nabla v + \nabla P(u^\ell, v) &= 0, \\ \operatorname{div} v &= 0, \\ v(0) &= u_0 \,, \end{aligned} \tag{3.1}$$

with initial approximation $u^0 = u_0$ and u_0 in an appropriate space. The solution v is constructed locally by a fixed-point argument and then is extended by a priori estimates. It is unique by construction. The bilinear operator P is defined by

$$P(u,v) = \sum_{j,k} R_j R_k (u_j v_k) \,,$$

where $u = (u_1, \ldots, u_n)$, $v = (v_1, \ldots, v_n)$ are functions from $\mathbf{R}^n$ to $\mathbf{R}^n$, and R_j denotes the Riesz transforms,

$$\widehat{[R_j f]}(\xi) = -i \frac{\xi_j}{|\xi|} \hat{f}(\xi) \quad \text{for } 1 \leq j \leq n \,.$$

When $u^\ell = v$ we recover the Navier–Stokes equations, since the pressure p and the velocity u of the Navier–Stokes equations are related by

$$\Delta p = -\sum_{j,k} \frac{\partial^2}{\partial x_j \partial x_k} (u_j u_k) \,,$$

hence

$$\hat{p}(\xi, t) = -\sum_{j,k} \frac{\xi_j \xi_k}{|\xi|^2} \widehat{u_j u_k}(\xi, t),$$

and

$$p = \sum_{j,k} R_j R_k (u_j u_k) = P(u,u).$$

The linearization (3.1) is of the type used by Caffarelli, Kohn, and Nirenberg in [1], by Kajikiya and Miyakawa in [4], by Leray in [9], and by Sohr, von Wahl, and Wiegner in [14]. The advantage of making the linearization explicit is that we can apply to the sequence $\{u^\ell\}$ well-known properties of the Riesz transforms, such as their boundedness in L^p-spaces (see [15]) and in weighted L^p-spaces satisfying the Muckenhoupt condition (see [3], [16]), in order to obtain bounds for the solutions of the Navier–Stokes equations and of their moments. We expect that our proofs for establishing bounds in weighted L^p-spaces, with some modifications, could be used for the approximating solutions constructed by Caffarelli, Kohn, and Nirenberg [1], by Kajikiya and Miyakawa [4], and by Sohr, von Wahl, and Wiegner [14].

In [11] we constructed the solution to (3.1) via a fixed-point method. We recall briefly the construction, referring to [11] for details. Let

$$F(x,t) = F(t)(x) = (4\pi t)^{-n/2} e^{-|x|^2/4t}$$

be the fundamental solution of the heat equation in n space variables and set

$$H(u,v) = u \cdot \nabla v + \nabla P(u,v)\,.$$

If v solves (3.1), then v has the expression

$$v(t) = F(t) * u_0 - \int_0^t F(t-s) * H(u^\ell, v)(s)\,ds. \tag{3.2}$$

For $u, \varphi \in L^2([0,T], H^1(\mathbf{R}^n)^n)$, we define

$$\begin{aligned} \mathcal{M}_u\varphi(t) &= \int_0^t F(t-s) * [u \cdot \nabla\varphi(s) + \nabla P(u,\varphi)(s)]\;ds \\ &= \int_0^t F(t-s) * H(u,\varphi)(s)\,ds \end{aligned} \tag{3.3}$$

and

$$\mathcal{L}_u\varphi(t) = F(t) * u_0 - \mathcal{M}_u\varphi(t). \tag{3.4}$$

The integral version (3.2) of (3.1) linearized with respect to u^ℓ becomes

$$v = \mathcal{L}_{u^\ell}(v);$$

that is, the solution to the linearized Navier–Stokes equation (3.1) can be obtained as a fixed point of the operator $\mathcal{L}_{u^\ell}$ (see [11]). We prove in [11] that for some $T > 0$, $T = \infty$ for small data, the sequence $\{u^\ell\}$ converges in $C([0,T], L^2 \cap L^r)$ to a weak solution of the Navier–Stokes equations, provided the data is in $L^2 \cap L^r$ and $r > n$. If the data is also in H^1 and is sufficiently small, the solution will be smooth. These are the solutions we will be interested in. Although Kato [5] has obtained smooth solutions with small data in $L^2 \cap L^n$, we do not use his construction because we want to ensure that the solutions also lie in the appropriate weighted space whenever the data belong to that space too. However, our solutions are clearly Hopf–Leray solutions (see [11, Theorem 2.4]); furthermore, in the notation used by Fabes, Jones, and Riviere in [2], they are in $L^{p,\tilde q}(\mathbb{R}^n \times (0,T))$ for every $T > 0$ for some $r > n, \tilde q > 2$

(see (3.6) below). By the uniqueness results of section V of [2] our solutions coincide with the solutions of Kato.

In [11] we needed to introduce numbers ν, q, r, r_1 satisfying the relations

$$0 \leq \nu < n, \quad 2 \leq r_1 \leq r, \quad 1 \leq q \leq \infty, \quad r > n\,, \tag{3.5}$$

$$\frac{1}{q} < \frac{\nu}{2} - \frac{n}{2r} + \frac{1}{2}, \quad \frac{1}{r} \leq \frac{1}{r_1} + \frac{\nu}{n} < 1 - \frac{1}{r}\,. \tag{3.6}$$

We recall Lemma 2.2 of [11], which we state here for convenience:

Lemma 3.1. *Assume the function u satisfies*

$$u \in C([0,T], W^{m,r}(\mathbf{R}^n)^n) \cap L^q([0,T], (W^{m,r_1}(\mathbf{R}^n)^n).$$

There exists a constant $K(T,u)$ of the form

$$K(T,u) = C(T)\left(\|u\|_{C_T(W^{m,r})} + \|u\|_{L^q_T(W^{m,r_1})}\right)$$

with $C(T)$ independent of u such that if $D^\alpha u_0 \in L^{r_1}_\nu \cap L^r(\mathbf{R}^n)^n$ for $|\alpha| \leq m$, then the fixed point v of $\mathcal{L}_u$ satisfies $D^\alpha v \in C([0,T], L^{r_1}_\nu(\mathbf{R}^n))$ for $|\alpha| \leq m$ and

$$\|D^\alpha v(t)\|_{L^{r_1}_\nu} \leq C(T)\left(\|u_0\|_{W^{m,r_1}} + \sum_{|\beta|\leq m} \|D^\beta u_0\|_{L^{r_1}_\nu}\right) + K(T,u)\left(\|u_0\|_{W^{m,r}} + \|u_0\|_{W^{m,r_1}}\right).$$

If u is a strong solution of the Navier–Stokes equations, then u is the fixed point of $\mathcal{L}_u$; moreover, $r, r_1 \geq 2$ so that $K(T,u)$ is finite if $2 \leq n \leq 5$ by (2.11) for every $T > 0$. The following corollary is immediate.

Corollary 3.2. *Assume $2 \leq n \leq 5$, conditions* (3.5), (3.6)*, and let u be a strong solution of the Navier–Stokes equations with data $u_0 \in W^{m,r} \cap W^{m,r_1} \cap H^1(\mathbf{R}^n)^n$. Then*

$$\|D^\alpha u(t)\|_{L^{r_1}_\nu} \leq C(T)C_0, \tag{3.7}$$

where C_0 depends only on appropriate norms of the data.

4. Decay of moments of derivatives. In order to obtain the decay of moments of derivatives, we will first need to establish uniform bounds. Once these are obtained, the decay will follow by a Hölder inequality between the (m,s) moments and the L^2 norm of the derivatives.

Theorem 4.1. *Let u_0 be as in Corollary* 3.2*. Let u be a strong solution of the Navier–Stokes equations with data u_0 satisfying*

$$\|u(t)\|_2 \leq C(t+1)^{-\mu}, \quad \textit{where } \mu > \frac{n}{4} - \frac{1}{2}\,. \tag{4.1}$$

Then

$$\tilde{M}_{s,m}(t) \leq C(t+1)^{-(2\mu+m)(1-\frac{s}{n})}\,, \tag{4.2}$$

for $m = 0, 1, 2, \ldots,\ s = 0, 1, \ldots, n$.

Proof. As before we note that if the data is sufficiently small then this solution u exists. In particular, if $u \in H^2 \cap L^\infty$, then all the derivatives of higher order are in L^6 (see [12]). Moreover, inequalities (2.9), (2.10), and (2.11) will hold.

For the proof, note that the case $s=0$ is covered by (2.9). Assuming $s>0$ from now on, we proceed by induction on m. Each induction step is dealt with following the approach of Theorem 4.1 of [11] (where the case $m=0$ is proved). As in [11, Theorem 4.1] the estimate for $0<s<n$ will follow from the estimates for $s=0$ and $s=n$ by Hölder interpolation. Indeed, let $1/p=(n-s)/n$, $1/p'=s/n$, and $|\alpha|=m$; we have

$$\begin{aligned} M_{s,\alpha}(t) = \int_{\mathbf{R}^n} |x|^s |D^\alpha u|^2\,dx &\le \left(\int_{\mathbf{R}^n} |D^\alpha u|^2\,dx\right)^{1/p} \left(\int_{\mathbf{R}^n} |x|^n |D^\alpha u|^2\,dx\right)^{1/p'} \\ &= M_{0,\alpha}(t)^{1-\frac{s}{n}} M_{n,\alpha}(u)(t)^{\frac{s}{n}} \le C(t+1)^{-(2\mu+m)(1-\frac{s}{n})} M_{n,\alpha}(u)(t)^{\frac{s}{n}}. \end{aligned}$$

Thus, if $M_{n,\alpha}(u)(t)$ is uniformly bounded, we have

$$\tilde{M}_{s,m}(t) \le C(t+1)^{-(2\mu+m)(1-\frac{s}{n})}\,.$$

It suffices thus to prove the estimate for $s=n$, which merely says that $\tilde{M}_{n,m}(t)$ is bounded uniformly with respect to t, for $t>0$. In other words, it suffices to prove

$$\sup_{t>0} \tilde{M}_{n,m}(t) < \infty \tag{4.3}$$

for $m=0,1,\dots$.

Let α be a multi-index with $|\alpha|=m$. For a function g and a multi-index β, we set $g_\beta = D^\beta g$. By Leibniz's product formula, differentiating (1), we obtain

$$u_{\alpha t} = \Delta u_\alpha - \sum_{\beta+\gamma=\alpha} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} u_\beta \cdot \nabla u_\gamma - \nabla p_\alpha;$$

dot multiplying by $|x|^s u_\alpha$ and using that $\operatorname{div} u = 0$ and $\operatorname{div} u_\alpha = 0$, we get, after some technical but straightforward manipulations,

$$\begin{aligned} |x|^s u_{\alpha t}\cdot u_\alpha = -|x|^s|\nabla u_\alpha|^2 &+ \frac{s}{2}(s-2+n)|x|^{s-2}|u_\alpha|^2 + \frac{s}{2}|x|^{s-2}(x\cdot u)|u_\alpha|^2 \\ &-|x|^s \sum_{\beta+\gamma=\alpha,\beta\neq 0} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} (u_\beta\cdot\nabla u_\gamma)\cdot u_\alpha + s|x|^{s-2}(x\cdot u_\alpha)p_\alpha \\ &+\operatorname{div} E_{s,\alpha}\,, \end{aligned}$$

where

$$E_{s,\alpha} = \frac{|x|^s}{2}\nabla(|u_\alpha|^2) - \frac{s}{2}|x|^{s-2}|u_\alpha|^2 x - \frac{|x|^s}{2}|u_\alpha|^2 u - |x|^s u_\alpha p_\alpha.$$

One can prove now, as in Lemma 6.1, Appendix B of [11], that

$$\liminf_{R\to\infty} \int_{|x|=R} |E_{s,\alpha}|\,dS = 0.$$

More precisely, the proof is a repetition of the arguments in the above mentioned lemma, where we replace u by u_α and use the appropriate estimates for the derivatives obtained in [12]. Thus

$$\int_{\mathbf{R}^n} \operatorname{div} E_{s,\alpha}\,dx = 0\,,$$

and we obtain

$$\frac{1}{2}\frac{d}{dt}M_{s,\alpha}(t) = A(t) + B(t) + C(t) + D(t), \tag{4.4}$$

where

$$A(t) = -\int_{\mathbf{R}^n} |x|^s |\nabla u_\alpha|^2\,dx + \frac{s}{2}(s-2+n)M_{s-2,\alpha}(t)\frac{s}{2}(s-2+n)M_{s-2,\alpha}(t),$$
$$B(t) = \frac{s}{2}\int_{\mathbf{R}^n} |x|^{s-2}(x\cdot u)|u_\alpha|^2\,dx,$$
$$C(t) = -\sum_{\beta+\gamma=\alpha,\beta\neq 0}\begin{pmatrix}\alpha\\ \beta\end{pmatrix}\int_{\mathbf{R}^n}|x|^s(u_\beta\cdot\nabla u_\gamma)\cdot u_\alpha\,dx,$$
$$D(t) = s\int_{\mathbf{R}^n}|x|^{s-2}(x\cdot u_\alpha)p_\alpha\,dx.$$

Assume $m=0$. Recall that we write M_s for $M_{s,0}$. We prove by induction on s that there exists $C\geq 0$ such that $M_s(u)(t)\leq C$ for all $t\geq 0$, $s=1,\ldots,n$. We begin considering the case $s=2$; the case $s=1$ follows by interpolation between the cases $s=0$ and $s=2$ and induction can then proceed in steps of 2; i.e., M_k bounded implies M_{k+2} bounded.

If A,B,C,D are as in (4.4) for $|\alpha|=m=0$, $s=2$, we get $A(t)\leq nM_{0,0}(u)(t) = n\|u(t)\|_2^2$, $C(t)=0$ and

$$B(t)\leq\int_{\mathbf{R}^n}|x||u|^3\,dx\leq M_2(u)(t)^{1/2}\|u(t)\|_4^2,$$
$$D(t)\leq M_2(u)(t)^{1/2}\|p(t)\|_2^2\leq CM_2(u)(t)^{1/2}\|u(t)\|_4^2$$

so that by (4.4)

$$\frac{d}{dt}M_2(u)(t)\leq CM_2(u)(t)^{1/2}\|u(t)\|_4^2 + n\|u(t)\|_2^2. \tag{4.5}$$

By (2.11) (with $j=0$ and $p=4$)

$$\|u(t)\|_4\leq C(t+1)^{-\mu-n/8};$$

it follows from this and (4.1)

$$\frac{d}{dt}M_2(u)(t)\leq n(t+1)^{-2\mu} + CM_2(u)(t)^{1/2}(t+1)^{-\delta}$$

with $\delta = 2\mu+n/4$. A bit of elementary arithmetic yields

$$\frac{d}{dt}M_2(u)(t)\leq n(t+1)^{-2\mu} + C(t+1)^{-\delta} + CM_2(u)(t)(t+1)^{-\delta}.$$

Since the moments are bounded (time dependent) and since $\delta>1$ it follows by Gronwall's inequality that

$$M_2(u)(t)\leq Ce^{\int_0^\infty (t+1)^{-\delta}dt}\leq \text{const}$$

proving the case $s = 2$. Assume now $s > 2$. In this case

$$A(t) \leq \frac{s}{2}(s+n-2)M_{s-2}(u)(t), \tag{4.6}$$

$$B(t) \leq \int_{\mathbf{R}^n} |x|^{s-1}|u|^3\,dx \leq M_s(u)(t)^{(s-1)/s}\|u(t)\|_{s+2}^{(s+2)/s}, \tag{4.7}$$

$$D(t) \leq \int_{\mathbf{R}^n} |x|^{s-1}|u||p|\,dx \leq CM_s(u)(t)^{(s-1)/s}\|u(t)\|_{s+2}^{(s+2)/s}. \tag{4.8}$$

Inequality (4.7) is an immediate consequence of Hölder's inequality (with exponents $s/(s-1)$ and s). For (4.8) notice first that, by Hölder's inequality,

$$\begin{aligned}\int_{\mathbf{R}^n} |x|^{s-1}|u||p|\,dx &\leq \left(\int_{\mathbf{R}^n} |x|^s|u|^2\,dx\right)^{1/2}\left(\int_{\mathbf{R}^n} |x|^{s-2}|p|^2\,dx\right)^{1/2}\\ &= M_s(u)(t)^{1/2}\|p\|_{L^2_\nu}\end{aligned}$$

with $\nu = s/2 - 1$. Then $n/(n-\nu) < 2$ (since $s < n+2$) hence the Riesz transforms are bounded in L^2_ν (see [11, Lemma 5.1]); since $p = -\sum_{j,k} R_j R_k(u_j u_k)$, we get

$$\begin{aligned}\|p\|_{L^2_\nu} &\leq C\|\,|u|^2\|_{L^2_\nu}\\ &= C\left(\int_{\mathbf{R}^n} |x|^{s-2}|u|^4\,dx\right)^{1/2} \leq M_s(u)(t)^{(s-2)/2s}\|u\|_{s+2}^{(s+2)/s},\end{aligned}$$

where we factored $|u|^4 = |u|^{2-4/s}|u|^{2+4/s}$ and used Hölder's inequality with exponents $s/(s-2)$, $s/2$. Inequality (4.8) follows. To continue estimating, we get by Hölder's inequality, (4.1), and (2.10) (for $j = 0$)

$$\|u(t)\|_{s+2} \leq \|u(t)\|_2^{2/(s+2)}\|u\|_\infty^{s/(s+2)} \leq C(t+1)^{-\mu-(ns)/(4s+8)};$$

by Hölder's inequality and (4.1)

$$M_{s-2}(u)(t) \leq M_s(u)(t)^{(s-2)/s}\|u(t)\|_2^{4/s} \leq C(t+1)^{-4\mu/s}M_s(u)(t)^{(s-2)/s};$$

whence combining with (4.6), (4.7), (4.8), and (4.4),

$$\begin{aligned}\frac{d}{dt}M_s(u)(t) &\leq C_1(t+1)^{-4\mu/s}M_s(u)(t)^{(s-2)/s}\\ &\quad + C_2(t+1)^{-\mu(s+2)/s-n/4}M_s(u)(t)^{(s-1)/s}.\end{aligned}$$

We estimate the two terms on the right-hand side (R.H.S.) using $M_s^\tau \leq 1 + M_s$ for $\tau = (s-2)/s$ and $\tau = (s-1)/s$, respectively; we get

$$\frac{d}{dt}M_s(u)(t) \leq C_1(t+1)^{-\rho} + C_2(t+1)^{-\rho}M_s(u)(t),$$

where

$$\rho = \min\left\{\frac{4\mu}{s}, \frac{s+2}{s}\mu + \frac{n}{4}\right\} > 1.$$

Integrating from 0 to t, considering that

$$\int_0^t (\sigma+1)^{-\rho}\,d\sigma \leq \int_0^\infty (\sigma+1)^{-\rho}\,d\sigma = \frac{1}{\rho-1} < \infty,$$

we get

$$M_s(u)(t) \le C\,(1 + M_s(u)(0)) + C\int_0^t (\sigma+1)^{-\rho} M_s(u)(\sigma)\,d\sigma.$$

By Gronwall's lemma,

$$M_s(u)(t) \le C\,(1 + M_s(u)(0))\,e^{\int_0^\infty (\sigma+1)^{-\rho}\,d\sigma} \le C_0 < \infty$$

which completes the proof of the case $m = 0$.

Assume now m is a positive integer and that the estimates (4.2) have been proved up to $m-1$; $s = 0, \dots, n$. Let $|\alpha| = m$. Time dependent bounds for $M_{s,\alpha}(u)(t)$ are easily established by induction on s, $0 \le s \le n$. In fact, the case $s = 0$ is (as already mentioned) immediate and the induction proceeds by means of energy estimates which are quite straightforward and as such will be omitted; the reader can refer to [11] for details of a similar proof. With this established, to obtain the uniform bound we proceed as follows. Let $A(t)$, $B(t)$, $C(t)$, $D(t)$ be as in (4.4) with $s = n$.

Bound for $A(t)$.

Notice first that if $n = 2$ then

$$A(t) \le 2M_{0,\alpha}(t) \le C_0(1+t)^{-2\mu-1}\,, \tag{4.9}$$

where $2\mu > n/2 = 1$. Suppose now that $3 \le n \le 5$; by Hölder's inequality and by (2.9),

$$M_{n-2,\alpha}(t) \le M_{n,\alpha}(t)^{(n-2)/n}\|u_\alpha(t)\|_2^{4/n} \le C(1+t)^{-\rho}M_{n,\alpha}(t)^{(n-2)/n},$$

with $\rho = (4/n)(\mu + m/2) > 1$. In general, from now on, ρ denotes a constant > 1, not the same one in all inequalities. By the definition of $A(t)$, using also

$$(1+t)^{-\rho}M_{n,\alpha}(t)^{(n-2)/n} \le \frac{2}{n}(1+t)^{-\rho} + \frac{n-2}{n}(1+t)^{-\rho}M_{n,\alpha}(t),$$

we prove that

$$|A(t)| \le C(1+t)^{-\rho}\left(1 + \tilde{M}_{n,m}(t)\right)\,. \tag{4.10}$$

Bound for $B(t)$.

$$\begin{aligned}|B(t)| &= \left|\frac{n}{2}\int_{\mathbf{R}^n} |x|^{n-2}(x\cdot u)|u_\alpha|^2\,dx\right| \le \frac{n}{2}\int_{\mathbf{R}^n} |x|^{n-1}|u||u_\alpha|^2\,dx \\ &\le \frac{n}{2}\|u_\alpha\|_2^{2/n}\|u\|_\infty\left(\tilde{M}_{n,m}(t)\right)^{(n-1)/n}\,,\end{aligned}$$

so that by (2.9) and (2.10),

$$\begin{aligned}|B(t)| &\le C(1+t)^{-\rho}\left(\tilde{M}_{n,m}(t)\right)^{(n-1)/n} \\ &\le C(1+t)^{-\rho}\left(1 + \tilde{M}_{n,m}(t)\right),\end{aligned}$$

where this time $\rho = (2/n)(\mu + m/2) + \mu + n/4 > 1$.

Bound for $C(t)$.

Note that C(t) is a sum in terms of α and β, where $|\beta|+|\gamma|=|\alpha|$ and $\beta\neq 0$. The general term in $C(t)$ can be estimated by

$$\int_{\mathbf{R}^n} |x|^n\,|(u_\beta\cdot\nabla u_\gamma)\cdot u_\alpha|\,dx \le \|D^j u\|_\infty \tilde M_{n,\ell}(t)^{1/2}\tilde M_{n,m}(t)^{1/2},$$

where $j=\min(|\beta|,|\gamma|+1)$, $\ell=\max(|\beta|,|\gamma|+1)$, so that $0\le j\le [m/2]$, $[(m+1)/2]\le \ell\le m$, and $j+\ell=m+1$. When $\ell=m$, and so $j=1$, (2.10) implies a bound of the form

$$C(1+t)^{-(\mu+n/4+1)}\tilde M_{n,m}(t).$$

The terms with $\ell<m$ are bounded, using the induction hypothesis and (2.10), by

$$C(1+t)^{-(\mu+n/4+j/2)}\tilde M_{n,m}(t)^{1/2},$$

and we obtain again an estimate of the form

$$|C(t)|\le C(1+t)^{-\rho}\left(1+\tilde M_{n,m}(t)\right), \tag{4.11}$$

where $\rho>1$.

Bound for $D(t)$.

Since the Riesz transforms are bounded in L^2_ν with $\nu=(n-2)/2$, and D^α commutes with the Riesz transforms, we can write

$$p_\alpha = D^\alpha p = \sum_{j,k} R_jR_k[D^\alpha(u_ju_k)] = \sum_{k,j,\beta+\gamma=\alpha}\begin{pmatrix}\alpha\\ \beta\end{pmatrix} R_jR_k(u_{\beta,j}u_{\gamma,k}),$$

and we have

$$\begin{aligned}|D(t)| &= \left|n\int_{\mathbf{R}^n}|x|^{n-2}(x\cdot u_\alpha)p_\alpha\,dx\right|\\ &\le C\int_{\mathbf{R}^n}|x|^{n-1}|u_\alpha||p_\alpha|\,dx \le C\tilde M_{n,m}(t)^{1/2}\|p_\alpha\|_{L^2_\nu}\\ &\le C\tilde M_{n,m}(t)^{1/2}\sum_{\beta+\gamma=\alpha}\||u_\beta||u_\gamma|\|_{L^2_\nu}.\end{aligned}$$

Then Hölder's inequality gives

$$\begin{aligned}\||u_\beta||u_\gamma|\|_{L^2_\nu} &= \left(\int_{\mathbf{R}^n}|x|^{n-2}|D^\beta u|^2|D^\gamma u|^2\,dx\right)^{1/2}\\ &\le C\|D^j u\|_\infty\|D^\ell u\|_2^{1/n}\tilde M_{n,\ell}(t)^{(n-2)/2n},\end{aligned}$$

with $j=\min(\beta,\gamma)$, $\ell=\max(\beta,\gamma)$ (so $0\le j\le m/2$). Once more we apply (2.9), (2.10) to get $\|D^j u\|_\infty\|D^\ell u\|_2^{1/n}\le C(1+t)^{-\rho}$ with $\rho=(1/n)(\mu+\ell/2)+\mu+n/4+j/2>1$. By the induction hypothesis $\tilde M_{n,\ell}(t)$ is bounded uniformly in t if $\ell<m$, so all terms with $\ell<m$ in the last estimate for D can be bounded by $C(1+t)^{-\rho}$ and the remaining term is bounded by

$$C(1+t)^{-\rho}\tilde M_{n,m}(t)^{(n-2)/2n}\le C(1+t)^{-\rho}\left(1+\tilde M_{n,m}(t)\right)$$

so that $D(t)$ has a bound of the same type as $A(t)$, $B(t)$, $C(t)$. Combining all the above estimates, we derive

$$\frac{d}{dt}\tilde{M}_{n,m}(t) \le C(1+t)^{-\rho} + C(1+t)^{-\rho}\tilde{M}_{n,m}(t)\,,$$

where $\rho > 1$. Hence, integrating in this inequality, we find

$$\tilde{M}_{n,m}(t) \le \left(\tilde{M}_{n,m}(0) + \frac{C}{\rho-1}\right) + C\int_0^t (s+1)^{-\rho}\tilde{M}_{n,m}(s)\,ds.$$

Then Gronwall's lemma implies

$$\tilde{M}_{n,m}(t) \le \left(\tilde{M}_{n,m}(0) + \frac{C}{\rho-1}\right)e^{c/(\rho-1)}\,,$$

thus proving that $\tilde{M}_{n,m}(t)$ is bounded uniformly with respect to t for $t > 0$. ☐

Note. We took some pains to avoid having to bound $\|D^j u\|_\infty$ for $j > [(m+1)/2]$. In this way, bounds on the L^2-norm of derivatives of order m will give (sometimes) all the needed L^∞ bounds on the $D^j u$'s.

The next theorem establishes the spatial and time decay of strong solutions to equations for which the moments decay.

THEOREM 4.2. *Let $2 \le n \le 5$. With the assumptions of Theorem 4.1, let u be a strong solution u of the Navier–Stokes equations with data u_0. Let $k \le n/2$. Then*

$$|D^\alpha u(x,t)| \le C_{k,m}\frac{1}{(t+1)^{\rho_0}(1+|x|^2)^{k/2}}\,, \tag{4.12}$$

where $\rho_0 = (\mu + m/2 + n/4)(1-2k/n)$ and $|\alpha| = m$.

Proof. Note that n is restricted to the values $2 \le n \le 5$ for which we have estimates for the moments. The main tools for the proof are Theorem 4.1 and the Gagliardo–Nirenberg inequality. Let

$$v(x,t) = (1+|x|^2)^{k/2}D^\alpha u(x,t)\,.$$

By Leibniz's formula, we have

$$D^s v = \sum_{j=0}^{s} c_j^s (1+|x|^2)^{\frac{k-j}{2}} D^{s-j}u_\alpha\,. \tag{4.13}$$

Together with the decay of the moments of derivatives given by Theorem 4.1, this formula implies that

$$\|D^s v\|_2 \le C_0\sum_{j=0}^{s}(1+t)^{-(\mu+m/2+(s-j)/2)(1-2(k-j)/n)}\,. \tag{4.14}$$

Since the function $f(j) = (\mu + m/2 + (s-j)/2)(1-2(k-j)/n)$ is increasing, it has a minimum at $j = 0$. Thus we have

$$\|D^s v\|_2 \le C_0(1+t)^{-(\mu+m/2+s/2)(1-2k/n)}\,. \tag{4.15}$$

In particular when $s = 0$,

$$\|v\|_2 \le C_0(1+t)^{-(\mu+m/2)(1-2k/n)}\,. \tag{4.16}$$

Let us apply the Gagliardo–Nirenberg inequality with $a = n/(2s) < 1$, provided $n/2 < s$, i.e., $s > [n/2]$, to get

$$\|v(\cdot,t)\|_\infty \le \|v(\cdot,t)\|_2^{1-a}\|D^s v(\cdot,t)\|_2^a\,. \tag{4.17}$$

Combining with (4.16) and (4.15) yields

$$|(1+|x|^2)^{k/2}D^\alpha u(x,t)| \le ||v(t)||_\infty \le C_0(1+t)^{-\rho_0}\,,$$

where

$$\begin{aligned}\rho_0 &= (1-2k/n)\left((\mu+m/2+s/2)n/(2s)+(1-n/(2s))(\mu+m/2)\right)\\ &= (1-2k/n)(\mu+m/2+n/4)\,.\end{aligned}$$

We note that the above value of ρ_0 is independent of s. Thus we could have obtained it using only the s derivative with $s > [n/2]$. In particular note that when $n = 3$, it suffices to use $s = 2$ and $\rho_0 = (\mu + m/2 + 3/4)(1 - 2k/3)$. The proof is complete. ∎

4.1. Comparison with the heat equation. It is easy to show that the fundamental solution of the heat equation,

$$E(x,t) = (4\pi t)^{-n/2}e^{-|x|^2/4t}\,,$$

which is the linear part of the Navier–Stokes equations, has the following asymptotic behavior:

$$|D^\alpha E(x,t)| \le c_0|x|^{-a}t^{-b},$$

where $a+2b = n+m$, with $m = |\alpha|$. It is also easy to show that there is a large class of solutions to the heat equation which will have the same type of decay. For instance solutions such that the data satisfies $u_0 \in \mathcal{K}$ where

$$\mathcal{K} = \{u_0 : u_0(y) \ge e^{-y^2/4t_0}\}$$

will have the above type of decay, provided we are considering $t \ge t_0 + \varepsilon$. In the case of solutions to the Navier–Stokes equations, if we take $\mu = n/4$, the relation that holds between the decay in space and in time is

$$2\rho_0 + 2k = m + n - \frac{2km}{n}\,.$$

For $k = 0$, we recover the decay of the heat equation, but this only gives decay in time. If $m = 0$ we recover the relation $2\rho_0 + 2k = n$; i.e., we have the same decay relation in space and in time as for solutions to the heat equation.

Final remarks. We expect that our results can be extended easily to dimensions 6 and 7 using the L^2 decay results, for derivatives of higher order, recently obtained by Wiegner [19].

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