Geodesic Flows on Diffeomorphisms of the Circle, Grassmannians, and the Geometry of the Periodic KdV Equation

**Maria E. Schonbek, Andrey N. Todorov, and
Jorge P. Zubelli**

UCSC, California, USA

schonbek@cats.ucsc.edu

UCSC and Bulgarian Academy of Sciences, Sofia

todorov@cats.ucsc.edu

IMPA, Rio de Janeiro, Brazil

zubelli@impa.br

Abstract

We start by constructing a Hilbert manifold $\mathcal{T}$ of orientation preserving diffeomorphisms of the circle (modulo the group of bi-holomorphic self-mappings of the disc). This space, which could be thought of as a completion of the universal Teichmüller space, is endowed with a right-invariant Kähler metric.

Using results from the theory of quasiconformal mappings we construct an embedding of $\mathcal{T}$ into the infinite dimensional Segal-Wilson Grassmannian. The latter turns out to be a very natural ambient space for $\mathcal{T}$. This allows us to prove that $\mathcal{T}$'s sectional curvature is negative in the holomorphic directions and by a reasoning along the lines of Cartan-Hadamard's theory that its geodesics exist for all time.

The geodesics of $\mathcal{T}$ lead to solutions of the periodic Korteweg-de Vries (KdV) equation by means of V. Arnold's generalization of Euler's equation. As an application, we obtain long-time existence of solutions to the periodic KdV equation with initial data in the periodic Sobolev space $\mathcal{H}_{\mathrm{per}}^{3/2}(\mathbb{R}, \mathbb{R})$.

1   Introduction

The interplay between the theory of infinite dimensional completely integrable systems and complex analysis has been extremely fruitful to both fields. We feel, however, that there are still many pieces of the picture missing, especially if one takes into account the development of the theory of Teichmüller spaces and quasiconformal mappings.

Our main object of study concerns geometric aspects of a certain class of orientation preserving diffeomorphisms of the circle modulo conformal diffeomorphisms of the disk. More precisely, this Teichmüller space $\mathcal{T}$ is endowed with a Hilbert manifold structure by means of a (unique up to constant) right-invariant Kähler metric and contains the Teichmüller space of C^∞ diffeomorphisms of the circle. The construction of such space $\mathcal{T}$ is the object of Section 2.

One of the motivations for such study is the remarkable fact that investigating the geodesics of $\mathcal{T}$ will give information about the behavior of solutions to the periodic Korteweg-de Vries equation

$$\partial_t u = \partial_x^3 u + 6u\partial_x u. \tag{1}$$

The connection between the geodesic flow on $\mathcal{T}$ and the KdV equation follows from V. Arnold's approach to Euler's equation [4, 5], which was applied by Khesin and Ovsienko [39] to one of the co-adjoint orbits of

the Bott-Virasoro group. We apply the same ideas to *a different* co-adoint orbit. As far as we know, this is a new remark (see also [46]). The upshot is the following result, which is proved in Appendix 1:

Theorem 1. *If $\mathcal{T}$ is endowed with the above mentioned right-invariant Kähler metric, then one can associate to each geodesic through the identity a solution to the KdV equation. Furthermore, in this case, the solution to the KdV equation exists for all time.*

To prove this, we shall need some preparation. Our first result, which seems to be proved in the literature [46] only for the case of C^∞ diffeomorphisms, is the following:

Theorem 2. *The space $\mathcal{T}$ embeds isometrically into the Segal-Wilson Grassmannian, when the latter is endowed with the Hilbert-Schmidt norm.*

By virtue of this embedding, we are in position of studying the curvature properties of $\mathcal{T}$ in a very convenient ambient space. This allows us to prove the following basic result, which is of interest on its own:

Theorem 3. *The sectional curvature of $\mathcal{T}$ in the holomorphic directions is bounded from above by a negative number.*

As a corollary, we can identify the Teichmüller space $\mathcal{T}$ with the infinite-dimensional Siegel disc. See Section 7.

Kirillov and Yurev [22, 23] have formulas for the curvature of $\mathcal{T}$, but we believe that this is the first time one shows the negativity of the sectional curvature in the holomorphic direction for the invariant Kähler metric of $\mathcal{T}$.

A consequence of this circle of ideas is that the existence of geodesics on $\mathcal{T}$ implies existence of periodic solutions to the KdV equation. More precisely, through an application of the Hopf-Rinow theorem [10, 13] to a certain two dimensional totally geodesic complete manifold we obtain long time existence of periodic solutions to the KdV equation with initial (real) data in the 2π-periodic Sobolev space $\mathcal{H}^{3/2}_{\text{per}}(\mathbb{R}, \mathbb{R})$.

This approach has a very different flavor from the standard proof of existence of solutions to the KdV equation, which can be found for

example in [14, 17, 18, 19, 20] and references therein. We feel that this gives a very promising geometrical picture of the problem. The hope being that this can be applied to more general problems.

A good part of the paper is dedicated to the computation of the curvature tensor of $\mathcal{T}$. We show that sectional curvature in the holomorphic directions is less than $-3/2$. This information is then used together with an extension to infinite dimensions of the Cartan-Hadamard theorem to show the Arnold exponential instability of the geodesic flow. The general approach for infinite dimensional case of the Cartan-Hadamard theory can be found in S. Lang's book [26].

This paper will be organized as follows:

In Sections 2 and 3 we construct a holomorphic equivariant map from the Teichmüller space

$$\mathcal{T} = \text{Diff}_+^{3/2}(S^1)/\text{PSU}_{1,1}$$

to the Segal-Wilson Grassmannian. Here, and throughout this paper, we denote by $\text{PSU}_{1,1}$ the sub-group of linear fractional transformations that send the unit disk into itself. The embedding is done by recalling that each $\phi \in \mathcal{T}$ can be expressed as the composition of two univalent functions f_0^{-1} and f_∞. The function f_∞ is defined in the exterior of the disc, which we call $\mathbb{D}_\infty$, and f_0^{-1} is defined on the complement of $f_\infty(\mathbb{D}_\infty)$. From the function f_∞ we define the Beltrami operator. Hence, we associate to each ϕ the space of solutions W_ϕ to the Beltrami equation with a certain complex dilation μ_ϕ.[1] The functions in W_ϕ when restricted to S^1 are naturally elements of the Hilbert space $H \overset{\text{def}}{=} L^2(S^1)$. Thus, W_ϕ can be identified with a subspace of H, which we denote by $\mathcal{W}_\phi$. However, more can be said. Recall the standard decomposition $H = H_+ \oplus H_-$ as the direct sum of non-negative indexed Fourier components and negative indexed Fourier components. We show in Section 3 that the projection $\text{pr}_- : \mathcal{W}_\phi \to H_-$ is Hilbert-Schmidt. Since from our construction it follows immediately that $\text{pr}_+ : \mathcal{W}_\phi \to H_+$ is an isomorphism we get that $\mathcal{W}_\phi$ is a point in the Segal-Wilson Grassmannian [47]. This Grassmannian, which plays an important role in theory of solitons, is used here in a totally novel way (for the traditional approach see [16, 47]). In our construction, the KdV flow appears as a geodesic

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[1]The idea of looking at the elements of the Grassmannian as the boundary values of solutions of differential equations is due to Witten [52].

flow in $\mathcal{T}$, which can be in turn embedded into the Segal-Wilson Grassmannian.

Section 4 is dedicated to endowing the space $\mathcal{T}$ with a Hilbert manifold structure.

In Section 6 we show that the Teichmüller space $\mathcal{T}$ equipped with the unique (modulo a constant) right-invariant Kähler metric has negative Gaussian curvature. The proof is based on the construction of the so-called Cartan coordinate system.

Section 7 is concerned with the description of the geodesics. Here we show that since the geodesics stay in a certain two-dimensional manifold, we can employ the Hopf-Rinow theorem to establish global existence in time of the geodesics. Hence, using the results reviewed in the Appendix 1 we are able to show global existence of solutions to the periodic KdV equation with initial data in $\mathcal{H}^{3/2}_{\mathrm{per}}$. Section 7 ends with an infinite dimensional analogue of Cartan-Hadamard's theorem [10, 13] showing Arnold's exponential spreading of the geodesics.

In Section 8 we give a procedure to construct periodic solutions to the KdV equation. This is done explicitly, modulo the Riemann mapping theorem, by using solutions of Beltrami's equation.

The paper ends with two appendices. On the first one, we give a review of Arnold's point of view on Euler's equation and the construction of geodesics. On the second one, we summarize the different results from Teichmüller theory used throughout the paper.

We close the introduction with a bit of notation. We shall denote by $\mathcal{H}^s_{\mathrm{per}}(\mathbb{R}; \mathbb{C})$ the set of (2π-periodic) distributions $\sum_{n \in \mathbb{Z}} a_n \exp(inx)$ such that $\sum_{n \in \mathbb{Z}} (1 + n^2)^s |a_n|^2 \leq \infty$. We shall often identify this space with $\mathcal{H}^s(S^1)$ by setting $z = \exp(ix)$.

2   Construction of a Holomorphic map from $\mathrm{Diff}_+^{3/2}(S^1)/\mathrm{PSU}_{1,1}$ to the Segal-Wilson Grassmannian

We recall that the Segal-Wilson Grassmannian is the set of closed subspaces W of $H = H_+ \oplus H_-$ such that the projection $\mathrm{pr}_+ : W \to H_+$ is Fredholm and the projection $\mathrm{pr}_- : W \to H_-$ is Hilbert-Schmidt, where H is the L^2 space of complex functions on S^1.

The goal of Sections 2 through 5 is to construct the space [2]

$$\mathcal{T} = \mathrm{Diff}_+^{3/2}(S^1)/\mathrm{PSU}_{1,1}$$

and simultaneously to embed it into the Segal-Wilson Grassmannian. This embedding will prove to be instrumental in the computation of the curvature of $\mathcal{T}$ and the construction of the Cartan coordinates. One may think of $\mathcal{T}$ as a completion of the space of $\mathrm{Diff}_+^\infty(S^1)/\mathrm{PSU}_{1,1}$ with respect to a right-invariant Kähler metric. This abstract definition, however, does not seem to make the embedding into the Grassmannian evident. We chose therefore to proceed in a more concrete way by constructing $\mathcal{T}$ and its embedding simultaneously.

This approach also highlights different but equally important aspects of $\mathcal{T}$. From one side one can think of $\mathcal{T}$ as a space of Beltrami differentials, from a second one, as an orbit space of the Virasoro group, and from a third one as a complete Kähler manifold isometrically embedded into the Segal-Wilson Grassmannian.

In the present section we limit ourselves to constructing a map

$$\Psi : \mathcal{T} \cap \mathcal{D} \to \widetilde{Gr},$$

where

$$\widetilde{Gr} = \{W \text{ closed subspace of } H | \mathrm{pr}_+ : W \to H_+ \text{ is an isomorphism}\},$$

and $\mathcal{D}$ is an open set close to the identity map of S^1.

In the next section we show that if ϕ is a diffeomorphism of the circle satisfying the additional assumptions 1 and 2, which we make explicit

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[2]The upper index $3/2$ in the above formula is to underscore that the metric on the tangent space is equivalent to a Sobolev $3/2$ norm.

bellow, then the image $W = \Psi(\phi)$ has the property that $\mathrm{pr}_- : W \to H_-$ is Hilbert-Schmidt.

The outline of the main steps is essentialy the following: From a diffeomorphism ϕ we construct the corresponding Beltrami differential. Given the Beltrami differential we construct an operator $\bar{\partial}_\phi$ on the plane, which agrees with the usual $\bar{\partial}$ operator outside the disk. We then construct a suitable (non-orthonormal) basis for $\ker \bar{\partial}_\phi$ and finally restrict ourselves to the unit circle to obtain $\Psi(\phi) = \ker \bar{\partial}_\phi$.

We will show that the vector space generated by elements in $\ker \bar{\partial}_\phi$ when restricted to the circle gives us a point in the Grassmannian.

We start with some notation. The open unit disc centered at the origin will be denoted by $\mathbb{D}_0$. The complement to the closure of $\mathbb{D}_0$ will be

$$\mathbb{D}_\infty \overset{\mathrm{def}}{=} \{z \in \widehat{\mathbb{C}} \mid |z| > 1\},$$

where $\widehat{\mathbb{C}}$ denotes the Riemann sphere $\mathbb{C} \cup \{\infty\}$. The set of compactly supported functions of class C^k on $\mathbb{C}$ will be denoted by $C_0^k(\mathbb{C})$. We denote by $\mathrm{PSU}_{1,1}$ the group of bi-holomorphic functions from the disc $\mathbb{D}_0$ into itself.

Our construction is based on the so called *sewing problem*, which also plays a role in other problems of complex analysis [22, 28, 29]. Given ϕ a quasisymmetric homeomorphism of the circle, find a pair of homeomorphisms f_0 and f_∞, such that

a) $f_0 : \overline{\mathbb{D}}_0 \longrightarrow f_0(\overline{\mathbb{D}}_0) \subset \widehat{\mathbb{C}}$ and $f_\infty : \overline{\mathbb{D}}_\infty \longrightarrow f_\infty(\overline{\mathbb{D}}_\infty) \subset \widehat{\mathbb{C}}$, where f_0 and f_∞ are conformal in the interior of their domains of definition,

b) The sets $f_0(\mathbb{D}_0)$ and $f_\infty(\mathbb{D}_\infty)$ are complementary Jordan domains, and

c) For every $z \in S^1$

$$\phi(z) = f_0^{-1} \circ f_\infty(z) \ ,$$

where the superindex -1 indicates the inverse function.

In the Appendix 2 we shall review the notion of quasisymmetric functions and cover some known results related to the theory of Teichmüller

spaces. For the time being we remark that the sewing problem stated above has a unique solution provided we require ϕ, f_0 and f_∞ to be normalized. One such normalization is achieved by requiring that they all fix the points -1, $-i$ and 1 of S^1. See [40, 28, 29].

We recall that the Schwarzian derivative of an analytic function f is defined by

$$\mathcal{S}(f) \stackrel{\text{def}}{=} \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2.$$

We are now ready to introduce a key object in our construction:

Definition 4. For ϕ a normalized quasisymmetric function we set

$$\mu_\phi(z) = \begin{cases} -\frac{1}{2}(1-|z|^2)^2 \mathcal{S}[f_\infty(1/z)](\bar{z}), & z \in \mathbb{D}_0 \\ 0, & z \in \mathbb{C} \setminus \mathbb{D}_0. \end{cases}$$

Since f_∞ is univalent, Nehari's Lemma (see [1]) implies that

$$\|\mu_\phi\|_\infty \leq 3.$$

We assume the following:

Assumption 1. The function ϕ is such that the solution of the sewing problem f_∞ satisfies

$$\|\mu_\phi\|_\infty < 1. \tag{2}$$

In Appendix 2 we show that if ϕ is sufficiently close to the identity map in the C^1 topology, then Assumption 1 is satisfied.

We define the Beltrami operator:

Definition 5. Let $\mu \in L^\infty(\mathbb{C})$. For $f \in C_0^\infty(\mathbb{C})$ we define the operator $\bar{\partial}_\mu$ by

$$\bar{\partial}_\mu f(z) \stackrel{\text{def}}{=} \bar{\partial} f(z) - \mu(z)\partial f(z).$$

We shall denote also by $\bar{\partial}_\mu$ the extension (in the weak-derivative sense) of this operator to more general functions, such as $f \in L^2(\mathbb{C})$ whose distributional derivatives ∂f and $\bar{\partial} f$ are locally in L^2.

Recall that H_+ denotes

$$H_+ = \left\{ f \in L^2(S^1) | f(z) = \sum_{n \geq 0} a_n z^n, z \in S^1 \right\},$$

and H_- denotes its orthogonal complement in $L^2(S^1)$. Hence, $H = H_+ \oplus H_-$.

Theorem 6. *The map $\phi \to \ker \bar{\partial}_{\mu(\phi)}|_{S^1} = W_\phi \subset H$ defines an isometric embedding of $\mathrm{Diff}_+^\infty(S^1)/\mathrm{PSU}_{1,1}$ into the Segal-Wilson Grassmannian endowed with the Hilbert-Schmidt norm. (The metric on $\mathrm{Diff}_+^\infty(S^1)/\mathrm{PSU}_{1,1}$ will be defined in Section 4.)*

The main step in the proof of this theorem, which will be given in Section 3, is to show that $pr_- : W_\phi \to H_-$ is Hilbert-Schmidt. The proof of this fact relies heavily on the construction of a special basis for W. The rest of this section is devoted to this construction.

We define some operators that will play an important role.

Definition 7. Let $h \in L^p(\mathbb{C})$, then

$$Ph(\zeta) \overset{\text{def}}{=} -\frac{1}{\pi} \int_{\mathbb{R}} \int_{\mathbb{R}} h(z) \left(\frac{1}{z - \zeta} \right) \, dxdy, \tag{3}$$

where $z = x + iy$.

Definition 8. Let $h \in C_0^2(\mathbb{C})$, then

$$Th(\zeta) \overset{\text{def}}{=} \lim_{\epsilon \to 0} -\frac{1}{\pi} \int \int_{|z-\zeta|^2 > \epsilon} \frac{h(z)}{(z - \zeta)^2} dxdy,$$

where $z = x + iy$.

It can also be shown that (Lemma 2 page 87 of [1])

$$\bar{\partial} Ph = h,$$

and

$$\partial Ph = Th.$$

From a result of Calderon and Zygmund it is known that T extends as a bounded operator from $L^p(\mathbb{C})$ into itself of norm C_p for any $p > 1$, where $C_p \to 1$ as $p \to 2$.

Construction of Special Solutions to Beltrami's Equation:

We define $\nu^{(n)}$ by

$$\nu^{(n)} = \sum_{k=1}^{\infty} T_k^n(\mu_\phi),$$

where $T_0^n(\mu_\phi) = nz^{n-1}$ and

$$T_k^n(\mu_\phi) = T(\mu_\phi(T_{k-1}^n(\mu_\phi)))$$

Now we set, for $n > 0$,

$$w^{(n)} \stackrel{\text{def}}{=} z^n + P(\mu_\phi(\nu^{(n)} + nz^{n-1})). \tag{4}$$

The following result is proved for the case of $n = 1$ in [1] (Theorem 1, page 91). The general case is a generalization of the proof therein and will be given in the Appendix B.

Theorem 9. *For any integer $n \geq 1$ we have that $w^{(n)}$ is the unique solution of the problem:*

$$\begin{cases} \bar{\partial}_{\mu_\phi} w^{(n)} = 0, \\ \partial_z w^{(n)} - nz^{n-1} \in L^p \text{ for some } p > 2, \\ \int_0^{2\pi} w^{(n)}(e^{i\theta})d\theta = 0. \end{cases} \tag{5}$$

Note that from equation (4)

$$\mu_\phi(\nu^{(n)} + nz^{n-1}) = \mu_\phi \sum_{k=0}^{\infty} (T \circ \mu_\phi)^k(nz^{n-1}),$$

where, by $T \circ \mu_\phi$ we mean the composition of the operator T with the operator of multiplication by μ_ϕ. So, with this notation, we get another way of writing $w^{(n)}$. It can be written as

$$w^{(n)} = z^n + P\left[\mu_\phi \sum_{k=0}^{\infty} (T \circ \mu_\phi)^k(nz^{n-1})\right]. \tag{6}$$

Lemma 10. *The restriction to S^1 of second term on the r.h.s. of equation (6) is an element H_-.*

Proof. Since $w^{(n)}|_{D_\infty}$ is complex analytic we have that

$$w^{(n)}\Big|_{D_\infty} = z^n + f(z) + \sum_{j>1} a_j z^{-j}, \qquad (7)$$

where $f(z)$ is an entire function. From the condition that $\partial_z w^{(n)} - nz^{n-1} \in L^p$ it follows that $f'(z)$ is zero. Hence, $f(z) = a_0$, where a_0 is a constant. From the chosen normalization for the definition of the operator P in equation (3) it is easy to see that

$$a_0 = \frac{1}{2\pi i} \int_{S^1} P\left[\mu_\phi\left(\nu^{(n)} + nz^{n-1}\right)\right](\zeta)\frac{d\zeta}{\zeta} = 0.$$

$\square$

Definition 11. The space $\mathcal{W}_\phi$ is defined as the graph of the closure of the operator $w_\phi : H_+ \to H_-$ that maps $z^n \mapsto v^{(n)}$, where for $n = 1, 2, \ldots$

$$v^{(n)} \overset{\text{def}}{=} P\left[\mu_\phi \sum_{k=0}^\infty (T \circ \mu_\phi)^k (nz^{n-1})\right]\Bigg|_{S^1}$$

and $v^{(0)} = 0$.

Remarks.

1. The space $\mathcal{W}_\phi$ is spanned by the set $w^{(i)} = z^i + v^{(i)}$, since it is the graph of the operator w_ϕ. The main result of the next section is that w_ϕ is Hilbert-Schmidt. This implies in particular that $\mathcal{W}_\phi$ is closed. From the definition of $\mathcal{W}_\phi$, it follows that $\text{pr}_+ : \mathcal{W}_\phi \to H_+$ is an isomorphism. Hence, $\text{pr}_- : \mathcal{W}_\phi \to H_-$ is a Hilbert-Schmidt operator. So the results of the next section show that

$$\phi \mapsto \mathcal{W}_\phi \qquad (8)$$

is a well defined map from $\mathcal{T}$ into the Grassmannian Gr.

2. In Section 4 we shall show that the map defined in equation (8) is holomorphic. Furthermore, we show that when restricted to $\phi \in C^\infty$ it coincides with the map defined in [36]. From the definition used in [36] it follows directly the equivariance of this map.

3   The Projection $\mathrm{pr}_- : \mathcal{W}_\phi \to H_-$ is Hilbert-Schmidt

The main goal of this section is to prove that under appropriate assumptions the operator w_ϕ is a Hilbert-Schmidt operator. In Section 4 we prove that all the operators w_ϕ in a certain neighborhood of the identity have such property.

We recall that for H_1 and H_2 Hilbert spaces, an operator $T : H_1 \to H_2$ is called Hilbert-Schmidt if for one orthogonal basis $\{e_i\}_{i \in I}$ of H_1 we have

$$\sum_{i \in I} \|Te_i\|^2 < \infty .$$

If this is the case for one basis of H_1 it is also the case for every basis of H_1.

For technical reasons, which will become clear bellow, we shall need to use the following hypothesis: [3]

Assumption 2. The function $g = -\frac{1}{2}\mathcal{S}[f_\infty(1/z)]$ is bounded in $\mathbb{D}_0$.

We can now state the main result of this section, namely:

Theorem 12. *If ϕ satisfies Assumptions 1 and 2 the operator w_ϕ of Definition 11 is a Hilbert-Schmidt operator.*

Proof of the Theorem.   The proof of the theorem is based on showing that under the hypothesis the sequence $\{v^{(n)}\}_{n \geq 1}$ of Definition 11 satisfies

$$\sum_{n=1}^{\infty} \|v^{(n)}\|^2 < \infty.$$

Since the space $H_- = span\{z^{-i}|i \geq 1\}$, we are going to show that

$$\sum_{n=1}^{\infty} \sum_{j=1}^{\infty} |\langle v^{(n)}, z^{-j}\rangle|^2 < \infty.$$

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[3]In fact, from the proof bellow, such assumption may be relaxed by asking that $g(z)$ belongs to the (inner) Hardy class $\mathsf{H}^{3/2}(\mathbb{D}_0)$, i.e., $\sup_{r<1} \int_{-\pi}^{\pi} |g(re^{i\theta})|^{3/2} \; d\theta < \infty$. This condition is obviously a consequence of the Assumption 2.

Let's massage a little bit the expression for $|\langle v^{(n)}, z^{-j}\rangle|^2$. We are going to use the convention that $z = \rho \exp(ix)$.

An elementary application of Stoke's theorem gives that

$$
\begin{aligned}
\langle v^{(n)}, z^{-j}\rangle &= \frac{1}{2\pi} \int_0^{2\pi} v^{(n)}(z, \bar{z})\bar{z}^{-j} \ dx \\
&= \frac{1}{2\pi i} \int_{S^1} v^{(n)}(z, \bar{z})z^j \frac{dz}{z} \\
&= \frac{1}{2\pi i} \int_{\mathbb{D}_0} dv^{(n)} \wedge (z^{j-1}dz) \\
&= \frac{1}{2\pi i} \int_{\mathbb{D}_0} \bar{\partial}v^{(n)} \ d\bar{z} \wedge (z^{j-1}dz).
\end{aligned}
$$

But, from the construction of $v^{(n)}$ we have that

$$
\begin{aligned}
\bar{\partial}v^{(n)} &= \bar{\partial}P\left[\mu_\phi \sum_{k=0}^{\infty}(T \circ \mu_\phi)^k(nz^{n-1})\right] \\
&= \mu_\phi \sum_{k=0}^{\infty}(T \circ \mu_\phi)^k(nz^{n-1}),
\end{aligned}
$$

where, by $T \circ \mu_\phi$ we mean the composition of the operator T with the operator of multiplication by μ_ϕ. So,

$$
\langle v^{(n)}, z^{-j}\rangle = \frac{1}{2\pi i} \int_{\mathbb{D}_0} \mu_\phi \sum_{k=0}^{\infty}(T \circ \mu_\phi)^k(nz^{n-1})z^{j-1} \ d\bar{z} \wedge dz. \qquad (9)
$$

We now recall that

$$
\mu_\phi(z) = (1 - |z|^2)^2 g(\bar{z}), \qquad (10)
$$

where

$$
g(z) = -\frac{1}{2}\mathcal{S}[f_\infty(1/z)](z). \qquad (11)
$$

Now, we are going to apply Cauchy-Schwarz (for the inner-product of $L^2(\mathbb{D}_0)$) to the right hand side of equation (9). Indeed, we write

$\mu_\phi = \mu_\phi^{1/4}\mu_\phi^{3/4}$, except for possibly a set of zero measure in $\mathbb{D}_0$, and we get

$$\left| \frac{1}{2\pi i} \int_{\mathbb{D}_0} \mu_\phi \sum_{k=0}^{\infty} (T \circ \mu_\phi)^k (nz^{n-1}) z^{j-1} d\bar{z} \wedge dz \right|^2$$

$$\leq \|\mu_\phi^{1/4} z^{j-1}\|^2 \left\| \sum_{k=0}^{\infty} (T \circ \mu_\phi)^k (nz^{n-1}) \mu_\phi^{3/4} \right\|^2, \qquad (12)$$

where the norms are taken in $L^2(\mathbb{D}_0)$. We are going to estimate each of the norms of equation (12). First, $\|\mu_\phi^{1/4} z^{j-1}\|^2$.

$$\begin{aligned}
\|\mu_\phi^{1/4} z^{j-1}\|^2 &= \frac{1}{\pi} \int_0^{2\pi} \int_0^1 [(1-\rho^2)^{2/4}]^2 |g(1/\bar{z})|^{2/4} \rho^{2j-1} \; d\rho \; dx \\
&\leq 2\|g\|_{L^\infty(\mathbb{D}_0)}^{1/2} \int_0^1 (1-\rho^2)\rho^{2j-1} \; d\rho \\
&= \|g\|_{L^\infty(\mathbb{D}_0)}^{1/2} \frac{1}{j(j+1)} \\
&= \mathcal{O}(1/j^2).
\end{aligned}$$

Now, we work on the term $\|\sum_{k=0}^{\infty}(T \circ \mu_\phi)^k (nz^{n-1})\mu_\phi^{3/4}\|^2$. We begin by noticing that

$$\mu_\phi^{3/4} \sum_{k=0}^{\infty} (T \circ \mu_\phi)^k (nz^{n-1}) = \sum_{k=0}^{\infty} (\mu_\phi^{3/4} \circ T \circ \mu_\phi^{1/4})^k (\mu_\phi^{3/4} nz^{n-1}).$$

Now, we proceed as in the construction of $w^{(n)}$ (Section 3) by using the fact that the Hilbert transform T is an isometry of $L^2(\mathbb{C})$, to get

$$\left\| \sum_{k=0}^{\infty} (\mu_\phi^{3/4} \circ T \circ \mu_\phi^{1/4})^k (\mu_\phi^{3/4} nz^{n-1}) \right\| \leq \sum_{k=0}^{\infty} \|\mu_\phi\|_{L^\infty(\mathbb{D}_0)}^k \|nz^{n-1}\mu_\phi^{3/4}\|.$$

So,

$$\left\| \sum_{k=0}^{\infty} (\mu_\phi^{3/4} \circ T \circ \mu_\phi^{1/4})^k (\mu_\phi^{3/4} nz^{n-1}) \right\|^2 \leq \frac{1}{(1-c_0)^2} \|nz^{n-1}\mu_\phi^{3/4}\|^2,$$

where $c_0 \overset{\text{def}}{=} \|\mu_\phi\|_{L^\infty(\mathbb{D}_0)} < 1$.

Now, we estimate

$$
\begin{aligned}
\|nz^{n-1}\mu_\phi^{3/4}\|^2 &= \frac{1}{\pi}\int_0^{2\pi}\int_0^1 n^2[(1-\rho^2)^{3/2}]^2|g(\bar{z})|^{3/2}\rho^{2n-1}\,d\rho\,dx \\
&\leq \left(2n^2\int_0^1(1-\rho^2)^3\rho^{2n-1}\,d\rho\right)\|g\|_{L^\infty(\mathrm{D}_0)}^{3/2} \\
&= \frac{6n^2}{n(n+1)(n+2)(n+3)}\|g\|_{L^\infty(\mathrm{D}_0)}^{3/2} \\
&= \mathcal{O}(1/n^2),
\end{aligned}
$$

where in the next to last step above we used the fact that

$$
\int_0^1(1-\rho^2)^3\rho^{2n-1}\,d\rho = \frac{3}{n(n+1)(n+2)(n+3)}.
$$

We conclude the proof using the two estimates above for the norms that appear in the right hand side of equation (12) to get

$$
\begin{aligned}
\sum_{n=1}^\infty\|v^{(n)}\|^2 &\leq \sum_{n=1}^\infty\sum_{j=1}^\infty\left(\frac{C_1C_2}{j^2n^2}\right) \\
&= C_1C_2\left(\sum_{n\geq 1}\frac{1}{n^2}\right)^2 < \infty,
\end{aligned}
$$

where C_1 and C_2 are constants that depend only on ϕ. $\qquad\square$

Corollary 13. *The projection* $\mathrm{pr}_- : \mathcal{W}_\phi \to H_-$ *is Hilbert-Schmidt.*

We remark that the constants C_1 and C_2 in the proof of Theorem 12 depended only on the norms $\|g\|_{L^\infty(\mathrm{D}_0)}$ and $\|\mu_\phi\|_{L^\infty(\mathrm{D}_0)}$ where g and μ_ϕ were defined by equations (11) and (10). A straightforward corollary of this proof is the following:

Theorem 14. *Let g be constructed according to Section 2. Suppose that g is an element of the Hardy class $\mathsf{H}^{3/2}(\mathbb{D}_0)$ and that*

$$
\mu_\phi(z) = (1-|z|^2)^2 g(\bar{z})
$$

satisfies

$$
\|\mu_\phi\|_{L^\infty(\mathrm{D}_0)} \leq c_0 < 1.
$$

Then, the operator w_ϕ of Definition 11 is Hilbert-Schmidt and

$$\|w_\phi\|_{HS}^2 \leq \frac{K}{(1-c_0)^2}\|g\|_{\mathsf{H}^{3/2}(\mathsf{D}_0)}\|g\|_{\mathsf{H}^{1/2}(\mathsf{D}_0)},$$

where K is independent of g.

4   The Hilbert Manifold Structure of $\mathrm{Diff}_+^{3/2}(S^1)/\mathrm{PSU}_{1,1}$

Our goal in this section is to endow the manifold $\mathcal{T}= \mathrm{Diff}_+^{3/2}(S^1)/\mathrm{PSU}_{1,1}$ with a Hilbert manifold structure. This will be done in several steps.

Step 1.   Give a Hilbert space structure to the tangent space of the manifold $\mathcal{T}$ at the identity, which will be denoted by $T_{id}\mathcal{T}$.

Let $\mathcal{T}^\infty$ be the space of C^∞ orientation preserving diffeomorphisms of S^1 modulo the group $PSU_{1,1}$. The space $\mathcal{T}^\infty$ has a complex structure. By that we mean, one has a global splitting of the complexified tangent bundle

$$T\mathcal{T}^\infty \otimes \mathbb{C} = (T\mathcal{T}^\infty \otimes \mathbb{C})^{(1,0)} \oplus (T\mathcal{T}^\infty \otimes \mathbb{C})^{(0,1)}$$

such that

$$\overline{(T\mathcal{T}^\infty \otimes \mathbb{C})^{(1,0)}} = (T\mathcal{T}^\infty \otimes \mathbb{C})^{(0,1)}$$

and that the bracket of two $(1,0)$ vector fields is also $(1,0)$. Now, the holomorphic part $(T_{id}\mathcal{T}^\infty \otimes \mathbb{C})^{(1,0)}$, can be identified by means of

$$(T_{id}\mathcal{T}^\infty)^{(1,0)} \cong \left\{ f \in C^\infty(S^1;\mathbb{C}) | f = \sum_{n \geq 2} a_n z^n \right\}$$

To be concrete, we may think of an orientation preserving diffeomorphism γ on S^1 as $\gamma(e^{ix}) = \exp(i\psi(x,t))$, where $\psi(x+2\pi) = \psi(x)+2\pi$. Using such parametrization, a tangent vector to $\mathcal{T}$ at the origin is given by

$$\dot{\psi} = \sum_{k \neq \pm 1, 0} a_k e^{ikx} \frac{d}{dx},$$

where $a_k = \overline{a_{-k}}$ since $\dot\psi$ is real. Then, the corresponding vector in $(T\mathcal{T}^\infty \otimes \mathbb{C})^{(1,0)}$ is given by $\sum_{k\geq 2} a_k z^k$.

We recall that a Hermitean metric on a complex manifold is called Kähler if its imaginary part is a symplectic form. In [43] it is shown that there exists, up to a constant, a unique right-invariant Kähler metric on $\mathcal{T}^\infty$ such that for any $f = \sum_{n\geq 2} a_n z^n \in (T_{id}\mathcal{T}^\infty)^{(1,0)}$ we have

$$\|f\|^2 = \sum_{n\geq 2} n(n^2 - 1)|a_n|^2. \tag{13}$$

We define the Hilbert space $T_{id}\mathcal{T}$ as the completion of the $T_{id}\mathcal{T}^\infty$ with respect to this metric. It is easy to see that this is a linear subspace of H_+ which is complete in the Sobolev $\mathcal{H}^{3/2}_{\mathrm{per}}$-norm.

Step 2. Define an "exponential map":

Set

$$e_k(z) = \frac{z^k}{\sqrt{k(k^2 - 1)}}.$$

Obviously, the set $\{e_k | k \geq 2\}$ is an o.n. basis of $(T_{id}\mathcal{T})^{(1,0)}$ with respect to the metric defined in (13). Therefore, any f in such space can be written as:

$$f = \sum_{k\geq 2} t_k e_k$$

with the vector $(t_2, t_3, \dots) \in \ell^2$. We associate to the above f

$$\mu_t(z) = \mu(z; t_2, t_3, \dots)$$
$$\overset{\mathrm{def}}{=} \begin{cases} (1 - |z|^2)^2 \sum_{k\geq 2} t_k e_k(\bar z)\bar z^{-2}, & z \in \mathbb{D}_0 \\ 0, & z \in \widehat{\mathbb{C}} \setminus \mathbb{D}_0. \end{cases}$$

The μ_t constructed this way is a Beltrami differential provided that

$$\|\mu_t(\cdot)\|_{L^\infty(\mathbb{C})} < 1 . \tag{14}$$

Note that the set $\mathcal{O}$ of such values of the parameters $t \overset{\mathrm{def}}{=} (t_2, t_3, \dots)$ is open in ℓ^2. Indeed, one can easily estimate

$$\|\mu_t(.)\|_{L^\infty} \leq C'\|t\|_{\ell^2}.$$

It is a well known fact that [1, 6] for each $t \in \mathcal{O}$ there exists a quasi-conformal diffeomorphism ω_t such that:

$$\begin{cases} \bar{\partial}\omega_t - \mu_t \partial \omega_t = 0, \\ \omega_t(\rho) = \rho \text{ for } \rho \in \{-1, -i, 1\}. \end{cases} \tag{15}$$

We remark that this is a bit different from the usual normalization

$$\begin{cases} \partial\omega_t(z) - 1 \in L^p \text{ for some } p > 2 \\ \omega_t(0) = 0, \end{cases}$$

but the two are related by a linear fractional transformation.

Since $\mu_t(z) \equiv 0$ on $\mathbb{D}_\infty$, the function ω_t is univalent and complex analytic on $\mathbb{D}_\infty$. Let

$$f_{\infty,t} \overset{\text{def}}{=} \omega_t|_{\overline{\mathbb{D}}_\infty}.$$

The Riemann Mapping Theorem yields that there exists a unique univalent complex analytic function $f_{0,t}$ such that

$$\begin{cases} f_{0,t} : \mathbb{D}_0 \mapsto \widehat{\mathbb{C}} \setminus f_{\infty,t}(\mathbb{D}_\infty), \\ f_{0,t}(\rho) = \rho \text{ for } \rho \in \{-1, -i, 1\} \end{cases} \tag{16}$$

Let

$$\exp(t) = f_{0,t}^{-1} \circ f_{\infty,t}|_{S^1} = \sigma_t,$$

where the super-index -1 means the inverse function.

For further use we make note of the following estimates.

Lemma 15. *If we set, for $z \in \mathbb{D}_0$*

$$g(z) = \sum_{k \geq 2} t_k \frac{\bar{z}^{k-2}}{\sqrt{k(k^2 - 1)}},$$

then

$$\|g\|_{L^\infty(\mathbb{D}_0)} \leq C\|t\|_{\ell^2}.$$

Also, for $0 \leq \epsilon < 1$

$$\|\mu_t\|_{C^{1+\epsilon}(\mathbb{C})} \leq C'_\epsilon \|t\|_{\ell^2}.$$

Since the function $\mu_t(\cdot)$ defined above is of class $C^{1+\epsilon}$ for any $0 < \epsilon < 1$, a straightforward adaptation of Theorem 1 on page 269 of [6] gives that:

Proposition 16. *The quasiconformal mapping $z \mapsto \omega_t(z)$ defined above is of class $C^{2+\epsilon}$ for every $0 < \epsilon < 1$.*

As a consequence of this last fact, we can use a result of Kellog-Warschawski [50] (see also page 49 of [41]) that ensures that the inner mapping $f_{0,t}$ defined above is of class $C^{2+\epsilon}$ on $\mathbb{D}_0$. Hence, our manifold $\mathcal{T}$ will be composed of class $C^{2+\epsilon}$ diffeomorphisms. In particular, σ_t is quasisymmetric.

As an application of the Ahlfors-Weill result (see Appendix 2, Theorem 51) it follows that

Proposition 17. *For t in a neighborhood of $0 \in \ell^2$ the mapping*

$$t \mapsto \exp(t) \in \mathrm{Diff}_+^{2+\epsilon}(S^1)/\mathrm{PSU}_{1,1}$$

is locally one-to-one.

Step 3. Around each point ϕ in $\mathcal{T}^\infty$ we define a new neighborhood $N_{\phi;\epsilon}$ obtained by right translation of $N_{\sigma_0;\epsilon}$ by the function ϕ, where σ_0 is the identity map. In such a way we define the Hilbert manifold $\mathcal{T}$.

Note that for each $\phi \in \mathcal{T}$, the tangent space at ϕ is given by the span of the following orthonormal set:

$$\{e_k \circ \phi | k \geq 2\}.$$

Indeed, this can easily be seen by looking at the tangent space as equivalence classes of smooth paths in the manifold.

Theorem 18. *The space $\mathcal{T}$ is a complete metric space with respect to the Kähler metric.*

Proof. Let $\{\phi_n\}$ be any Cauchy sequence. Using the homogeneity of $\mathcal{T}$ we can assume that ϕ_n is contained in the set $N_0 = \exp(B(0; \epsilon))$, where $B(0, \epsilon)$ is a sufficiently small ball in ℓ^2. Since the exponential is 1-1 and a local diffeomorphism, it follows that the limit of $\exp^{-1}(\phi_n)$ exists. Hence, $\{\phi_n\}$ has a limit. $\square$

Remarks.

1. The estimate of Lemma 15 together with Theorem 14 and the Ahlfors-Weill result gives that the mapping

$$t \in \mathcal{O} \subset \ell^2 \mapsto \mathcal{W}_{\sigma_t} \in \mathrm{Gr}$$

   is continuous, provided we endow Gr with the Kähler metric of Section 7.8 of [42].

2. Along the lines of the Ahlfors-Bers' theorem [1]: If the Beltrami differential depends holomorphically on the parameter t then the solutions of equation (5) also depend holomorphically on t. Combining this theorem with the construction of the exponential map it follows easily that the map from $\mathcal{T}$ into Gr defined above is holomorphic.

We know that

$$\left\{ e_k = \frac{z^k}{\sqrt{k(k^2 - 1)}} \,\middle|\, k \geq 2 \right\}.$$

is an orthonormal basis in the tangent space to $id \in \mathcal{T}$. In the next Theorem we will compute the identification between the tangent and cotangent spaces with respect to the canonical Kähler metric on $\mathcal{T}$ which we will call the Weil-Petersson metric.

Theorem 19. *The identification of tangent space to the identity of the Universal Teichmüller space $\mathcal{T}$ with the cotangent space at the identity of $\mathcal{T}$ with respect to the Weil-Petersson Kähler metric is given by the following formula for $k \geq 2$:*

$$\mathcal{A}(e_k) = \mathcal{A}\left(\frac{z^k}{\sqrt{k(k^2 - 1)}} \frac{\partial}{\partial z} \right) = \left(\sqrt{2k(k^2 - 1)} z^{k-2} \right) dz.$$

Proof. We know that the tangent space of the Universal Teichmüller space $\mathcal{T}$ can be identified with the Beltrami differential on the unit disk, i.e. with tensors of the type $\mu \, \overline{dz} \otimes \frac{\partial}{\partial z}$. By using the Poincare metric $\mathcal{P} = \frac{1}{2\pi}(1 - |z|^2)^{-2} \, dz \wedge \overline{dz}$ on the unit disk we can identify canonically

the space of Beltrami differentials $\mu \, \overline{dz} \otimes \frac{\partial}{\partial z}$ with the space of quadratic differentials by the standard map:

$$\mu \overline{dz} \otimes \frac{\partial}{\partial z} \to \mu(1 - |z|^2)^{-2} \left(\overline{dz}\right)^{\otimes 2}.$$

So the cotangent bundle of the Universlal Teichmüller space can be canonically identified with the quadratic differentials on the unit disc restricted to the unit circle. The above expression should be invariant under the action of the group $\mathrm{PSU}_{1,1}$ on the unit disk. So we have that if $\omega = f(z)(dz)^{\otimes 2}$, is a quadratic diffenerial then

$$f(\gamma z) = \left(\frac{d\gamma}{dz}\right)^2 f(z),$$

where $\gamma \in \mathrm{P}SU_{1,1}$. The inner product $\langle \omega_1, \omega_2 \rangle_{W.P.}$ on the quadratic differentials $\omega_1 = f_1(z)(dz)^{\otimes 2}$ and $\omega_2 = f_2(z)(dz)^{\otimes 2}$ defined by the Weil-Petersson metric is given as follows:

$$\begin{aligned}
\langle \omega_1, \omega_2 \rangle_{W.P.} &= \frac{1}{2\pi i} \int_{\mathrm{D}_0} (1 - |z|^2)^4 f_1(z) \overline{f_2(z)} (1 - |z|^2)^{-2} dz \wedge \overline{dz} \\
&= \frac{1}{2\pi i} \int_{\mathrm{D}_0} (1 - |z|^2)^2 f_1(z) \overline{f_2(z)} dz \wedge \overline{dz}.
\end{aligned}$$

An easy computation shows that

$$\|z^n\|_{W.P.}^2 = \frac{1}{2\pi i} \int_{\mathrm{D}_0} (1 - |z|^2)^2 |z|^{2n} dz \wedge \overline{dz} = \int_0^1 (1 - r^2)^2 r^{2n+1} dr,$$

(17)

where $z = r \exp(i\theta)$. So by integrating twice by parts we get that

$$\int_0^1 (1 - r^2)^2 r^{2n+1} dr = \frac{1}{2(n+1)(n+2)(n+3)}.$$

From here, we deduce that under the canonical identification between the tangent and cotangent bundle on the universal Teichmüller space $\mathrm{Diff}_+(S^1)/\mathrm{P}SU_{1,1}$ is given by the following formulas

$$\Gamma(\mathrm{Diff}_+(S^1)/\mathrm{P}SU_{1,1}, T_{\mathrm{Diff}_+(S^1)/\mathrm{P}SU_{1,1}})$$
$$\longrightarrow \Gamma\left(\mathrm{Diff}_+(S^1)/\mathrm{PSU}_{1,1}, T^*_{\mathrm{Diff}_+(S^1)/\mathrm{PSU}_{1,1}}\right)$$
$$\frac{z^n}{\sqrt{n(n^2 - 1)}} \longmapsto \sqrt{2n(n^2 - 1)} z^{n-2}.$$

Our Theorem 19 is proved. $\square$

5   Conclusion of the Proof of the Embedding of $\mathcal{T}$ into the Grassmannian

As announced in Section 2 we shall now conclude the proof that $\mathcal{T}$ is embedded in the Segal-Wilson Grassmannian. This will be done by first showing that if $\|t\|_2$ is sufficiently small, then Assumptions 1 and 2 of Sections 2 and 3 are satisfied, and then by using that the manifold $\mathcal{T}$ is constructed by right translation of neighborhoods of the identity by C^∞ diffeomorphisms of the circle. Before that, we start with some notation and general remarks.

For $\mathcal{W}$ a subspace of $L^2(S^1)$ define

$$\mathcal{W} \circ \phi \stackrel{\text{def}}{=} \{h \circ \phi | h \in \mathcal{W}\}.$$

Note that if ϕ is a diffeomorphism of class C^1, then the map

$$R_\phi : L^2(S^1) \ni f \mapsto f \circ \phi \in L^2(S^1)$$

is a bounded linear transformation. We shall show that if $\phi \in C^{2+\epsilon}$ then R_ϕ maps the Segal-Wilson Grassmannian into itself. More precisely, R_ϕ belongs to the group $\mathrm{GL_{res}}(L_2(S^1))$. We recall [42] the definition of $\mathrm{GL_{res}}(L_2(S^1))$. A linear operator $A \in \mathrm{GL_{res}}(L_2(S^1))$ iff the following conditions are satisfied:

1. A is an invertible bounded linear operator of $L^2(S^1)$ onto itself.

2. If we write A in block matrix form with respect to the decomposition $L^2(S^1) = H_+ \oplus H_-$ as

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \tag{18}$$

   then, the operators b and c are Hilbert-Schmidt operators.

As in Section 2, we use the notation

$$\mu_\phi(z) = \begin{cases} -\frac{1}{2}(1 - |z|^2)^2 \mathcal{S}[f_\infty(1/z)](\bar{z}), & z \in \mathbb{D}_0 \\ 0, & z \in \mathbb{C} \setminus \mathbb{D}_0, \end{cases}$$

where (f_0, f_∞) is a solution of the normalized sewing problem associated to ϕ. (Note that $\phi \in \mathrm{Diff}_+^{3/2}(S^1)/\mathrm{PSU}_{1,1}$ and hence we can always choose the representative which is normalized.)

Lemma 20. *Let σ_t be as defined in Step 2 of the construction of Section 4. Then, σ_t is quasisymmetric and there exists $\epsilon > 0$ such that $\|t\|_2 < \epsilon$ implies*

$$\|\mu_{\sigma_t}\|_{L^\infty} < 1. \tag{19}$$

Let $(f_{0,t}, f_{\infty,t})$ denote the solution of the sewing problem associated to σ_t. Then, there exists $\epsilon' > 0$ such that $\|t\|_2 < \epsilon'$ implies that there exists M such that

$$|\mathcal{S}[f_{\infty,t}(1/z)]| < M, \ \forall z \in \mathbb{D}_\infty. \tag{20}$$

Hence, for all t in a sufficiently small neighborhood of 0 the Assumptions 1 and 2 are satisfied.

Proof. The fact that σ_t is quasisymmetric is a simple consequence of the fact that it is a $C^{2+\epsilon}$ diffeomorphism, as we remarked above. To prove inequalities (19) and (20), we shall apply the Ahlfors-Weill result, which we review in the Appendix 2. Notice that if we define

$$\varphi(z) \stackrel{\text{def}}{=} \left(\sum_{k \geq 2} \frac{t_k (1/z)^{k-2}}{\sqrt{k(k^2-1)}} \right) \left(-\frac{2}{z^4} \right),$$

then φ is holomorphic inside $\mathbb{D}_\infty$. Furthermore, for

$$\|t\|_2 < \epsilon_0,$$

where ϵ_0 is sufficiently small, we have that

$$\sup_{z \in \mathbb{D}_\infty} \left| -2\frac{(1-|z|^2)^2}{z^4} \sum_{k \geq 2} \frac{t_k (1/z)^{k-2}}{\sqrt{k(k^2-1)}} \right| < 2.$$

Let's consider the Beltrami equation

$$\bar{\partial} F - \mu_t \partial F = 0, \tag{21}$$

where μ_t is the Beltrami differential

$$
\mu_t = \begin{cases} -\frac{1}{2}\frac{(1-|z|^2)^2}{\bar{z}^4}\varphi(1/\bar{z}) = (1-|z|^2)^2 \sum_{k\geq 2}\frac{t_k \bar{z}^{k-2}}{\sqrt{k(k^2-1)}}, & z \in \mathbb{D}_0 \\ 0, & z \in \mathbb{D}_\infty. \end{cases}
$$

Because of the Ahlfors-Weill result we have

$$
\varphi(z) = \mathcal{S}[F] \ , \ z \in \mathbb{D}_\infty
$$

where F is a quasiconformal mapping satisfying the Beltrami equation (21). Now, by the construction of σ_t, we have that the pair $(f_{0,t}, f_{\infty,t})$ is the unique solution of the sewing problem for σ_t. Furthermore, $f_{\infty,t}$ coincides with F on $\mathbb{D}_\infty$ provided F is given the normalization (15).

Therefore,

$$
\varphi(z) = \mathcal{S}[f_{\infty,t}](z), \ z \in \mathbb{D}_\infty.
$$

In other words,

$$
\mathcal{S}[f_{\infty,t}](z) = \left(-\frac{2}{z^4}\right)\sum_{k\geq 2}\frac{t_k(1/z)^{k-2}}{\sqrt{k(k^2-1)}}, \ z \in \mathbb{D}_\infty.
$$

Taking $\|t\|_2$ sufficiently small we then have

$$
\left\| -\frac{1}{2}(1-|z|^2)^2 \mathcal{S}[f_{\infty,t}(1/z)] \right\|_{L^\infty(\mathbb{D}_0)} \leq c_0 < 1,
$$

and that $\mathcal{S}[f_{\infty,t}(1/z)]$ is bounded in $\mathbb{D}_0$. $\square$

The goal now is to extend the embedding of the neighborhood of the identity of $\mathcal{T}$ in the Grassmannian to all the manifold $\mathcal{T}$. To perform this extension we will need the following:

Lemma 21. *Let $\phi \in \mathrm{Diff}_+^\infty(S^1)$ and σ_t in the image of the exponential map for t in a sufficiently small neighborhood of 0 so that Assumptions 1 and 2 are satisfied. Take*

$$
\chi \overset{\text{def}}{=} \mu_{\sigma_t \circ \phi}
$$

and

$$\nu \stackrel{\text{def}}{=} \mu_{\sigma_t}.$$

Then,

$$\ker \bar{\partial}_\chi\Big|_{S^1} = \ker \bar{\partial}_\nu\Big|_{S^1} \circ \phi.$$

Proof. Take $\psi = \sigma_t \circ \phi$. Let $\widetilde{\psi}$ and $\widetilde{\sigma}_t$ be the Beurling-Ahlfors extensions of ψ and σ_t, respectively. We are going to show that $\widetilde{\psi}^{-1} \circ \widetilde{\sigma}_t$ maps solutions of

$$\bar{\partial}_\chi G = 0 \tag{22}$$

into solutions of

$$\bar{\partial}_\nu \widetilde{G} = 0 \tag{23}$$

by means of

$$\widetilde{G} = G \circ (\widetilde{\psi}^{-1} \circ \widetilde{\sigma}_t).$$

Let F_χ and F_ν be the unique solutions of equations (22) and (23) with the conditions of say fixing the points -1, $-i$, and 0. (Theorem 43 of Appendix 2.)

Claim. For points in the interior of $\mathbb{D}_0$ we have

$$\bar{\partial}_\nu F_\chi \circ \widetilde{\psi}^{-1} \circ \widetilde{\sigma}_t = 0.$$

Indeed, since F_χ and $\widetilde{\psi}$ both have the same Beltrami coefficient inside $\mathbb{D}_0$ it follows that $\bar{\partial} F_\chi \circ \widetilde{\psi}^{-1} = 0$. So, $F_\chi \circ \widetilde{\psi}^{-1}$ is analytic inside $\mathbb{D}_0$. The claim follows since $\bar{\partial}_\nu f \circ \widetilde{\sigma}_t = 0$ inside $\mathbb{D}_0$ for any analytic function f and because $\widetilde{\sigma}_t$ maps $\mathbb{D}_0$ on itself.

Take now G any solution of (22), then by Claim (β) of page 258 of [6] it follows that there exists an analytic function g such that $G = g \circ F_\chi$. The same argument employed to prove the Claim gives that inside $\mathbb{D}_0$

$$\bar{\partial}_\nu g \circ f \circ \widetilde{\sigma}_t = 0.$$

For points $z \in S^1$ we have that $\widetilde{\psi}^{-1} \circ \widetilde{\sigma}_t(z) = \phi^{-1}(z)$. We conclude the proof by remarking that the function $\widetilde{\psi}^{-1} \circ \widetilde{\sigma}_t$ admits a quasiconformal extension Γ to $\widehat{\mathbb{C}}$ with $\bar{\partial}\Gamma = 0$ in $\mathbb{D}_\infty$. Hence, if $\bar{\partial}_\chi G = 0$ in $\mathbb{D}_\infty$, then $\widetilde{G} = G \circ \Gamma$ is a solution of $\bar{\partial}_\nu \widetilde{G} = 0$. The continuity of Γ on the boundary and the fact that for $z \in S^1$ we have

$$\phi^{-1}(z) = \widetilde{\psi}^{-1} \circ \widetilde{\sigma}_t(z) = \Gamma(z)$$

implies that

$$\ker \bar{\partial}_\nu \Big|_{S^1} \supset \ker \bar{\partial}_\chi \Big|_{S^1} \circ \phi^{-1}.$$

A similar argument writing $\sigma_t = \sigma \circ \phi^{-1}$, and using ϕ in the role of ϕ^{-1} gives the equality. $\qquad\square$

As a consequence of the previous Lemma it follows that every point in the manifold $\mathcal{T}$ is associated to an element of the Grassmannian Gr. More precisely, we take $\sigma \in \mathcal{T}$ and write $\sigma = \sigma_t \circ \phi$ with t in a sufficiently small neighborhood provided by Lemma 20. For σ_t we know from Theorem 12 that the space $\mathcal{W}_{\sigma_t}$ of Definition 11 is a point in the Grassmannian. We now define

$$\mathcal{W}_\sigma = \mathcal{W}_{\sigma_t} \circ \phi.$$

Theorem 22. *Let $\sigma \in \mathcal{T}$, then $\mathcal{W}_\sigma$ is a point of the Segal-Wilson Grassmannian* Gr.

Proof. We write $\sigma = \sigma_t \circ \phi$ with t in a sufficiently small neighborhood provided by Lemma 20. Note that $\ker \bar{\partial}_{\mu_{\sigma_t}} \Big|_{S^1}$ is dense in $\mathcal{W}_{\sigma_t}$. From the Lemma 21 we have that

$$\ker \bar{\partial}_{\mu_\sigma} \Big|_{S^1} = R_\phi \left(\ker \bar{\partial}_{\mu_{\sigma_t}} \Big|_{S^1} \right),$$

where R_ϕ was defined in the beginning of the present section. The proof reduces to showing that R_ϕ belongs to the group $\mathrm{GL}_{\mathrm{res}}(L_2(S^1))$, Now, the operator

$$R_\phi : L^2(S^1) \rightarrow L^2(S^1)$$

can be written as

$$R_\phi = M_\phi \cdot C_\phi,$$

where

$$M_\phi[f](z) \overset{\text{def}}{=} \frac{f(z)}{\sqrt{\phi'(z)}},$$

and

$$C_\phi[f](z) \overset{\text{def}}{=} f \circ \phi^{-1}(z)\sqrt{\phi'(z)}.$$

Since $\phi \in \text{Diff}_+^\infty(S^1)$, it is shown on page 91 of [42] (see also [45]) that C_ϕ belongs to $\text{GL}_{\text{res}}(L_2(S^1))$. As for M_ϕ the difficulty lies on the fact that the loop

$$z \mapsto (1/\sqrt{\phi'(z)})$$

is not necessarily continuous. However, it has finitely many discontinuity points. An easy adaptation of the argument on page 83 of [42] gives that M_ϕ also belongs to $\text{GL}_{\text{res}}(L_2(S^1))$. An alternative way is to follow directly the method of [42], namely, to consider the operator

$$J : H_+ \oplus H_- \ni (f_+, f_-) \mapsto (f_+, -f_-) \in H_+ \oplus H_-,$$

and to show that the kernel of the integral operator that represents the commutator $[R_\phi, J]$ is Hilbert-Schmidt. This shows that the off-diagonal terms of the decomposition of R_ϕ in the form (18) are Hilbert-Schmidt operators. $\square$

Remark. The map $\Pi : \mathcal{T}^\infty \to Gr$, where

$$\Pi : \sigma \mapsto N_\sigma = \text{span}\{1, \sigma, \sigma^2, \dots\}.$$

was defined in [36]. Based on the argument of this section it is easy to see that this map coincides with our map $\sigma \mapsto W_\sigma$ restricted to $\mathcal{T}^\infty$. The proof is a consequence of the Lemma 21 of Section 5.

6   The Curvature Computation

In this section we show that the Teichmüller space $\mathcal{T} = \mathrm{Diff}_+^{3/2}(S^1)/\mathrm{PSU}_{1,1}$ equipped with the unique invariant Kähler metric has negative curvature in holomorphic directions.[4] More precisely, we will show that the curvature is negative and uniformly bounded away from zero in holomorphic directions. This combined with an extension of the Hopf-Rinow Theorem will be used in the next section to yield the existence of geodesics for all time.

In the Appendix 1 we will prove that the geodesics of $\mathcal{T}$ yield solutions to the Korteweg-de Vries equation. See also G. Segal's paper [46].

We start reviewing a few facts about the invariant Kähler metric of the Grassmannian. We follow closely the exposition and notation of Section 7.8 of [42]. The first step in the construction of the Kähler metric is to define it at the point $H_+ \equiv H_+ \oplus \{0\} \in \mathrm{Gr}$. Note that

$$T_{H_+}\mathrm{Gr} = HS(H_+, H_-) \ ,$$

where $HS(H_+, H_-)$ denotes the space of Hilbert-Schmidt operators from H_+ into H_-. Hence, it is natural to define[5] for ψ and χ in $T_{H_+}\mathrm{Gr}$

$$\langle \psi, \chi \rangle \stackrel{\mathrm{def}}{=} \mathrm{Tr}(\psi^*\chi).$$

To extend the definition to the rest of the Grassmannian one uses the fact that $GL_{res}(H)$ acts transitively on Gr. (in fact $U_{res}(H)$ already acts transitively). Let $A \in GL_{res}(H)$ be a transformation sending $H = H_+ \oplus H_-$ into $H = W \oplus W^\perp$, and preserving the direct sum decomposition. Then, if $\widetilde{\psi}$ and $\widetilde{\chi}$ are elements of $T_W\mathrm{Gr} \equiv HS(W, W^\perp)$ we have that $\psi \stackrel{\mathrm{def}}{=} A^{-1}\widetilde{\psi}A$ and $\chi \stackrel{\mathrm{def}}{=} A^{-1}\widetilde{\chi}A$ belong to $T_{H_+}\mathrm{Gr}$. Hence, to have a right-invariant metric we must set

$$
\begin{aligned}
\langle \widetilde{\psi}, \widetilde{\chi} \rangle_W &= \langle \psi, \chi \rangle_{H_+} && (24) \\
&= \mathrm{Tr}(A^*\widetilde{\psi}^*(AA^*)^{-1}\widetilde{\chi}A) && (25) \\
&= \mathrm{Tr}(\widetilde{\psi}^*(AA^*)^{-1}\widetilde{\chi}AA^*), && (26)
\end{aligned}
$$

---

[4]After a first draft of this article was written we learned of the work of Misiolek [35], where formulae for the curvature are also given. We remark, however, that he is working in a different orbit of the Bott-Virasoro group.

[5]Our definition differs from the one used in [42] by a factor of 2.

where in the last equality we used a well known fact. Namely, that $\operatorname{Tr}(AB) = \operatorname{Tr}(BA)$ for any A bounded and B trace-class. Note that as a particular case, if $A \in U_{res}(H)$ then $\langle \widetilde{\psi}, \widetilde{\chi} \rangle_W = \operatorname{Tr}(\widetilde{\psi}^* \widetilde{\chi})$.

It was shown in [36] that the embedding of $\mathcal{T}^\infty$ into the Grassmannian is isometric. Since $T_{id}\mathcal{T}^\infty$ is dense in $T_{id}\mathcal{T}$, and our embedding coincides with that of [36] in $\mathcal{T}^\infty$, it follows that the embedding of $\mathcal{T}$ into the Grassmannian we obtained is also isometric. Here, however, we have at both sides Hilbert manifolds.

Theorem 23. *Let $\mathcal{T} = \operatorname{Diff}_+^{3/2}(S^1)/\operatorname{PSU}_{1,1}$ be equipped with the unique right-invariant Kähler metric and isometrically embedded in the Grassmannian* Gr. *Let $\{\psi_k\}$ be an orthonormal basis of*

$$T_{H_+}\operatorname{Gr} = HS(H_+, H_-).$$

Then,

(a) *The component of the curvature tensor with respect to the orthonormal basis $\{\psi_k\}$ is given by:*

$$\begin{aligned} R_{i\bar{j},k\bar{l}} = &-\delta_{i\bar{j}}\delta_{k\bar{l}} - \delta_{i\bar{l}}\delta_{k\bar{j}} \\ &+ \operatorname{Tr}(\psi_i\psi_j^* \wedge \psi_k\psi_l^*) + \operatorname{Tr}(\psi_i\psi_l^* \wedge \psi_k\psi_j^*) \ \text{ for } (i,j) \neq (k,l) \end{aligned}$$

and

$$R_{i\bar{j},i\bar{j}} = -2\delta_{i\bar{j}} + \operatorname{Tr}(\psi_i\psi_j^* \wedge \psi_i\psi_j^*).$$

(b) *For any complex direction ψ, if we denote by K_ψ the Gaussian sectional curvature in the two-dimensional space defined by $\operatorname{Re}\psi$ and $\operatorname{Im}\psi$, we have*

$$K_\psi = -2 + \operatorname{Tr}(\psi\psi^* \wedge \psi\psi^*) < -3/2.$$

Proof. The proof is based on the construction of the so-called Cartan coordinate system [12]. By that we mean a holomorphic coordinate system $(x^1, x^2, \dots)$ in which the components of the metric tensor $g_{i,\bar{j}}$ is given by the formula:

$$g_{i,\bar{j}} = \delta_{i,\bar{j}} + r_{i\bar{j},k\bar{l}}x^k\bar{x}^l + \mathcal{O}(|x|^3).$$

Cartan [44] proved that if the coordinate system satisfies this last equation, then

$$r_{i\bar{j},k\bar{l}} = -R_{i\bar{j},k\bar{l}},$$

where R is the curvature tensor.

Construction of the Cartan's coordinates.

Define the exponential map as follows: [6]

$$\exp : HS(H_+, H_-) \to \mathrm{Gr}.$$

We assign to $\varphi_t = \sum t_i \psi_i \in HS(H_+, H_-)$, the subspace W_t of H spanned by the set

$$\{1 + \varphi_t(1), z + \varphi_t(z), \dots, z^n + \varphi_t(z^n), \dots\}$$

Obviously, W_t is the graph of the operator φ_t. Here, $t = (t_1, t_2, \dots)$ defines the local coordinates.

Lemma 24. *In the above coordinates the following expansion near $t = 0$ holds*

$$
\begin{aligned}
-&\frac{\partial^2}{\partial t_i \partial \bar{t}_j} \log \det(id - \sum t_i \bar{t}_j \psi_i \psi_j^*) \\
&= \delta_{i\bar{j}} + (2\delta_{i\bar{j}} - \mathrm{Tr}(\psi_i \psi_j^* \wedge \psi_i \psi_j^*)) t_i \bar{t}_j \\
&\quad + \sum_{(k,l) \neq (i,j)} \left[\delta_{i\bar{j}} \delta_{k\bar{l}} + \delta_{i\bar{l}} \delta_{k\bar{j}} - \mathrm{Tr}(\psi_i \psi_j^* \wedge \psi_k \psi_l^*) - \mathrm{Tr}(\psi_i \psi_l^* \wedge \psi_k \psi_j^*) \right] t_k \bar{t}_l \\
&\quad + \text{h.o.t.}
\end{aligned}
$$

Proof. Let

$$f(t) = \det\left(id - \sum t_i \bar{t}_j \psi_i \psi_j^*\right), \tag{27}$$

then

$$\frac{\partial^2}{\partial t_i \partial \bar{t}_j} \log f = \frac{\partial^2 f}{\partial t_i \partial \bar{t}_j} f^{-1} - \frac{\partial f}{\partial t_i} \frac{\partial f}{\partial \bar{t}_j} f^{-2}. \tag{28}$$

---

[6]This is the same map constructed by Nag in [36].

From the definition of the determinant, we have that

$$\det\left(id - \sum t_i\bar{t}_j\psi_i\psi_j^*\right)$$
$$= 1 - \sum_{i,j} t_i\bar{t}_j\mathrm{Tr}(\psi_i\psi_j^*) + \sum_{i,j,k,l} t_i\bar{t}_j t_k\bar{t}_l\mathrm{Tr}(\psi_i\psi_j^* \wedge \psi_k\psi_l^*) + \text{h.o.t.}$$

Hence,

$$\frac{\partial^2}{\partial t_i\partial\bar{t}_j}\det\left(id - \sum t_i\bar{t}_j\psi_i\psi_j^*\right)$$
$$= -\delta_{i\bar{j}} + \mathrm{Tr}(\psi_i\psi_j^* \wedge \psi_i\psi_j^*))t_i\bar{t}_j$$
$$+ \sum_{(k,l)\neq(i,j)} \mathrm{Tr}\left[\psi_i\psi_j^* \wedge \psi_k\psi_l^* + \psi_i\psi_l^* \wedge \psi_k\psi_j^*\right] t_k\bar{t}_l + \text{h.o.t.},$$

$$\frac{\partial}{\partial t_i}\det\left(id - \sum t_i\bar{t}_j\psi_i\psi_j^*\right) = -\sum_l \mathrm{Tr}(\psi_i\psi_l^*)\bar{t}_l + \text{h.o.t.},$$

and

$$\frac{\partial}{\partial\bar{t}_j}\det\left(id - \sum t_i\bar{t}_j\psi_i\psi_j^*\right) = -\sum_k \mathrm{Tr}(\psi_k\psi_j^*)t_k + \text{h.o.t.}$$

Furthermore,

$$1/f = 1 + \sum t_i\bar{t}_j\mathrm{Tr}(\psi_i\psi_j^*) + \text{h.o.t.}$$

and

$$1/f^2 = 1 + 2\sum t_i\bar{t}_j\mathrm{Tr}(\psi_i\psi_j^*) + \text{h.o.t.}$$

Substituting the last 5 equations into equation (28) we obtain the result. This completes the proof of the lemma. $\square$

From [36] it follows that $\log f(t)$, with f defined by (27), is the potential of the Kähler metric. Hence, Lemma 24 gives that the coordinates $(t_1, t_2, \dots)$ forms a Cartan coordinate system. Thus, for $(i, j) \neq (k, l)$ we get:

$$R_{i\bar{j},k\bar{l}} = -\delta_{i\bar{j}}\delta_{k\bar{l}} - \delta_{i\bar{l}}\delta_{k\bar{j}} + \mathrm{Tr}(\psi_i\psi_j^* \wedge \psi_k\psi_l^* + \psi_i\psi_l^* \wedge \psi_k\psi_j^*),$$

and

$$R_{i\bar{j},i\bar{j}} = -2\delta_{i\bar{j}} + \mathrm{Tr}(\psi_i\psi_j^* \wedge \psi_i\psi_j^*).$$

This concludes the proof of part (a) of the theorem. To prove part (b) remark that the Gaussian curvature in the direction ψ_i is given by

$$K_{\psi_i} = R_{i\bar{i},i\bar{i}} = -2 + \mathrm{Tr}(\psi_i\psi_i^* \wedge \psi_i\psi_i^*).$$

Now, if $\psi_1 = \varphi$, we complete the set $\{\psi_1\}$ to an orthonormal set in the Hilbert space $T_{H_+}\mathrm{Gr}$. Hence, from the previous discussion it follows that

$$K_\varphi = -2 + \mathrm{Tr}(\varphi\varphi^* \wedge \varphi\varphi^*).$$

Lemma 25. *If* $\mathrm{Tr}(\varphi\varphi^*) = 1$, *then* $\mathrm{Tr}(\wedge^2\varphi\varphi^*) < 1/2$.

Proof. Note that $\varphi\varphi^*$ is a compact positive operator and hence its nonzero eigenvalues are all positive. Let $\{\lambda_i\}$ be the set of such nonzero eigenvalues. Since $\varphi\varphi^*$ is trace-class and $\|\varphi\|^2 = \mathrm{Tr}\varphi\varphi^* = 1$ it follows that $\mathrm{Tr}\varphi\varphi^* = \sum \lambda_i = 1$. A simple argument with tensor products gives

$$Tr(\wedge^2\varphi\varphi^*) = \sum_{i<j} \lambda_i\lambda_j.$$

Now we remark that

$$1 = \left(\sum \lambda_i\right)^2 = 2\sum_{i<j} \lambda_i\lambda_j + \sum \lambda_i^2.$$

From here it follows that

$$Tr(\wedge^2\varphi\varphi^*) < 1/2.$$

Using the formula for the curvature in holomorphic direction φ it follows that

$$K_\varphi < -3/2 < 0.$$

$\square$

7 Description of the Geodesics

In this section we shall give a description of the geodesics passing through the identity of $\mathcal{T} \hookrightarrow Gr$ in the complex direction ψ. We recall from Section 6 the definition of the exponential map. It was defined, for each $s \in \mathbb{C}$ and $\psi \in HS(H_+, H_-)$ as the subspace of H spanned by the set of vectors

$$\{1 + s\psi(1), z + s\psi(z), \dots\}.$$

Our immediate goal is to show that:

Theorem 26. *The complex curve $\gamma_\psi(s) = \exp(s\psi)$ is a totally geodesic 2-real-dimensional submanifold of $\mathcal{T}$.*

We recall that a submanifold S of a Riemannian manifold M is called geodesic at p if each M-geodesic passing through p in a tangent direction to S remains in S for all time. If S is geodesic at all its points, then it is called totally geodesic [13].

The proof of the Theorem is a consequence of the following two results:

Lemma 27. *Let $\psi \in HS(H_+, H_-)$ be such that $\|\psi\|^2 = 1$ and $s(t) = s_0 + e^{i\theta}t$, with $\theta \in \mathbb{R}$. Then, the norm of the tangent vector $\dot{\gamma}_\psi$ to the the path $t \mapsto \gamma_\psi(s(t))$ is given by*

$$\|\dot{\gamma}_\psi\| = 1.$$

Proof. Call $\mathcal{W}_s$ the subspace $\gamma_\psi(s) \subset H$. With respect to the decomposition $H = H_+ \oplus H_-$ define the block matrix

$$A_s = \begin{bmatrix} id & -\bar{s}\psi^* \\ s\psi & id \end{bmatrix}.$$

From the definition of $\mathcal{W}_s$ we have

$$\mathcal{W}_s = A_s H_+.$$

Let $T : H \to H$ be any bounded operator. Using that $\mathrm{Graph} T$ is perpendicular to $\mathrm{Graph}' T^*$, where

$$\mathrm{Graph}' T \overset{\text{def}}{=} \{(-Tx, x) \mid x \in \mathrm{dom}(T)\},$$

we get

$$\mathcal{W}_s^\perp = A_s H_-.$$

Hence, the matrix A_s maps $H = H_+ \oplus H_-$ into $W_s \oplus W_s^\perp$ and preserves the direct sum decomposition.

It is well known that the operators $(id + s\bar{s}\psi\psi^*)$ and $(id + s\bar{s}\psi^*\psi)$ are invertible. The following relations can be checked easily:

$$(id + s\bar{s}\psi^*\psi)^{-1}\psi^* = \psi^*(id + s\bar{s}\psi\psi^*)^{-1}, \tag{29}$$

and

$$(id + s\bar{s}\psi\psi^*)^{-1}\psi = \psi(id + s\bar{s}\psi^*\psi)^{-1}. \tag{30}$$

Therefore,

$$A_s^{-1} = \begin{bmatrix} id & \bar{s}\psi^* \\ -s\psi & id \end{bmatrix} \begin{bmatrix} (id + s\bar{s}\psi^*\psi)^{-1} & 0 \\ 0 & (id + s\bar{s}\psi\psi^*)^{-1} \end{bmatrix} \tag{31}$$

$$= \begin{bmatrix} (id + s\bar{s}\psi^*\psi)^{-1} & 0 \\ 0 & (id + s\bar{s}\psi\psi^*)^{-1} \end{bmatrix} \begin{bmatrix} id & \bar{s}\psi^* \\ -s\psi & id \end{bmatrix}. \tag{32}$$

To compute the norm $\|\dot{\gamma}_\psi(s)\|$, we use the identification of $T_{\mathcal{W}_s}\mathrm{Gr}$ with $HS(\mathcal{W}_s, \mathcal{W}_s^\perp)$. The latter, is mapped onto $HS(H_+, H_-)$ by means of

$$\widetilde{X} \mapsto A_s^{-1}\widetilde{X}A_s\Big|_{H_+}.$$

From the invariance of the metric we have that the norm of $\widetilde{X} \in T_{\mathcal{W}_s}\mathrm{Gr}$ is given by

$$\|\widetilde{X}\|^2 = \mathrm{Tr}\left(\left(A_s^{-1}\widetilde{X}A_s\Big|_{H_+}\right)^* \left(A_s^{-1}\widetilde{X}A_s\Big|_{H_+}\right)\right). \tag{33}$$

Since,

$$\dot{A} = \begin{bmatrix} 0 & -e^{-i\theta}\psi^* \\ e^{i\theta}\psi & 0 \end{bmatrix},$$

a technical but straightforward computation with $\widetilde{X} = \dot{A}(s)$ taking into account equations (29), (30), and (32) yields the lemma. □

Lemma 28. *Set*

$$\dot{\gamma}_\psi(s) = \frac{d}{ds}\gamma_\psi(s)$$

and assume that

$$\|\dot{\gamma}_\psi(s)\|^2 = 1.$$

Let ∇ be the covariant derivative in $\mathcal{T}$ given by the Levi-Civita connection. Then,

$$\nabla_{\dot{\gamma}_\psi(s)}\dot{\gamma}_\psi(s) = 0.$$

Proof. Let $\gamma_\psi(s)$ be a geodesic in our Kähler manifold. For each point s of the geodesic $\gamma_\psi(s)$ we define a complex direction as follows:

$$\dot{\gamma}_\psi(s) + i\mathcal{I}\dot{\gamma}_\psi(s).$$

For each s in $\gamma_\psi(s)$ we consider a geodesic $\mathcal{I}\dot{\gamma}_\psi(s)$ and so each point τ on the geodesic from s with direction $\mathcal{I}\dot{\gamma}_\psi(s)$ we have two tangent vectors. The first, $\alpha(\tau)$, which is the parallel transport of $\dot{\gamma}_\psi(s)$ and the second $\beta(\tau)$ which is given by

$$\beta(\tau) = \frac{d}{ds}\mathcal{I}\dot{\gamma}_\psi(s),$$

and is the parallel transport of $\mathcal{I}\dot{\gamma}_\psi$. Now, the Kähler condition gives that $[\dot{\gamma}(s), \mathcal{I}\dot{\gamma}(s)] = 0$. Hence, it follows from Frobenius theorem that there exist a surface S such that the tangent space is spanned by $\alpha(s)$ and $\beta(s)$. Since, $\mathcal{I}(\alpha(s) + i\beta(s)) = i(\alpha(s) + i\beta(s))$ it follow that S is a complex analytic curve and we can take z as a complex analytic coordinate associated to the point $z = \exp(x\dot{\gamma}_\psi(s) + iy\mathcal{I}\dot{\gamma}_\psi(s))$. Let's write

$$\dot{\mu}(z) = \dot{\gamma}_\psi(s) + i\mathcal{I}\dot{\gamma}_\psi(s).$$

From the properties of the Levi-Civita connection it follows that

$$0 = \frac{d}{dz}\|\dot{\mu}(z)\|^2\Big|_{s_0} = \langle \nabla_z\dot{\mu}(z), \dot{\mu}(z)\rangle = 0. \tag{34}$$

and so

$$\frac{d^2}{d\bar{z}dz}\|\dot{\mu}(z)\|^2\Big|_{s_0} = \|\nabla_z\dot{\mu}(z)\|^2 - R\|\dot{\mu}(z)\|^2 = 0.$$

Since $R < 0$, we have

$$\nabla_z\dot{\mu}(z) = 0.$$

Restrict $\nabla_z\dot{\mu}(z)$ on the real part at s_0 and we get that

$$\nabla_z\dot{\gamma}_\psi(z) = 0.$$

Hence, taking $\mathrm{Re}z = \dot{\gamma}_\psi$, we have

$$\nabla_{\dot{\gamma}_\psi(s)}\dot{\gamma}_\psi(s)\Big|_{s_0} = 0.$$

$\square$

Remark. As a consequence of the previous theorem, we can characterize a geodesic in the space $\mathcal{T}$ by considering this space embedded in the Grassmannian Gr. Given a real direction v define

$$\psi = v + i\mathcal{I}v,$$

where $\mathcal{I}$ is the complex structure operator. Clearly v is contained in the 2-real-dimensional plane spanned by ψ. Since $\exp(s\psi)$ is a totally geodesic submanifold of $\mathcal{T}$ it follows that the geodesic in the real direction v will be contained in $\{\exp(s\psi)|s \in \mathbb{C}\}$.

The above remarks allow us to state:

Theorem 29. *For each point ϕ in $\mathcal{T}$ and each direction v in the tangent space $T_\phi\mathcal{T}$ the geodesic $\gamma(s)$ passing through ϕ in the direction v exists for all $s \in \mathbb{R}$.*

Proof. Given the homogeneity of the space $\mathcal{T}$ without loss of generality we may assume that ϕ is the identity map of S^1. Because of the completeness of $\mathcal{T}$ it follows that $\exp(s\psi)$ is a complete closed submanifold. Now, we can apply Hopf-Rinow Theorem [10] to the *finite dimensional* manifold $\{\exp(s\psi)|s \in \mathbb{C}\}$ to conclude that the geodesic exists for all time. $\square$

Corollary 30. *The periodic solution to KdV equation with initial data*

$$u_0(x) = \sum a_n e^{inx} \in \mathcal{H}^{3/2}_{\mathrm{per}}(\mathbb{R}, \mathbb{R}) \tag{35}$$

exists for all time.

Proof. Consider the pairing between (real) quadratic forms on S^1 and (real) vector fields. It is given by

$$\left\langle q(x)dx^{\otimes 2} \middle| v(x)\frac{d}{dx} \right\rangle = \int_0^{2\pi} q(x)v(x)dx$$

Using the Weil-Petersson metric we have an identification between the tangent and cotagent spaces. Let's denote by $\mathcal{A} : T_{id}\mathcal{T} \to T_{id}\mathcal{T}^*$ such identification. Given an initial condition u_0 as in equation (35) we take $\psi = \mathcal{A}^{-1}u_0$.

Let $\gamma_\psi(t, z)$ be the geodesic whose existence is guaranteed for all real time by Theorem 29. From the results in the Appendix 1 it follows that the function

$$u(t, x) = \mathcal{A}\left(\frac{\dot{\gamma}_\psi(t, \exp(ix))}{\gamma'_\psi(t, \exp(ix))} \right),$$

where $'$ denotes the derivative w.r.t. z, is a solution to the KdV equation. Since $\gamma_\psi(t, \cdot)$ exists for all time, so does $u(t, \cdot)$. $\qquad\square$

We close this section with an infinite dimensional analogue of Hadamard's theorem, which will be used in the next section to show exponential spreading of the geodesic flow on $\mathcal{T}$.

Theorem 31. *The exponential map gives a global diffeomorphism between the tangent space $T_{id}\mathcal{T}$ and $\mathcal{T}$.*

Proof. First we prove that for fixed ψ, the exponential map restricted to the two-real dimensional space $\{t\psi | t \in \mathbb{C}\}$ and taking values in the totally geodesic submanifold D_ψ is a covering map. To show this we use that the curvature in the holomorphic direction is negative, and so it implies that the norm

$$\|d\exp(tv)v\| \geq \|v\|.$$

From a standard result in (finite dimensional) differential geometry it follows that the map is covering. Furthermore, if $t_1 \neq t_2$ then $\mathcal{W}_{t_1\psi} \neq \mathcal{W}_{t_2\psi}$. Hence, $\exp(t_1\psi) \neq \exp(t_2\psi)$. Suppose that ψ_1 and ψ_2 are linearly independent vectors, from the construction of exp it follows that complex curves $\{\exp(t\psi_1)|t \in \mathbb{C}\}$ and $\{\exp(t\psi_2)|t \in \mathbb{C}\}$ intersect only at the identity. These arguments imply that exp is a differentiable inclusion from $T_{id}\mathcal{T}$ to $\mathcal{T}$. Since $T_{id}\mathcal{T}$ is open and closed, $\mathcal{T}$ will be the image of exp. From here it follows that the exponential map is surjective. Finally, to show that the inverse map to the exponential map is differentiable, all we have to do is to remark that

$$d\exp_{tv} = L_{\sigma(t)} : T_{id}\mathcal{T} \to T_{\exp(tv)}\mathcal{T},$$

where $L_{\sigma(t)}$ is the parallel transport along the geodesic $\sigma(t)$ connecting the id to the point $\exp(tv)$. Obviously this map is invertible, and hence because of the inverse function theorem we have an isomorphism between the tangent spaces. $\square$

Remarks.

1. The totally geodesic manifold $D_\psi = \{\gamma_\psi(s)|s \in \mathbb{C}\}$ is isometric to the disc D_R of radius $R = |4/K_\psi|$ endowed with the Poincaré metric

$$\frac{4R^4 du^2}{(R^2 - |u|^2)^2}.$$

2. The negativity of the curvature shows the instability of the geodesic flow in the sense of Arnold [5].

3. The analogue of the Cartan-Hadamard result shown in Theorem 31 allows us to conclude that the $\text{Diff}^+(S^1)$ is homotopically equivalent to S^1.

4. The infinite-dimensional Siegel disc is by definition the set of Hilbert-Schmidt operators $T : H_+ \to H_-$ such that $\det(I - TT^*) > 0$. As a consequence of our results, it follows that the space $\mathcal{T}$ can be isometrically identified with the infinite-dimensional Siegel disc. Indeed, any two points in both spaces can be joined by a unique geodesic and we have just shown that $\mathcal{T}$ is geodesically complete.

8 Construction of Solutions to KdV from the Beltrami Equation

In this section we will describe a procedure to construct solutions to the periodic KdV equation. The construction will be based on the following theorem:

Theorem 32. *Let $\mu \in T_{id}\mathcal{T}^*$ be such that*

$$\mu = \begin{cases} -\frac{1}{2}(1 - |z|^2)^2 \varphi(\bar{z}), & z \in \mathbb{D}_0 \\ 0, & z \in \widehat{\mathbb{C}} \setminus \mathbb{D}_0 \end{cases}$$

and $\|\mu\|_\infty < 1$. For s in a neighborhood of 0 take $f_{\infty,s}$ and $f_{0,s}$ as in the construction of Section 4.

Then, $\exp_\mu(s) = f_{0,s}^{-1} \circ f_{\infty,s}$ extends analytically as a totally geodesic two-real-dimensional manifold in $\mathcal{T}$.

Proof. The idea is to show that in a neighborhood of 0, there exists a 1-1 correspondence between $\exp_\mu(s)$ and the totally geodesic submanifold constructed in Section 7. We recall the embedding of $\mathcal{T}$ into the Grassmannian given by

$$\mathcal{T} \ni \sigma_s \mapsto \mathcal{W}_s \in Gr,$$

where

$$\mathcal{W}_s \stackrel{\text{def}}{=} \overline{(\ker \bar{\partial}_{s\mu})|_{S^1}}.$$

In order to complete the proof of the theorem we need the following:

Lemma 33. *For s sufficiently small, there exists a linear operator $L_\mu : H_+ \to H_-$ such that*

$$\mathcal{W}_s = \mathrm{Graph}(sL_\mu).$$

Proof. We denote by $\Gamma(\mathbb{D}_0, \Omega^{1,0})$ the set of smooth $(1,0)$-one-forms on $\mathbb{D}_0$ of the form $f dz$ such that f extends to the circle, and mutatis

mutandis $\Gamma(\mathbb{D}_0, \Omega^{0,1})$ the set of $(0,1)$ one-forms. For sufficiently small s let

$$sL_\mu : \Gamma(\mathbb{D}_0, \Omega^{1,0}) \to \Gamma(\mathbb{D}_0, \Omega^{0,1}),$$

be a family of linear operators defined by

$$dz \mapsto s\mu_\varphi d\bar{z}.$$

Let

$$W_{\mathbb{D}_0, s\mu} \overset{\text{def}}{=} \mathrm{Graph}(sL_\mu) \subset \Gamma(\mathbb{D}_0, \Omega^{1,0}) \oplus \Gamma(\mathbb{D}_0, \Omega^{0,1}).$$

We will show that $\mathcal{W}_s = \overline{W_{\mathbb{D}_0, s\mu}|_{S^1}}$. Note that f is a solution of

$$\bar{\partial}_{s\mu} f = 0$$

iff

$$df \in W_{\mathbb{D}_0, s\mu} = \mathrm{Graph}(sL_\mu).$$

This follows directly from the following computation:

$$
\begin{aligned}
df &= \partial f dz + \bar{\partial} f d\bar{z} \\
&= \partial f \left(dz + \frac{\bar{\partial} f}{\partial f} d\bar{z} \right)
\end{aligned}
$$

So, $df \in W_{\mathbb{D}_0, s\mu}$ iff

$$\frac{\bar{\partial} f}{\partial f} = s\mu.$$

From here the lemma follows just by restricting to the points of S^1 and using that the solutions of $\bar{\partial}_{s\mu} f = 0$ are of the form $f(z) = G(w(z))$ with G analytic and w an $s\mu$-quasiconformal homeomorphism of the plane on itself. $\square$

From Lemma 33 it follows that there exists a 1-1 correspondence between $\exp_\mu(s)$ and the totally geodesic submanifold constructed in Section 7, namely $\exp(sL_\mu)$.

We are now ready to state a method for the construction of the solutions to the KdV equation starting with initial data of the form (35)

We summarize this discussion in the following procedure of constructing periodic solutions of KdV:

1. Given u_0 as in equation (35) take

$$\phi(\exp ix) = \sum_{n \geq 0} a_n \exp(inx)$$

and construct $s\mu_\phi = s(1 - |z|^2)^2 \phi(\bar{z})$ for $z \in \mathbb{D}_0$ and $s\mu_\phi = 0$ outside the disk $\mathbb{D}_0$.

2. Let $\omega_{s\phi}$ be the solution of the Beltrami equation

$$\begin{cases} \bar{\partial}\omega - s\mu_\phi(z)\partial\omega = 0, \\ \omega(\rho) = \rho \text{ for } \rho \in \{-1, -i, 1\}. \end{cases}$$

3. We define $f_{\infty,s}(z) = \omega_{s\phi}(z)$ for $z \in \mathbb{D}_\infty$. Set $f_{0,s}$ as in equation (16).

4. Let for $z \in S^1$

$$g(t, z) = f_{0,s(t)}^{-1} \circ f_{\infty,s(t)}(z).$$

5. Choose a parameterization $s = s(t)$ so that

$$\left\| \frac{d}{dt} g \right\|_{WP} = 1,$$

where the $\| \ \|_{WP}$ is the norm w.r.t. the Weil-Petersson metric, i.e., the right-invariant metric defining the Riemannian structure of $\mathcal{T}$.

6. If we take

$$u(t, x) = \mathcal{A}\left(\frac{\dot{g}(t, \exp(ix))}{g'(t, \exp(ix))} \right),$$

where $'$ denotes the derivative w.r.t. z, $\cdot$ the time derivative and $\mathcal{A}$ is the identification used in Corollary 30, then u satisfies the KdV equation by the Corollary 41 in the Appendix 1.

9   Final Remarks

1. As this manuscript was being finalized we became aware of a beautiful paper by Nag and Sullivan [37] where the diffeomorphisms of the circle modulo rotations are endowed with a Hilbert manifold structure modeled on the Sobolev space $\mathcal{H}^{1/2}$. We emphasize the difference that in this paper we are concerned with the Hilbert manifold structure of $\mathrm{Diff}_+^{3/2}(S^1)/\mathrm{PSU}_{1,1}$. Hence our manifold is modeled upon $\mathcal{H}^{3/2}$.

2. The idea of the curvature computation is closely related to the curvature computation of the Weil-Petersson metric on the moduli space of Calabi-Yau manifolds [49]. Other formulae for the curvature of some of the orbits of the Bott-Virasoro group with a natural Kähler metric were obtained by Kirillov and Yurev [22, 23]. We expect in a future work to obtain explicit formulae for the curvature in terms of the Green function for the Laplace operator on the disk with the Poincaré metric (following some ideas of Siu).

3. The relation between metrics of constant negative curvature and completely integrable systems is also present in the work of S.S. Chern and K. Tenenblat (see [9]).

4. A number of deep analytic results concerning existence and uniqueness for the periodic KdV in Sobolev space were recently obtained by J. Bourgain [8]. We remark that even the problem of local existence for periodic KdV in Sobolev spaces is a non-trivial one when $s \leq 3/2$. See [20]. Our method gives long time existence for the solutions of KdV. We remark that our techniques are geometric ones and can be applied to other orbits of the Bott-Virasoro group. We are currently investigating the relation between existence results for L^2 initial data and the co-adjoint orbit of the Bott-Virasoro group isomorphic to $\mathrm{Diff}(S^1)$. (The classification of the co-adjoint orbits of the Bott-Virasoro group was obtained in a result of Kuiper's [24].)

5. The relation between the KdV (or more generally the KP) hierarchy and the Grassmannian is a standard fact from the theory of solitons [16, 47, 32]. We remark, however, that the relation

between the KdV equation and the Grassmannian in the present work is totally distinct from the one in [47].

6. The study of the periodic KdV equation is naturally associated to Hill's operator and to the theory of theta functions (see [38] and references therein). It would be very interesting to connect the present results with specific known examples, in particular with the results in [33, 34, 31, 15]. We are currently working on such examples.

A Appendix 1: Arnold's point of view on Euler equations

In the first part of this appendix we describe Arnold's ideas about the connection between Euler equation and geodesic flows on finite dimensional Lie groups. On the second part we describe the infinite dimensional analogue between Euler's equation and geodesic flows due to Arnold. For an introductory exposition to the material of this appendix we refer the reader to [30].

- *Arnold's approach* [5] *to Euler's Equation in a Finite Dimensional Space*: Let G be a finite dimensional Lie group, and let $\mathcal{G}$ be its Lie Algebra. As usual we denote by $L_g h = gh$ the left multiplication (resp. $R_g h = hg$ the right-multiplication), and by L_{g*} the derivative of this transformation L_g (respec. R_{g*}). Let

$$A : \mathcal{G} \to \mathcal{G}^* = \mathrm{Hom}(\mathcal{G}, \mathbb{R})$$

be a positive symmetric operator. By positive symmetric operator we mean

$$\langle Au | u \rangle \; > \; 0 \quad \forall u \neq 0$$

and

$$\langle Av | u \rangle \; = \; \langle Au | v \rangle,$$

where $\langle | \rangle$ denotes the duality bracket between $\mathcal{G}^*$ and $\mathcal{G}$, i.e., $\langle F | \xi \rangle = F(\xi)$ for $F \in \mathcal{G}^*$ and $\xi \in \mathcal{G}$.

The operator A defines a right-invariant metric on G. Let $g(t)$ be a geodesic w.r.t. to this right-invariant metric defined by A. From the right-invariance of the metric, we have that

$$R_{g^{-1}*}\dot{g}(t) = \dot{g}(0) = u_0,$$

which is equivalent to

$$R_{g(t)*}u_0 = \dot{g}(t).$$

Let $u(t) = L_{g^{-1}(t)*}\dot{g}(t) = L_{g^{-1}(t)*}R_{g(t)*}u_0$. Since, $A : \mathcal{G} \to \mathcal{G}^*$ is an isomorphism we will look at $\mathcal{G}$ as identified with $\mathcal{G}^*$ by choosing an orthonormal basis in $\mathcal{G}$. The vector $m = Au_0$ is called the generalized angular momentum and $M(t) = Au(t)$ is called the relative angular momentum. We have the conservation law

$$\frac{dm}{dt} = 0 \ ,$$

meaning that m is a constant of motion. From here one obtains (see Section D of Appendix 2, [5]) Arnold-Euler's equation for $M(t)$, namely:

$$\frac{dM}{dt} = \{u, M\},$$

where $\{,\}$ is the infinitesimal co-adjoint action. We recall that the infinitesimal co-adjoint action for $M \in \mathcal{G}^*$ and $\xi, \eta \in \mathcal{G}$ is given by:

$$\{\xi, M\}(\eta) \stackrel{\text{def}}{=} (\text{ad}_\xi^* M)(\eta) = \langle M | [\xi, \eta] \rangle$$

- *Arnold's approach to Euler's Equation in an Infinite Dimensional Space*: In this case we restrict ourselves to the Bott-Virasoro group $\widetilde{G}$. Bott proved that (see [7]):

Theorem 34. *There exists a group $\widetilde{G}$ that is a central extension of $G = \text{Diff}_+(S^1)$ given by Bott's co-cycle $B : \text{Diff}_+(S^1) \times \text{Diff}_+(S^1) \to \mathbb{R}$, where*

$$B(\sigma_1, \sigma_2) = \int_{S^1} \log(\sigma_1(\sigma_2))' \ d(\log \sigma_2)'$$

Remark. The central extension $\widetilde{G}$ above is called the Bott-Virasoro group.

Definition 35. The Virasoro Lie Algebra is a vector space $\widetilde{\mathcal{G}} = \mathbb{C} \oplus \mathcal{G}$ where $\mathcal{G}$ is the Lie-algebra of complex vectors-fields on S^1 and the Lie-bracket is defined by

$$[(\lambda_1, f\partial_x), (\lambda_2, g\partial_x)] = (c_0(f, g), [f\partial_x, g\partial_x])$$

where

$$c_0(f, g) = \frac{1}{2\pi i} \int_{S^1} f''' g \; dz \tag{36}$$

Note that this co-cycle extends as a *bounded skew-symmetric form* on $\mathcal{H}_{\text{per}}^{3/2}(S^1)$. Indeed, if $f \in \mathcal{H}_{\text{per}}^{3/2}(S^1)$, then $f''' \in \mathcal{H}_{\text{per}}^{-3/2}(S^1) = (\mathcal{H}_{\text{per}}^{3/2}(S^1))'$. We remark that the tangent space at the identity to $\widetilde{G}$ is $\widetilde{\mathcal{G}}$. The construction of Arnold-Euler's equation in the finite-dimensional case leads us to the following natural definition in the infinite-dimensional context:

Definition 36. Let A be a positive operator from $\widetilde{\mathcal{G}}$ to $\widetilde{\mathcal{G}}^*$, and suppose that $M(t)$ is a curve in $\widetilde{\mathcal{G}}^*$ satisfying

$$\frac{dM}{dt} = \{A^{-1}M(t), M(t)\}, \tag{37}$$

where the infinitesimal co-adjoint action is given by:

$$\{\xi, M\}(\eta) = \langle M | [\xi, \eta] \rangle.$$

Then, equation (37) will be called the Arnold-Euler equation.

The following result is well known (see [39, 46, 45]) for the special coadjoint orbit of the Bott-Virasoro group which is isomorphic to $\text{Diff}_+^\infty(S^1)$.

Theorem 37. *The periodic KdV equation*

$$u_t = u_{xxx} + 6uu_x$$

coincides with the Arnold-Euler equation on the Kähler co-adjoint orbits of the Bott group.

Proof. First we will compute the co-adjoint action of the Bott-Virasoro group on $\widetilde{\mathcal{G}}^*$ following Kirillov (see [21]). For that we use the following:

Theorem 38 (Kirillov). *The dual Lie algebra $\widetilde{\mathcal{G}}^*$ as a $\widetilde{G}$-module is isomorphic to the $\widetilde{G}$-module*

$$Z^2(\mathcal{G}) \stackrel{\text{def}}{=} \{c : \mathcal{G} \times \mathcal{G} \to \mathbb{R} \mid c([\xi, \eta], \zeta) + c([\eta, \zeta], \xi) + c([\zeta, \xi], \eta) = 0\}$$

The idea of the proof of Kirillov's result is based on two facts. The first is the Gelfand-Fuchs theorem, which states that

$$Z^2(\mathcal{G}) = \{\lambda c_0 + \delta \mid \lambda \in \mathbb{R}, \quad c_0 \text{ as in eq. (36), and}$$
$$\delta[\xi, \eta] \stackrel{\text{def}}{=} \langle \delta | [\xi, \eta] \rangle \text{ for } \delta \in \mathcal{G}^* \}.$$

The second idea is the computation of the action of $\widetilde{G}$ on $Z^2(\mathcal{G})$. Let

$$q \in \mathcal{G}^* = \{q(x) \, dx^{\otimes 2}\}.$$

Kirillov proved that

$$\mathrm{Ad}^*(\lambda, \varphi)(t, q) = (t, \mathrm{Ad}^*(\varphi)(q) + th(\varphi)), \tag{38}$$

where

$$h(\varphi) = \mathcal{S}(\varphi) \circ \varphi^{-1}$$

and for $\eta \in \mathcal{G}$

$$\langle \mathrm{Ad}^*(\varphi) q | \eta \rangle = \langle q | \eta \circ \varphi^{-1} \rangle.$$

Here, $\mathcal{S}$ denotes the Schwarzian derivative defined in Section 2 (but here taken with respect to x). The proof of Kirillov's theorem is a direct consequence of these two remarks.

By a distribution of weight α we mean a tensor field $f(z)(dz)^\alpha$ that changes according to the standard rules, i.e.,

$$f(z)(dz)^\alpha = g(w)(dw)^\alpha$$

for a change of coordinate $w = w(z)$. We now recall the following result of Lazutkin [27] and Kirillov's.

Lemma 39 (Lazutkin, Kirillov). *The set $Z^2(\mathcal{G}) \equiv \widetilde{\mathcal{G}}^*$ can be identified with the $\mathcal{G}$-module of Hill's operators*

$$\{\lambda\partial_x^2 + q\}$$

acting on distributions of weight $-1/2$ as $\widetilde{G}$-modules.

The idea of the proof is based on the following fact: Suppose that η_1 and η_2 are two linearly independent solutions of Hill's equation

$$(\lambda\partial_x^2 + q)\eta = 0,$$

then

$$q = \mathcal{S}\left(\frac{\eta_1}{\eta_2}\right).$$

If we perform a change of variables $x \mapsto \sigma^{-1}(x)$ and use the property of the Schwarzian derivative

$$\mathcal{S}(\varphi \circ \psi) = (\mathcal{S}(\varphi) \circ \psi)(\psi_x)^2 + \mathcal{S}(\psi),$$

we get that $\eta_i \circ \sigma^{-1}(x)$ will satisfy the Hill's equation

$$(\lambda\partial_x^2 + \widetilde{q})\eta = 0,$$

with

$$\begin{aligned}
\widetilde{q}(x) &= \mathcal{S}\left(\frac{\eta_1(\sigma^{-1}(x))}{\eta_2(\sigma^{-1}(x))}\right) \\
&= q \circ \sigma^{-1}(x)((\sigma^{-1}(x))_x)^2 + \mathcal{S}(\sigma^{-1}(x)).
\end{aligned}$$

From here and the Kirillov formula (38) the lemma follows.

- *Computation of the co-adjoint action of $\xi \in \mathcal{G}$ on $\widetilde{\mathcal{G}}^*$:*

  We start by noticing, from the discussion above, that

$$\widetilde{\mathcal{G}}^* \cong \{(\lambda, qdx^{\otimes 2})\} .$$

  The computation will be done in three steps.

1. Computation of the infinitesimal action of ξ on quadratic differentials. We proceed in the standard way. Let $g_t(x)$ be a one-parameter family of diffeomorphisms of the circle and such that

$$g_t(x) = x + t\xi + \text{h.o.t.}$$

We will compute the derivative δb of the family of quadratic differentials $b(g_t(x))(d(g_t(x)))^{\otimes 2}$ at $t = 0$. Hence,

$$
\begin{aligned}
&b(g_t(x))(d(g_t(x)))^{\otimes 2} \\
&= b(x + t\xi + \text{ h.o.t.})(d(x + t\xi + \dots))^{\otimes 2} \\
&= (b(x) + tb_x(x)\xi + \text{ h.o.t})(dx + t\xi_x dx + \text{ h.o.t.})^{\otimes 2} \\
&= b(x)(dx)^{\otimes 2} + t(b_x\xi + 2\xi_x b)(dx)^{\otimes 2} + \text{ h.o.t.}
\end{aligned}
$$

Hence,

$$\delta b = (b_x\xi + 2\xi_x b)(dx)^{\otimes 2}.$$

2. Computation of the pairing of $\widetilde{\mathcal{G}}$ and $\widetilde{\mathcal{G}}^*$. We start by recalling that the dual $\widetilde{\mathcal{G}}^*$ can be identified to the set

$$\{(t, b) | t \in \mathbb{R} \text{ and } b \text{ is a quadratic differential}\}.$$

Here, by quadratic differential, we mean an element of the form $b(x)dx^{\otimes 2}$, where b for a local coordinate-chart is an element of $\mathcal{H}_{\text{per}}^{-3/2}$. So that the pairing between a point $(\mu, \eta) = (\mu, qd/dx) \in \widetilde{\mathcal{G}}$ and a point $(s, b(dx)^{\otimes 2})$ is given by

$$s\mu + \int_{S^1} bqdx.$$

This last integral being performed in the sense of the pairing between $\mathcal{H}_{\text{per}}^{3/2}$ and $\mathcal{H}_{\text{per}}^{-3/2}$. We also notice that this is independent of the parametrization of the circle. Now we compute

$$
\begin{aligned}
&\langle (t, b) | \left[(\lambda, \xi), (\mu, \eta) \right] \rangle \\
&= \left\langle (t, b) \left| \left(\frac{1}{2\pi i} \int_{S^1} \xi_{xxx} \eta dx, [\xi, \eta] \right) \right. \right\rangle \\
&= \frac{t}{2\pi i} \int_{S^1} \xi_{xxx} \eta dx + \frac{1}{2\pi i} \int_{S^1} [\xi, \eta] b dx.
\end{aligned}
$$

3. Computation of the infinitesimal co-adjoint action of ξ on $\widetilde{\mathcal{G}}^*$. We will combine the two items above to compute

$$(t, \widetilde{b}) \overset{\text{def}}{=} \text{ad}^*_{(\lambda, \xi)}(t, b).$$

Hence,

$$(t, \widetilde{b})(\mu, \eta) = \langle\, (t, b) \mid [(\lambda, \xi), (\mu, \eta)]\, \rangle.$$

From the above results we get

$$\widetilde{b} = \frac{1}{2}\lambda \xi_{xxx} + 2\xi_x b + \xi b_x. \tag{39}$$

The crucial point now is that for $b \in \mathcal{H}^{3/2}_{\text{per}}$ the mapping

$$\xi \mapsto \widetilde{b}$$

defined in equation (39) is a linear (unbounded) operator from the dense subset $\mathcal{H}^{+3/2}_{\text{per}}$ of $\mathcal{H}^{-3/2}_{\text{per}}$ into $\mathcal{H}^{-3/2}_{\text{per}}$.

- *Computation of the Arnold-Euler equation*:

We now define $A : \widetilde{\mathcal{G}} \to \widetilde{\mathcal{G}}^*$ a positive symmetric operator, with the help of Kirillov's form. It is defined at a given point $u \in \widetilde{\mathcal{G}}^*$ by

$$\langle A\xi|\eta\rangle = \Omega(\mathcal{I}\xi, \eta),$$

where $\mathcal{I}$ is the complex structure, and Kirillov's formula is given by

$$\Omega(\xi, \eta)(u) = \langle u|[\xi, \eta]\rangle,$$

for ξ and η in $\widetilde{\mathcal{G}}$. We remark that all these objects extend by right translation to the whole orbit. Furthermore,

$$\Omega(\mathcal{I}\xi, \xi) \geq 0$$

and

$$\Omega(\mathcal{I}\xi, \eta) = \Omega(\mathcal{I}\eta, \xi).$$

Finally, the above defined symmetric form realizes the Weil-Petersson metric. At this point we choose a parametrization of the

circle, and with the help of such parametrization $x : \mathbb{R} \to S^1$ we have an orthonormal basis given by

$$e_n(x) = \frac{z^{n-2}}{\sqrt{n^3 - n}}, \quad n \geq 2,$$

where $z = \exp(ix)$.

We remark that in a local system of coordinates, the function q above belongs to $\mathcal{H}_{\mathrm{per}}^{3/2}$ and the second component of the map A^{-1} has image in $\mathcal{H}_{\mathrm{per}}^{-3/2}(\mathbb{R}, \mathbb{R})$. Now we use equation (39) where $\lambda = 1$, $\xi = q d/dx$ and $b = q dx^{\otimes 2}$. This yields in the sense of $\mathcal{H}_{\mathrm{per}}^{-3/2}$ (modulo linear changes of variables)

$$\frac{dq}{dt} = \left(\frac{1}{2} q_{xxx} + 3 q q_x \right) . \tag{40}$$

Hence, the Arnold-Euler equation gives the KdV equation. The Arnold-Euler equation (as remarked by Ovsienko and Khesin in [39]) is the Hamiltonian flow on the symplectic co-adjoint orbits in $\widetilde{\mathcal{G}}^*$ generated by the Hamiltonian function

$$q \mapsto \frac{1}{2} \langle (1, q(dx)^{\otimes 2}) | A^{-1}(1, q(dx)^{\otimes 2}) \rangle.$$

According to Kirillov (see also [51]) our manifold $\mathcal{T}$ is a co-adjoint orbit of the Bott group. $\square$

Repeating the argument of Arnold as presented on the first part of this Appendix we conclude:

Theorem 40. *The geodesic flow on the Kähler manifold $\mathcal{T}$ with respect to the right invariant Kähler metric defined in equation (13) induces the KdV flow on $T_{id}\mathcal{T}$ by means of left translations to the identity.*

Proof. Let $g(t) = g(t; \cdot)$ be a geodesic w.r.t. the Kähler metric on $\mathcal{T}$. From the right-invariance of the Kähler metric on $\mathcal{T}$ we have that

$$R_{g^{-1}*}\dot{g}(t) = \dot{g}(0) = u_0.$$

As before, this is equivalent to

$$R_{g(t)}u_0 = \dot{g}(t).$$

Let $u(t, x) = L_{g^{-1}*}\dot{g}(t)$. By choosing an o.n. basis we will identify the tangent and the cotangent spaces at the identity of $\mathcal{T}$. Let $A : T_{id}\mathcal{T} \to T_{id}\mathcal{T}^*$ be this identification. Using the result of Arnold concerning the geodesic flow, proved in Appendix 2 of Reference [5], we conclude that

$$\frac{dAu(t, x)}{dt} = \{u(t, x), Au(t, x)\}.$$

From formula (40), it follows that this last equation is equivalent to KdV. $\square$

Corollary 41. *Let $g(t, \exp(ix)) = \exp(i\phi(t, x))$ be a geodesic in $\mathcal{T}$ with initial velocity $\dot{\phi}(0, x)$ where*

$$A\dot{\phi}(0, x) = \sum a_n e^{inx} \in \mathcal{H}_{\mathrm{per}}^{3/2}(\mathbb{R}; \mathbb{R}).$$

Then,

$$q(t, x) = A\left(\frac{\dot{\phi}(t, x)}{\phi_x(t, x)}\right),$$

is a solution of the KdV equation in $\mathcal{H}_{\mathrm{per}}^{3/2}$.

Proof. We have shown that the geodesic flow along $\mathcal{T}$ extends for all time in the holomorphic directions. Since the geodesic g extends for all time it follows that so does the velocity vector $\dot{g}$. Hence, $q(t, x) = A(\dot{\phi}(t, \exp(ix))/\phi_x(t, \exp(ix)))$ is in $\mathcal{H}_{\mathrm{per}}^{3/2}$ for all time and satisfies the KdV equation because of Theorem 40. $\square$

B Appendix 2: Review of Some Basic Facts of Complex Analysis

B.1 Introduction

In this Appendix we shall summarize some well known facts from the Theory of Teichmüller spaces. The purpose is two-fold. First, we hope

that this will make the results of the paper accessible to a wider audience. Second, it will make the notation and the definitions used easily available. We stress, however, that this appendix is not intended to be a survey of the theory. For that please refer to [1, 28, 29, 11, 41, 36].

In what follows we shall denote by $\mathbb{H}$ the upper-half-plane and by $\widehat{\mathbb{R}}$ the one point compactification of the real line.

B.2   Quasiconformal Mappings

A *quasiconformal mapping* $f : \mathcal{D} \to f(\mathcal{D}) \subset \mathbb{C}$ is a homeomorphism that is absolutely continuous on lines, the partial derivatives are locally square-integrable, and f satisfies a.e. the Beltrami equation

$$\bar{\partial} f - \mu \partial f = 0,$$

for some measurable function μ such that

$$k \stackrel{\text{def}}{=} \sup_{z \in \mathcal{D}} |\mu(z)| < 1.$$

We shall say that f is K-quasiconformal if

$$K = \frac{1 + k}{1 - k}.$$

The complex valued function μ is called the *complex dilation* of the quasiconformal mapping f.

We introduce the Banach space B_p of complex functions $w(z)$, $z \in \mathbb{C}$ which vanish at $z = 0$ satisfy a Hölder condition with exponent $1 - (2/p)$ and have generalized derivatives in L_p. The norm in B_p is given by:

$$\|w\|_{B_p} = H_{1-2/p}[w] + \|\partial w\|_{L^p} + \|\bar{\partial} w\|_{L^p},$$

where

$$H_{1-2/p}[w] \stackrel{\text{def}}{=} \sup_{z_1 \neq z_2} \frac{|w(z_1) - w(z_2)|}{|z_1 - z_2|^{1-2/p}}.$$

The existence of a quasiconformal homeomorphisms for a given measurable function μ in a region $\Omega \subset \mathbb{C}$ is consequence of the following result: (See [6], page 269.)

Theorem 42. *Let $\mu \in L^\infty$ be such that $\|\mu\|_{L^\infty} < 1$. Then, for every $p > 2$ sufficiently close to 2 the equation*

$$\bar{\partial} w - \mu \partial w = \sigma$$

has a unique solution in B_p.

As a consequence of the result above one gets: [6]

Theorem 43. *Given μ measurable on the plane such that $\|\mu\|_{L^\infty} < 1$, there exists a unique quasiconformal homeomorphism with complex dilation μ of the plane onto itself with fixed points 0, 1 and ∞. Furthermore, if μ vanishes outside a compact subset of $\mathbb{C}$, then there exists a unique quasiconformal homeomorphism f^μ with complex dilation μ such that $f^\mu(0) = 0$ and $\partial f^\mu - 1 \in L^p(\mathbb{C})$ for some p.*

An analogous result for the disk $\mathbb{D}_0$ is

Theorem 44. *Given μ measurable on the disk $\mathbb{D}_0$ such $\|\mu\|_{L^\infty} < 1$, there exists a unique quasiconformal homeomorphism with complex dilation μ of the disk onto itself with fixed points 0 and 1.*

B.3   Quasisymmetric Functions

A quasisymmetric function on S^1 is a homeomorphism $h : S^1 \to \mathbb{C}$ for which there exists M such that for all z_1, z_2, and z_3 in S^1

$$|z_1 - z_2| = |z_2 - z_3| \Rightarrow |h(z_1) - h(z_2)| \leq M|h(z_2) - h(z_3)|.$$

Quasisymmetric functions play an important role in Teichmüller theory [1]. We shall say that a quasisymmetric function $\phi : S^1 \to S^1$ is *normalized* if it fixes the points -1, $-i$, and 1.

It is usual to consider quasisymmetric functions on the real line, in which case an equivalent definition is given. A strictly increasing homeomorphism $h : \widehat{\mathbb{R}} \to \widehat{\mathbb{R}}$, satisfying $h(\infty) = \infty$ is called λ-*quasisymmetric* if for every $x \in \mathbb{R}$ and $t > 0$ we have

$$\frac{1}{\lambda} \leq \frac{h(x+t) - h(x)}{h(x) - h(x-t)} \leq \lambda. \tag{41}$$

One of the most important results of the theory is that quasisymmetry of h is the necessary and sufficient condition for h to be a boundary function of a quasiconformal self-mapping of the upper-half plane $\mathbb{H}$ fixing ∞. This is the content of the next two results, the second of which is due to Beurling and Ahlfors. (See [28] pages 31 and 33.)

Theorem 45. *Let f be a K-quasiconformal mapping of $\mathbb{H}$ onto itself such that $f(\infty) = \infty$. Then, f can be extended to a continuous function $\widetilde{f}$ on the closure $\widehat{\mathbb{R}} \cup \mathbb{H}$ of $\mathbb{H}$. The boundary function $h = \widetilde{f}|_{\mathbb{R}}$ satisfies the inequalities in equation (41). Furthermore, the dependence of λ on K in equation (41) is continuous and $\lambda(1) = 1$.*

The converse of this result is given by

Theorem 46 (Beurling-Ahlfors). *Let h be λ-quasisymmetric, then for $y \geq 0$ and $x \in \mathbb{R}$*

$$f(x + iy) \stackrel{\text{def}}{=} \frac{1}{2} \int_0^1 (h(x + ty) + h(x - ty)) \, dt$$
$$+ i \int_0^1 (h(x + ty) - h(x - ty)) \, dt,$$

is a K-quasiconformal mapping of $\mathbb{H}$ onto itself. Furthermore, it agrees with h for $y = 0$, it is continuous, and $f(\infty) = \infty$. The maximal dilation K is bounded by $K(\lambda)$ that depends continuously on λ and tends to 1 as λ goes to 1.

We now remark:

1. Let h_1 and h_2 be quasisymmetric functions and f_1 and f_2 their quasiconformal Beurling-Ahlfors extensions. Then, the quasisymmetry constant of $f_2 \circ f_1^{-1}$ tends to 1 if so does the quasisymmetry constant λ of $h_2 \circ h_1^{-1}$.

2. Suppose $h = h(\cdot, \xi)$ depends analytically on the parameter ξ and $\partial_\xi h(\cdot, \xi)$ is integrable. Then, the Beurling-Ahlfors extension given above also depends analytically on this parameter.

B.4 The Sewing Problem

A number of problems in complex analysis reduce to the following *sewing problem*[28, 29, 22]: Given ϕ a quasisymmetric homeomorphism of the circle find a pair of homeomorphisms (f_0, f_∞) such that

(a) $f_0 : \overline{\mathbb{D}}_0 \longrightarrow f_0(\overline{\mathbb{D}}_0) \subset \widehat{\mathbb{C}}$ and $f_\infty : \overline{\mathbb{D}}_\infty \longrightarrow f_\infty(\overline{\mathbb{D}}_\infty) \subset \widehat{\mathbb{C}}$, where f_0 and f_∞ are conformal in the interior of their domains of definition,

(b) The sets $f_0(\mathbb{D}_0)$ and $f_\infty(\mathbb{D}_\infty)$ are complementary Jordan domains, and

(c) For every $z \in S^1$

$$\phi(z) = f_0^{-1} \circ f_\infty(z).$$

It is obvious that the problem, as stated above has many solutions if it has one. Indeed, let (f_0, f_∞) be a solution of the problem. Then, left composition by any linear fractional transformation T is such that $(T \circ f_0, T \circ f_\infty)$ is also a solution of the sewing problem. In order to solve this problem of nonuniqueness we introduce the following:

Definition 47. We shall say that a homeomorphism $\phi : S^1 \to S^1$ is normalized if it fixes the points -1, $-i$, and 1. We shall say that a solution (f_0, f_∞) of the sewing problem is normalized if both f_0 and f_∞ fix -1, $-i$, and 1.

We remark that the choice of the points -1, $-i$, and 1 is mere convenience. The sewing problem stated above is usually considered in the context of the upper half plane $\mathbb{H}$ and its boundary $\widehat{\mathbb{R}}$. In that context, the normalization imposed consists of fixing 0, 1 and ∞. Obviously, we are translating all such notions to the context of $\mathbb{D}_0$ and S^1 by means of the linear fractional transformation

$$z = \frac{\zeta - i}{\zeta + i}.$$

The following result settles the sewing problem [40, 28, 29]:

Theorem 48. *Let ϕ be a normalized quasisymmetric function. Then, the sewing problem for ϕ has a unique normalized solution pair (f_0, f_∞).*

The main steps in the proof of this result are: First, extend ϕ continuously to the interior of the disk as a quasiconformal mapping $\widetilde{\phi}$ using the Beurling-Ahlfors result. Second, let $\kappa[\widetilde{\phi}]$ be the complex dilation of $\widetilde{\phi}$ and construct F the unique *normalized* solution of the problem

$$\bar{\partial}F - \mu\partial F = 0, \tag{42}$$

where

$$\mu(z) = \begin{cases} \kappa[\widetilde{\phi}](z), & z \in \mathbb{D}_0 \\ 0, & z \in \widehat{\mathbb{C}} \setminus \mathbb{D}_0. \end{cases} \tag{43}$$

Third, let

$$f_0 \stackrel{\text{def}}{=} F|_{\overline{\mathbb{D}}_0} \circ \widetilde{\phi}^{-1},$$

and

$$f_\infty \stackrel{\text{def}}{=} F|_{\overline{\mathbb{D}}_\infty}.$$

It is easy to check that $f_0^{-1} \circ f_\infty(z) = \phi(z)$ for $z \in S^1$ and that both f_0 and f_∞ are analytic inside their domains of definition. The normalization of the solution pair (f_0, f_∞) follows from that of F and of ϕ. Uniqueness (as well as the above argument) can be found in [28] page 101.

The next proposition was used in Section 2 to guarantee that μ_ϕ as in Definition 4 corresponds to a complex dilation of some quasiconformal mapping.

Proposition 49. *Given $\epsilon > 0$, there exists $\delta > 0$ such that if ϕ is λ-quasisymmetric with $|\lambda - 1| < \delta$, then*

$$\|\mu_\phi\|_\infty = \sup_{z \in \mathbb{D}_0}(1 - |z|^2)^2|\mathcal{S}[f_\infty](1/\bar{z})| < \epsilon.$$

Proof. We follow the notation of Theorems 46 and 48. The univalent conformal function f_∞ is the restriction of a quasiconformal mapping F of the extended plane satisfying equation (42) with complex dilation given by (43). It is known that[7] if g is a quasiconformal mapping of

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[7]See for example Theorem 3.2 page 72 of [28].

the plane of complex dilation $\widetilde{\mu}$ and such that g restricted to the disc $\mathbb{D}_0$ is conformal, then

$$\sup_{z \in \mathbb{D}_0} (1 - |z|^2)^2 |\mathcal{S}[g](z)| \leq 6\|\widetilde{\mu}\|_\infty.$$

Writing $g(z) = F(1/z)$ and using properties of the Schwarzian derivative, it follows that

$$\sup_{z \in \mathbb{D}_0} |(1 - |z|^2)^2 \mathcal{S}[f_\infty](1/\bar{z})| \leq 6 \sup_{z \in \mathbb{D}_0} |\kappa[\widetilde{\phi}](z)|.$$

Using Theorem 46 we get that if ϕ is λ-quasisymmetric, then

$$\sup_{z \in \mathbb{D}_0} |\kappa[\widetilde{\phi}](z)| \leq \frac{K(\lambda) - 1}{K(\lambda) + 1},$$

where $K(\lambda) \to 1$, when $\lambda \to 1$. This proves that μ_ϕ given by

$$\mu_\phi(z) = \begin{cases} (1 - |z|^2)^2 \mathcal{S}(f_\infty)(1/\bar{z}), & z \in \mathbb{D}_0 \\ 0, & z \in \mathbb{C} \setminus \mathbb{D}_0 \end{cases}$$

can be made arbitrarily small in the L^∞ norm by taking ϕ a λ-quasisymmetric homeomorphism with λ close to 1. $\qquad\square$

As a simple corollary of the above result we get

Corollary 50. *If ϕ is sufficiently close to the identity in the C^1 topology, then μ_ϕ as in Definition 4 has $\|\mu_\phi\|_\infty < 1$.*

B.5   The Result of Ahlfors and Weill

The next result due to Ahlfors and Weill [2] plays an important role in the theory of Teichmüller spaces. It is usually stated in the context of $\mathbb{H}$ and $\mathbb{H}^*$ [11]. For convenience we are rewriting it in the context of $\mathbb{D}_0$ and $\mathbb{D}_\infty$.

Theorem 51 (Ahlfors-Weill). *Let $\varphi : \mathbb{D}_\infty \to \widehat{\mathbb{C}}$ be holomorphic such that*

$$(1 - |z|^2)^2 |\varphi(z)| < 2 \ , \ \forall z \in \mathbb{D}_\infty.$$

Take,

$$\mu(z) = \begin{cases} -\frac{1}{2}\frac{(1-|z|^2)^2}{\bar{z}^4}\varphi(1/\bar{z}), & z \in \mathbb{D}_0 \\ 0, & z \in \mathbb{D}_\infty. \end{cases}$$

Then, for any quasiconformal mapping F on the Riemann sphere satisfying

$$(\bar{\partial} - \mu\partial)F = 0, \tag{44}$$

we have that

$$\varphi(z) = \mathcal{S}[F](z).$$

We shall not repeat the proof of this result here. It could be found in page 100 of [11]. We shall, however, highlight the main points in the proof. This, we believe, will give some insight in the connection between Teichmüller's theory and the Schrödinger equation. The argument hinges upon the fact that the Schwarzian derivative of ratios of independent solutions of the Schrödinger equation

$$y'' + \frac{1}{2}\varphi y = 0 \ , \tag{45}$$

gives back the potential φ. More precisely, we have the following

Lemma 52. *Let $\varphi : \mathcal{D} \to \mathbb{C}$ be holomorphic. Set*

$$\mathcal{R}[\varphi] \stackrel{\text{def}}{=} \left\{ w = \frac{y_1}{y_2} \mid y_1 \text{ and } y_2 \text{ l.i. solutions of (45) on } \mathcal{D} \right\},$$

and

$$\mathcal{S}^{-1}[\varphi] \stackrel{\text{def}}{=} \{ w : \mathcal{D} \to \mathbb{C} \text{ holomorphic} \mid \mathcal{S}[w] = \varphi \}.$$

Then,

$$\mathcal{R}[\varphi] = \mathcal{S}^{-1}[\varphi].$$

Now, the idea of the proof is based on the explicit construction of a quasiconformal mapping satisfying (44) (at least under some simplifying assumptions), namely:

$$F(z) = \begin{cases} \frac{y_1(1/\bar{z}) + (z - \frac{1}{\bar{z}})y_1'(1/\bar{z})}{y_2(1/\bar{z}) + (z - \frac{1}{\bar{z}})y_2'(1/\bar{z})}, & z \in \mathbb{D}_0 \\ \frac{y_1(z)}{y_2(z)}, & z \in \widehat{\mathbb{C}} \setminus \mathbb{D}_0, \end{cases}$$

where, y_1 and y_2 are solutions of equation (45). Once one establishes that F as defined above is a solution of (44) it follows from the Lemma 52 that

$$\varphi = \mathcal{S}[y_1/y_2].$$

The result now follows using the invariance of the Schwarzian derivative given by the formula

$$\mathcal{S}(T \circ F) = \mathcal{S}(F),$$

where T is any linear fractional transformation, and the fact that different quasiconformal solutions of (44) are related by a linear fractional transformations.

Remark. It is well known the important role played by the Schrödinger operator in the study of the KdV equation. It is also known the importance of the Riemann-Hilbert boundary value problem in the solution of many completely integrable systems. The connection of the Schwarzian derivative with the Schrödinger equation described above seems to indicate the fact that the Schrödinger equation is entering the picture once more. Here, in the form of an explicit way to produce solutions to the Beltrami equation.

B.6   The Construction of the Normalized Solutions to the Beltrami Equation

In this part of the appendix we show how to construct a solution to the Beltrami equation with zero initial condition such that derivative of the solution minus certain monomials are in L^p with $p \geq 2$.

We recall that if $f : A \to \mathbb{C}$ has derivatives in $L^1(A)$ and D is a domain such that $\bar{D} \subset A$, then

$$f(z) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(\zeta)}{\zeta - z} dS(\zeta) - \frac{1}{\pi} \iint_D \frac{\bar{\partial} f(\zeta)}{\zeta - z} d\xi d\eta. \qquad (46)$$

This is the generalized Cauchy formula, which is an immediate consequence of Green's formula. Hence, if $f(z) \to 0$, as $z \to \infty$, and if f has

derivatives in $L^1(\mathbb{C})$ it follows that

$$f(z) = -\frac{1}{\pi} \iint_{\mathbb{C}} \frac{\bar{\partial} f(\zeta)}{\zeta - z} d\xi d\eta. \qquad (47)$$

In other words, $f = P\bar{\partial} f$, where P was defined in equation (3). With this information in hand we proceed to find a solution for Beltrami's equation with the properties described above. More precisely,

Theorem 53. *For any integer $n \geq 1$ there exists at least one solution $\omega^{(n)}$ of*

$$\begin{cases} \bar{\partial}_{\mu_\phi} \omega^{(n)} = 0 \\ \partial_z \omega^{(n)} - n z^{n-1} \in L^p \quad \text{for some } p > 2 \\ \int_{S^1} \omega^{(n)} = 0. \end{cases} \qquad (48)$$

Proof. We first find a solution to

$$\begin{cases} \bar{\partial}_{\mu_\phi} f^{(n)} = 0 \\ \partial f^{(n)} - n z^{n-1} \in L^p \text{ for some } p > 2. \end{cases} \qquad (49)$$

The solution is not unique. It will become unique when we impose the third condition in (48). We will suppose $f(\infty) = \infty$. Then, in a neighborhood of ∞, we have

$$f^{(n)}(z) = g(z) + \sum_{i=0}^{\infty} b_i z^{-i} = z^n + G(z) + \sum_{i=0}^{\infty} b_i z^{-i}, \qquad (50)$$

where $g(z)$ and $G(z)$ are entire. This is a consequence of $\mu_\phi = 0$ in $\mathbb{C} \setminus \mathbb{D}_0$. Since $\partial_z f^{(n)} - n z^{n-1} \in L^p$ for some $p > 2$, we have $G(z) = A_0$, with A_0 a constant to be determined below. Thus,

$$f^{(n)}(z) = z^n + A_0 + \sum_{i=0}^{\infty} b_i z^{-i},$$

Since $f^{(n)}$ satisfies the Beltrami equation, its derivatives are in L^2 locally ([28]) and hence the derivatives of $f^{(n)} - z^n - A_0 \in L^2(\mathbb{C})$. Note that $L^2(K) \subset L^1(K)$ for any K compact and by (50) in a neighborhood

of ∞ it follows that $f^{(n)} - z^n - A_0$ has derivatives in L^1. Hence their derivatives are in $L^1(\mathbb{C})$ and applying the Cauchy generalized formula it follows that

$$F = f^{(n)} - z^n - A_0 = P\bar{\partial}f^{(n)} = P\bar{\partial}F.$$

Thus,

$$\partial f^{(n)} - nz^{n-1} = \partial P\bar{\partial}f^{(n)} = T\bar{\partial}f^{(n)}.$$

From where,

$$\bar{\partial}f^{(n)} = \mu nz^{n-1} + \mu T\bar{\partial}f^{(n)}.$$

Suppose now $\|\mu\|_\infty \|T\|_p < 1$. Then the solution of the last equation can be obtained by a Neumann series. More precisely, let $\phi_1 = \mu nz^{n-1}$, and for $j > 1$, $\phi_j = \mu T\phi_{j-1}$. Hence,

$$\|\phi_j\|_p \leq (\pi)^{1/p}\|t\|_p^{j-1}\|\mu\|_\infty^j.$$

Hence $\sum_{i=1}^\infty \phi_i$ converges and $\bar{\partial}f^{(n)} = \sum_{i=1}^\infty \phi_i$ and moreover $\bar{\partial}f^{(n)} \in L^p$.

This establishes the existence of the $f^{(n)}$. To construct a solution satisfying the initial condition $\int_{S^1} \omega^{(n)} = 0$ we need the following:

Lemma 54. *Let f be a quasiconformal mapping of the plane with complex dilatation μ of compact support, satisfying*

$$\lim_{z\to\infty} f(z) - z^n - A_0 = 0.$$

Then,

$$f(z) = z^n + A_0 + P\sum_{i=1}^\infty \phi_i = z^n + A_0 + \sum_{i=1}^\infty P\phi_i.$$

For a proof, see Theorem 4.3, page 27 of [28]. Although the proof there is for $n = 1$ and $A_0 = 0$, with minor modifications it gives the result we need. (Using the ϕ_j as we defined here.)

Hence, it it is easy to see, using Cauchy's theorem, that $\omega^{(n)} = z^n + P\sum_{i=1}^\infty \phi_i(z)$, satisfies all the three conditions in (48). This concludes the proof of the theorem. $\qquad\square$

Acknowledgments

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