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Decay of the total variation and Hardy norms of solutions to parabolic conservation laws

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1. Introduction

Scalar parabolic conservation laws and systems of equations of this kind arise in continuum mechanics where they serve as models for elastic solids and ideal gases. This paper is concerned with the large-time behaviour of solutions to scalar parabolic conservation laws and systems of parabolic conservation laws in several space dimensions. The first part of the paper focuses on the derivation of decay estimates in the Hardy space $\mathscr{H}^1$; in the second part, we use these bounds to quantify the decay rate for the total variation of the solution.

The scalar parabolic conservation law under consideration has the form

$$\partial_t u + \sum_{j=1}^{n} \frac{\partial}{\partial x_j} F^j(u) = \sum_{i,j=1}^{n} \frac{\partial^2 u}{\partial x_i \partial x_j}, \tag{1}$$

after a linear change of independent variables this is transformed into the nonlinear partial differential equation

$$\partial_t u + \sum_{j=1}^{n} \frac{\partial}{\partial x_j} f^j(u) = \Delta u. \tag{2}$$

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The systems considered here are of the form

$$\partial_t u + F_j(u)_{x_j} = B_{jk} u_{x_j x_k}, \tag{3}$$

where we used the convention of summing over repeated indices and

$$F_j(u)_{x_j} = \frac{\partial}{\partial x_i} F_i(u) = \nabla F_i(u) \frac{\partial}{\partial x_i}(u - c) \tag{4}$$

for any constant vector $c = (c_1, c_2, \ldots, c_n)$.

Precise conditions on f_j, $F = (F_1, \ldots, F_n)$ and B are given in Theorem 4 in Section 2. The main purpose of the paper is to show that if the solution to the underlying linear part, namely the heat equation, decays in the Hardy norm, or the total variation norm does so, then so do the solutions to the parabolic conservation law if started with the same initial datum.

We recall that for the scalar case the L^1 norm remains bounded if the data is in L^1; consequently, L^1 decay and total variation decay can be easily obtained. For systems it is not known whether the L^1 norm remains bounded. Nevertheless, we show here that if the initial data are bounded in $\mathscr{H}^1$ then so is the solution for all time and, for appropriate data, it decays in $\mathscr{H}^1$. This information then enables us to deduce the decay of the total variation in the case of a system of parabolic conservation laws.

The decay of solutions to scalar parabolic conservation laws in L^p norms and Hilbertain Sobolev norms has been investigated in [6,7,12]; see also [1] for the analysis of the long-time behaviour of solutions to scalar multi-dimensional parabolic conservation laws in bounded domains. For systems of parabolic conservation laws on $\mathbb{R}^n$, decay estimates in $W^{k,p}(\mathbb{R}^n)$, $p \in [2, \infty]$, have been established in the work of Hoff and Zumbrum [2].

The present paper is structured as follows. In Section 2 we recall some decay theorems which will be necessary later on in our analysis. In Section 3 we study the decay of solutions to scalar parabolic conservation laws in the Hardy space $\mathscr{H}^1$. Section 4 establishes analogous decay results in $\mathscr{H}^1$ for systems of parabolic conservation laws. Section 5 then exploits these Hardy-norm estimates to quantify the rate of decay of the total variation norm for solutions to scalar parabolic conservation laws and systems of such equations. In particular, we shall show that, under suitable hypotheses on the initial datum and the flux function, the total variation of the solution decays to zero at the algebraic rate of $t^{-1/2}$, as $t \to \infty$. We shall suppose throughout that the solutions under consideration are regular; for scalar parabolic conservation laws the regularity of solutions is known in the case of large smooth initial data, while for systems this has only been established for small smooth initial data (see, for example, [3,2]).

In order to establish the existence of solutions which have bounded norm in $\mathscr{H}^1$, we have to impose growth conditions on the gradient of the flux function. Functions meeting these conditions are easy to construct, for example, powers of u are included. Thus we do not consider this condition as very restrictive.

2. Preliminaries

We shall use the multi-index notation $\alpha=(\alpha_1,\dots,\alpha_n)$, $|\alpha|=\alpha_1+\cdots+\alpha_n$ with $\alpha_i \ge 0$, and write

$$D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1}\dots\partial x_n^{\alpha_n}}.$$

Further, we define

$$C_0^m = C_0^m(\mathbb{R}^n) = \{u \in C^m(\mathbb{R}^n):\ \lim_{|x|\to\infty} D^\alpha u(x) = 0,\ |\alpha| \le m\}.$$

The L^2 norm (or energy norm) will be denoted by

$$\|u\| = \left(\int_{\mathbb{R}^n} |u(x)|^2\,\mathrm{d}x\right)^{1/2},$$

where $x=(x_1,x_2,\dots,x_n)$, $\mathrm{d}x = \mathrm{d}x_1\,\mathrm{d}x_2\dots\mathrm{d}x_n$. More generally, we consider the L^p norm, for $1 \le p < \infty$, denoted by

$$\|u\|_p = \left(\int_{\mathbb{R}^n} |u(x)|^p\,\mathrm{d}x\right)^{1/p}$$

and the L^∞ norm,

$$\|u\|_\infty = \operatorname{ess}\sup_{x\in\mathbb{R}^n} |u(x)|.$$

The Sobolev H^m norm is defined by

$$\|u\|_{H^m} = \left(\int_{\mathbb{R}^n} \sum_{|\alpha|\le m} |D^\alpha u(x)|^2\,\mathrm{d}x\right)^{1/2}.$$

Next, we recall the definition of Hardy spaces on $\mathbb{R}^n$, denoted $\mathscr{H}^p$ for $0 < p < \infty$ (see [10,11]). Let $\mathscr{S}$ denote the Schwarz space of rapidly decreasing functions on $\mathbb{R}^n$. Let $\phi \in \mathscr{S}$ satisfy $\int_{\mathbb{R}^n}\phi(x)\,\mathrm{d}x = 1$. A tempered distribution f belongs to $\mathscr{H}^p$ whenever

$$f^+(x) = \sup_{t>0} |\phi_t * f)| \in L^p(\mathbb{R}^n),$$

where $\phi_t(x) = t^{-n}\phi(x/t)$. The Hardy norm of f is defined by

$$\|f\|_{\mathscr{H}^p} = \|f^+\|_p \quad \text{for } p \ge 1.$$

We recall that $\mathscr{H}^1$ is a Banach space strictly contained in L^1 and that $L^p \sim \mathscr{H}^p$ for $p > 1$.

We list some known decay estimates in the L^p and H^m norm which will be required in subsequent sections. We begin by considering the scalar case.

Theorem 1. *Let $f_j \in C^1(\mathbb{R},\mathbb{R})$, $u_0 \in L^1 \cap C^1 \cap H^1(\mathbb{R}^n)$. Assuming that u is a solution of* (2) *with data u_0, we have that*

$$\|u(t)\|_p \le C_p(1+t)^{-(n/2)(1-1/p)}, \quad 1 \le p \le \infty. \tag{5}$$

For a proof of this result, see [6,7,12].

Theorem 2. *Let* $f_j \in C^{m+1}(\mathbb{R},\mathbb{R})$ *and* $u_0 \in L^1 \cap C^1 \cap H^m(\mathbb{R}^n)$. *Further, let u be a solution of* (2) *with data* u_0; *then*

$$\|u(t)\|_{H^m} \le C_m(t+1)^{-n/4}. \tag{6}$$

Moreover, if $m > [n/2]$ *then*

$$\sum_{|\alpha|<m-[n/2]} \|D^\alpha u(t)\|_\infty \le C(t+1)^{-n/2}. \tag{7}$$

Proof. We first note that the solution can be written as an integral equation involving the heat kernel. Since f is smooth and by the last theorem the solution u is in L^∞ we can put derivatives on the kernel inductively and hence we know that we have a smooth solution. For details on the proof of this result for smooth solutions, see [6,7,12].

We note that it is very easy to improve the above result in order to obtain the same decay as for solutions to the heat equation in higher-order Sobolev spaces. Specifically, we have the following result.

Corollary 3. *Let* $f_j \in C^{m+1}(\mathbb{R},\mathbb{R})$ *and* $u_0 \in L^1 \cap C^1 \cap H^m(\mathbb{R}^n)$. *Further, let u be a solution of* (2) *with data* u_0; *then*

$$\|u(t)\|_{H^m} \le C_m(t+1)^{-n/4-m/2}. \tag{8}$$

Moreover, if $m > [n/2]$ *then*

$$\sum_{|\alpha|<m-[n/2]} \|D^\alpha u(t)\|_\infty \le C(t+1)^{n/2-|\alpha|/2}. \tag{9}$$

Proof. The argument proceeds by induction; we only sketch the main ideas. The key ingredient of the proof of (8) is to use a Fourier splitting method [6–9]. The aim is to obtain an integral inequality of the type

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^n} |D^\alpha u|^2\,\mathrm{d}x \le G(t) - \int_{\mathbb{R}^n} |D^{\alpha+1}u|^2\,\mathrm{d}x = \mathrm{I} + \mathrm{II}, \tag{10}$$

where $G(t) \le (t+1)^{-\gamma}$, with $\gamma \ge m/2+n+1$. We also require that the Fourier transform of u has finite L^∞ norm in the vicinity of the origin; this follows from the hypotheses on the data. Once these two steps are accomplished, the decay at the appropriate rate follows by the Fourier splitting method. We note that in order to obtain (10) we multiply the parabolic conservation law by u and integrate. Term II on the right-hand side arises through integration by parts of the diffusion term, while $G(t)$ originates from the convective term. It is here that we have to apply some bootstrapping arguments of the type used to establish the decay for H^m norms of solutions to the Navier–Stokes equations (see [9]) combined with the decay results from Theorem 2.

To derive the second estimate, we use the first estimate combined with the Gagliardo–Nirenberg inequality. □

We now recall a theorem regarding the decay of solutions to systems of parabolic conservation laws with small initial data. This result was obtained by Hoff and Zumbrum [2]. In what follows, it will be supposed that system (3) has a convex entropy S satisfying

$$S'' > 0 \quad \text{on } \overline{B_{r0}(0)},$$

here $B_{r0}(0) = \{y : |y| < r_0\}$. Letting A_j denote the Jacobian of F_j at zero, it will be assumed that $S''(0)$ is a positive-definite symmetric matrix such that $S''(0)A_j$ is symmetric and there is a positive number ε such that

$$(\xi_j \xi_k S''(0) B_{jk} w, w) \geq \varepsilon |\xi|^2 |\omega|^2, \tag{11}$$

where $\xi \in \mathbb{R}^n$ and $\omega \in \mathbb{R}^n$.

Theorem 4. *Suppose that there is positive integer k such that*

$$F_j \in C^{k+2}(\overline{B_{x0}(0)}). \tag{12}$$

Let S be an entropy for system (3) *which satisfies the conditions described above. Let $\hat{G}^B(\xi, t) = \mathrm{e}^{-\xi_j \xi_k B_{jk} t}$ and suppose that the B_{jk} are such that there exists a constant C_1 with*

$$\|\hat{G}^B(\cdot, t)\|_1 = C_1.$$

Suppose also that $\xi_j A_j$ and $\xi_j \xi_k B_{jk}$ commute for all $\xi \in \mathbb{R}^n$. Then, given $r < r_0$, there are positive constants C and μ such that if

$$\|u_0\| \leq \frac{r}{C_1} \quad \textit{and} \quad \|u_0\| \leq \mu,$$

then there is a unique global solution u of system (3), *and u satisfies*

$$u \in C((0, \infty); W^{k,p}(\mathbb{R}^n)), \quad p \in [2, \infty], \tag{13}$$

$$\|D_x^\alpha u(\cdot, t)\|_p \leq c t^{n/2(1/p-1) - |\alpha|/2} \|u_0\|_1. \tag{14}$$

For a proof, see [2].

3. Scalar case: existence and decay of solutions in Hardy spaces

In this section we study the existence and decay of the solutions to scalar conservation laws in many dimensions in the $\mathscr{H}^1$ norm.

The main estimate arises from a bound on the flux term in $\mathscr{H}^1$. This estimate is based on a proof in [4] for bounding the convective term in the Navier–Stokes equations. Note that unlike the analysis in [4] where the index q is supposed to be strictly less than n, here we let $1 < q < \infty$. Specifically, we obtain estimates for the derivatives of a nonlinear function in the $\mathscr{H}^T$ norm in terms of L^p and L^q norms.

Assumption 1. We shall say that f satisfies $A1_k$ if there exist positive constants C and k such that $|f'(v)| \leq C|v|^k$ for all real v.

Lemma 5. *Let u_0 and f_j be as in Corollary* 3, *and that f_j, $j=1,\ldots,n$, satisfy $A1_k$ for some $k>1$. Suppose that $n\geq 2$ and $r\geq 1$, and let $p\geq 1$ and $q\geq 1$ be such that $1/r=1/p+1/q$. Then, the solution of Eq.* (2) *with flux function $f=(f_1,\ldots,f_n)$ and initial condition u_0 satisfies the bound*

$$\left\|\frac{\partial}{\partial x_i}f_i(u)\right\|_{\mathscr{H}^r}\leq C\|\,|u|^k\|_p\|\nabla u\|_q, \tag{15}$$

where $C>0$ depends only on the norms of the data. As usual, repeated indices mean summation over these indices.

Proof. We note first that the requirements on the data were only imposed so that we meet the conditions of Theorem 2 and Corollary 3. In reality, we could have worked with solutions which satisfy $u\in L^\infty$ with $\nabla u\in L^p\cap L^q$.

We deduce from Corollary 3 that $\nabla u\in L^\infty$ and $u\in H^2$; further it follows from Theorem 2 that $u\in L^q$. Now recall that by hypothesis

$$|f_j'(u)|\leq C|u|^k \quad \text{for some } k\geq 1. \tag{16}$$

Thus by Young's inequality it follows that

$$\left\|\left[\phi_t*\left(\frac{\partial}{\partial x_i}f_j'(u)\right)\right](x)\right\|_r\leq c\|\,|u|^k\nabla u\|_r. \tag{17}$$

By the choice of p and q it follows by Hölder's inequality that

$$\left\|\left[\phi_t*\left(\frac{\partial}{\partial x_i}f_j'(u)\right)\right](x)\right\|_r\leq C\|\,|u|^k\|_p\|\nabla u\|_q \tag{18}$$

and hence the conclusion of the lemma follows. □

As consequence of last lemma we have the existence of solutions in $\mathscr{H}^1$.

Theorem 6. *Let u_0 and f_j be as in Corollary* 3. *Suppose there exists a constant vector $c=(c_1,\ldots,c_n)$ such that $f_j(u)-c_ju$ satisfies $A1_k$ with $k>1$, and assume that u is a solution of* (2) *with initial datum u_0 and flux function $f=(f_1\ldots,f_n)$. If in addition $u_0\in\mathscr{H}^1$ then the solution $u(\cdot,t)$ belongs to $\mathscr{H}^1$ for $t\geq 0$.*

Proof. Let $g_j(u)=f_j(u)-c_ju$; then g_j satisfies $A1_k$ for a certain $k>1$. Since $f_i'(u)=g_i'(u)+c_j$, we can rewrite the parabolic conservation law as

$$\partial_t u+\sum_{j=1}^n\frac{\partial}{\partial x_j}g^j(u)+\sum_{j=0}^n c_j\frac{\partial u}{\partial x_j}=\Delta u. \tag{19}$$

Now let us introduce the differential operator

$$\Gamma=\sum_{j=0}^n c_j\frac{\partial}{\partial x_j}=-\Delta.$$

With this notation the differential equation can be written as

$$\partial_t u + \sum_{j=1}^{n} \frac{\partial}{\partial x_j} g_j(u) + \Gamma u = 0. \tag{20}$$

The linear part of the last equation is given by

$$\partial_t v + \Gamma v = 0. \tag{21}$$

It is easy to see that (21) can be solved by convolving a translated heat kernel with the data; that is, the solution is given by

$$v = \mathscr{K} * u_0$$

where

$$\mathscr{K}(x,t) = \frac{1}{(4\pi 2)^{n/2}} \exp\left\{ -\frac{|x_i - tc_i|^2}{4t} \right\}.$$

We will call the corresponding semigroup e^{-tB}. It is easy to show, performing minor modifications of the proof of Miyakawa [4,5] for the heat semigroup, that

$$\|e^{-Bt} g\|_{\mathscr{H}^p} \le C \|g\|_{\mathscr{H}^p} \tag{22}$$

for $0 < p < \infty$, where C is a positive constant. We rewrite (20) as the integral equation

$$u(\cdot,t) = e^{-Bt} u_0 - \int_0^t e^{-(t-s)B} g_j'(u) u_{x_j} \, ds. \tag{23}$$

Using (22), i.e. the proposed modification to Miyakawa's bound [4] yields $e^{-Bt}u_0 \in \mathscr{H}^1$. Further, by the previous lemma, $g_j'(u(\cdot,s))u_{x_j}(\cdot,s) \in \mathscr{H}^1$. Applying (22) again gives

$$\left\| \int_0^t e^{-(t-s)B} g_j'(u(\cdot,s)) u_{x_j}(\cdot,s) \right\|_{\mathscr{H}^1} ds \le C \int_0^t \|g_j'(u(\cdot,s)) u_{x_j}(\cdot,s)\|_{\mathscr{H}^1} \, ds.$$

Recalling that by hypothesis $|g_j'(u(\cdot,s))| \le C|u|^k$, the right-hand side of the last expression can be bounded by using Young's inequality as follows:

$$\begin{aligned} \int_0^t \|\phi_t\|_{L^1} \|g_j'(u(\cdot,s)) u_{x_j}(\cdot,s)\|_{L^1} \, ds &\le C \int_0^t \| \, |u|^k \nabla u\|_{L^1} \, ds \\ &\le C \int_0^t (t+1)^{-\nu} \, ds \end{aligned}$$

where $\nu = kn/2 + 1/2 > 1$, since $n \ge 2$ and $k > 1$. Hence the last integral is bounded, and thereby both terms on the right-hand side of (23) belong to $\mathscr{H}^1$, so the same is true of $u(\cdot,t)$ for $t \ge 0$. $\square$

We use Lemma 5, combined with Theorems 1 and 2, to show that solutions to parabolic conservation laws with appropriate data decay in the $\mathscr{H}^1$ norm. The decay will be in part a consequence of the decay of the solution to the heat equation established by Miyakawa [4]; specifically, he showed the following result.

Lemma 7. *Let* $n \geq 2$ *and suppose that*

$$u_0 \in L^1 \quad \textit{and} \quad \int_{\mathbb{R}^n} u_0(x)\,\mathrm{d}x = 0, \tag{24}$$

then

$$\lim_{t\to\infty} \|\mathrm{e}^{-tA}u_0\|_1 = 0. \tag{25}$$

Further, *if* $u_0 \in \mathscr{H}^1$, *then*

$$\lim_{t\to\infty} \|\mathrm{e}^{-tA}u_0\|_{\mathscr{H}^1} = 0. \tag{26}$$

Here, *as usual*, e^{-At} *denotes the heat semi-group. If*, *in addition to* (24), *it is assumed that*

$$\int_{\mathbb{R}^n} |x|^\beta |u_0(x)|\,\mathrm{d}x < \infty \quad \textit{for some } 0 < \beta \leq 1, \tag{27}$$

then we have the estimate

$$\|\mathrm{e}^{-tA}u_0\|_1 \leq Ct^{-\beta/2}\left(\int_{\mathbb{R}^n} |x|^\beta |u_0(x)|\,\mathrm{d}x\right) \quad \textit{for all } t > 0.$$

Remark 8. We remark that Miyakawa's proof of the last lemma can be applied with minor modifications to e^{-Bt}, the shifted heat semigroup.

Thus we have the following result.

Lemma 9. *Under the same hypothesis of Lemma* 7 *it follows that*

$$\lim_{t\to\infty} \|\mathrm{e}^{-tB}u_0\|_1 = 0. \tag{28}$$

Further, *if* $u_0 \in \mathscr{H}^1$, *then*

$$\lim_{t\to\infty} \|\mathrm{e}^{-tB}u_0\|_{\mathscr{H}^1} = 0. \tag{29}$$

The plan now is to use this last lemma combined with Theorems 1, 2 and Lemma 5 to establish the $\mathscr{H}^1$ decay.

Theorem 10. *Let* $n \geq 2$, *let* u_0 *and* f_j *be as in Theorem* 2 *and Lemma* 5. *Let* p, q, r *be in the same relation as in Lemma* 5 *and let* $k > 1$. *Suppose also that* $\int_{\mathbb{R}^n} u_0(x)\,\mathrm{d}x = 0$. *Let* u *be the global solution in* $\mathscr{H}^1$ *whose existence is ensured by Theorem* 6. *Then*

$$\lim_{t\to\infty} \|u(\cdot,t)\|_{\mathscr{H}^1} = 0.$$

Proof. As before, we rewrite the equation with the new linear part and hence we have to deal with the translation of the heat semigroup.

We shall show that, given any $\varepsilon > 0$, there is T_0 such that for all $t > T_0$ we have $\|u(\cdot,t)\|_{\mathscr{H}^1} \leq \varepsilon$. To do so, recall the integral expression for the solution u:

$$u(\cdot,t) = \mathrm{e}^{-B(t-T_0)}u(\cdot,T_0) - \int_{T_0}^{t} \mathrm{e}^{-(t-s)B} g_j'(u)u_{x_j}\,\mathrm{d}s, \tag{30}$$

where T is defined below. Hence, for $t > T_0$,

$$\|u(\cdot,t)\|_{\mathscr{H}^1} \leq \|e^{-B(t-T_0)}u(\cdot\,,T_0)\|_{\mathscr{H}^1} + \int_{T_0}^{t} \|e^{-(t-s)^B} g_j'(u)u_{x_j}\|_{\mathscr{H}^1}\,ds$$
$$= \mathrm{I}(t) + \mathrm{II}(t). \tag{31}$$

First, we bound $\mathrm{II}(t)$. Let us recall that by (22) and (8)

$$\|e^{-Bt}g\|_{\mathscr{H}^1} \leq C\|g\|_{\mathscr{H}^1}.$$

Thus by Lemma 5 we have, with $r = 1$.

$$\mathrm{II}(t) \leq + \int_{T_0}^{t} \|g_j'(u)u_{x_j}\|_{\mathscr{H}^1}\,ds \leq C\int_{T_0}^{t} \|u^k\|_p\|\nabla u\|_q\,ds. \tag{32}$$

Next, we make use of Theorem 2 and Corollary 3; then, an easy calculation shows that the integrand on the right-hand side can be bounded by $C(t+1)^{-\mu}$ where $\mu = kn/2 + n/2 + 1/2 - n/2(1/p + 1/q) = n/2(k+1) + 1/2 - n/2(1/r) > n/2 + 1/2 \geq 1$, so that

$$\mathrm{II}(t) \leq -\int_{T_0}^{t} (1+s)^{-\mu}\,ds \leq C(1+T_0)^{-\mu+1}. \tag{33}$$

Now choose T_0 so large that $C(1+T_0)^{-\mu+1} \leq \varepsilon/2$. Thus, for all $t > T_0$ it follows that

$$\mathrm{II}(t) \leq \varepsilon/2. \tag{34}$$

To estimate $\mathrm{I}(t)$, we proceed as follows. First note that the hypotheses $u_0 \in \mathscr{H}^1$ and $\int_{\mathbb{R}^n} u_0\,dx = 0$ yield that $u(T_0) \in \mathscr{H}^1$ and the average $\int_{\mathbb{R}^n} u(x,T)\,dx = 0$. Thus, just as in the Lemma 7, we deduce with e^{-Bt} instead of e^{-At} that

$$\lim_{t\to\infty} \|e^{-(t-T)B}u(\cdot,T)\|_{\mathscr{H}^1} = 0. \tag{35}$$

Specifically, the last limit follows as in Miyakawa's proof of Lemma 7, replacing u_0 by $u(.,T)$ and using the semigroup corresponding to the shifted heat semigroup; see (8). We also use that $u(\cdot,T) \in \mathscr{H}^1$. Let $w = t - T$; since if t tends to infinity so does $t - T$, it follows that

$$\lim_{w\to\infty} \|e^{-Aw}u(\cdot\,,T)\|_{\mathscr{H}^1} = 0.$$

Thus we choose T_0 so that for $t \geq T_0$ (i.e. for $w + T \geq T_0$)

$$\mathrm{I} \leq \varepsilon/2. \tag{36}$$

Hence $\mathrm{I} + \mathrm{II} \leq \varepsilon$, and the proof is complete. □

The next theorem establishes, for data described in (24), that the L^1 norm of the solution decays in the same way as the solution to the heat equation starting with the same initial datum. As mentioned in the introduction, we have only been able to show this result for the scalar equation, and not for systems.

Theorem 11. *Let u_0 be as in* (24) *and suppose that it belongs to H^m, with $m > [n/2]$, $n \geq 2$. Let f_j be as in the last theorem. If u is an L^1 solution with data u_0, then*

$$\lim_{t \to \infty} \|u(\cdot, t)\|_1 = 0.$$

If, in addition to (24), *we assume that*

$$\int |x|^\beta |u_0(x)| \, \mathrm{d}x < \infty \quad \textit{for some } 0 < \beta \leq 1,$$

then we have the estimate

$$\|u(\cdot, t)\|_1 \leq Ct^{-\beta/2} \quad \textit{for all } t > 0.$$

Proof. As before, the solution can be expressed in the integral form (41) and thus

$$\|u(x,t)\|_1 \leq \|\mathrm{e}^{-Bt} u_0\|_1 + \int_0^{t/2} \|\mathrm{e}^{-(t-s)B} g_j'(u) u_{x_j}\|_1 \, \mathrm{d}s$$
$$+ \int_{t/2}^t \|\mathrm{e}^{-(t-s)B} g_j'(u) u_{x_j}\|_1 \, \mathrm{d}s = J_1(t) + J_2(t) + J_3(t). \tag{37}$$

Note that by Lemma 9 it follows that $J_1(t)$ decays to zero. Hence, we analyse $J_2(t)$ and $J_3(t)$ only. The bounds on the flux term and the decay of the L^p norm of the solution yield

$$\begin{aligned} J_2(t) &\leq K_0 C_0 (t/2)^{-n/2} \int_0^{t/2} \|u(s)^k\|_2 \|\nabla_u(s)\|_2 \, \mathrm{d}s \\ &\leq K_0 C_0 (t/2)^{-n/2} \int_0^{t/2} (1+s)^{-\beta} \, \mathrm{d}s \\ &\leq C_* (t/2)^{-\beta+1-n/2}, \end{aligned} \tag{38}$$

where $\beta + n/2 = kn/2(1-1/2k) + n/4 + 1/2 + n/2 = kn/2 + n/4 + kn/2 + n/4 + 1/2 > 1$. Thus the decay of J_2 follows. Finally, for $J_3(t)$ we have, using Young's and Hölder inequalities, that

$$\begin{aligned} J_3(t) &\leq \int_{t/2}^t \|u(s)\|_k^k \|\nabla_u(s)\|_\infty \|\mathrm{e}^{-B(t-s)}\|_{L^1 \to L^1} \, \mathrm{d}s \\ &\leq C_*(t+1)^{-nk/2+1/2}. \end{aligned} \tag{39}$$

Here we used that the L^1 norm of the heat kernel is 1 and we also used the respective decays of the L^p norm of the solution and the L^∞ norm of the gradient of the solution. These decay results are formulated in Theorem 2 and Corollary 3. Hence $J_3(t)$ decays since $k > 1$ and $n \geq 2$. The second part of the Theorem is proved in the same way, using the second part of Lemma 7. □

4. Decay of $\mathscr{H}^1$ norms of systems of parabolic conservation laws

We note briefly that the proofs we have given for the scalar parabolic conservation law in the last two sections, with the exception of Theorem 11, where L^1 decay was obtained, can be extended to solution of systems satisfying the conditions of Theorem 4. In the case we have to replace the decay results for the L^p norms and H^m norms obtained in [6,7], by the decay results given in [2], and these are only valid for small data. If such decay results were available for large data then the $\mathscr{H}^1$ norm could also be proved to decay for such systems using the method presented here. An important difference between the scalar case and systems is that for the scalar case it is known that the L^1 norm is bounded for all time provided that the L^1 norm of the initial datum is finite. On the other hand, in order to show that the L^1 norm of the solution to a system is bounded we seem to be able to proceed only by showing that the $\mathscr{H}^1$ norm is bounded. Similarly, to show decay for systems we seem to require decay in the $\mathscr{H}^1$ norm.

To obtain existence and decay in $\mathscr{H}^1$ we need a basic bound for the flux term, just as in the scalar case. We shall state the corresponding theorem and give only a sketch of the proof since it is very similar to that in the scalar case.

Theorem 12. *Let u_0 and F_j be as in Theorem 4. Let $n \geq 2$ and $r \geq 1$. Assume that the flux vector $F = (F_1, \ldots, F_n)$ has components F_j such that $|\nabla F_j| \leq C|u|^k$, $j = 1, \ldots, n$, for some $k \geq 1$ and a positive constant C. Let us consider $p \geq 1$ and $q \geq 1$ such that $1/r = 1/p + 1/q$. Then*

$$\left\| \frac{\partial}{\partial x_i} F_i(u) \right\|_{\mathscr{H}^r} \leq C \|\nabla u\|_p \|\nabla u\|_q, \tag{40}$$

where $C > 0$ is a constant independent of u, p and q; as usual, repeated indices mean summation over these indices.

As in the scalar case, the proof follows by Young's inequality applied to each individual component $(\partial/\partial x_i)F_i(u)$, $i = 1, \ldots, n$.

In the remainder of the section we proceed to show the existence of solutions and their decay for systems of parabolic conservation laws. The proofs are based on the flux estimate given in Theorem 12.

Theorem 13. *Let u_0 and F be as in Theorem 4. In addition, let $u_0 \in \mathscr{H}^1$ and let the flux function F be as in Theorem 12; then the solution $u(x,t)$ constructed in Theorem 4 belongs also to $\mathscr{H}^1$.*

Proof. The proof is, with the appropriate modifications, the same as in the scalar case, except that we replace the scalar flux bound theorem with Theorem 12. □

Theorem 14. *Let u_0 and F be as in last theorem let $u_0 \in \mathscr{H}^1$, with average zero. In addition, let the flux function F be as in Theorem 12; then the solution $u(x,t)$ constructed in Theorem 4 decays to zero in $\mathscr{H}^1$.*

Proof. The proof, with minor modifications, proceeds along the same lines as in the scalar case. As in the last theorem, we replace the scalar-flux bounds from Lemma 5 with the bounds for the flux for systems obtained in Lemma 12. □

Given the $\mathscr{H}^1$ bounds, decay in $\mathscr{H}^1$ follows as for scalar laws.

5. Decay of the total variation for parabolic conservation laws

In this section we study the decay of the total variation for solutions to (3). We shall work with smooth solutions; that is, we shall suppose that the initial datum is smooth. To show the decay of the total variation we proceed by estimating the terms in the integral representation of the solution.

Theorem 15. *Suppose that $u_0 \in L^1 \cap H^m(\mathbb{R}^n)$ and let $f_j \in C^{m+1}$ with $m > [n/2]$. Suppose that there exists a constant vector $c = (c_1, \ldots, c_n)$ such that $f_j - c_j$ satisfies $A1_k$, with $k > 1$. Let $u(x,t)$ be a solution to (2) corresponding to the initial datum u_0. Then*

$$\int_{\mathbb{R}^n} |\nabla u(x,t)|\,\mathrm{d}x \leq C_0 t^{-1/2},$$

where C_0 depends only on norms of u_0.

Proof. By virtue of Duhamel's principle, if we let $f_j' = c_j + g_j'$, it follows that

$$\begin{aligned} u(x,t) = & \int_{\mathbb{R}^n} u(y,t/2) \frac{\mathrm{e}^{-(x+c-y)^2/2t}}{(2t\pi)^{n/2}}\,\mathrm{d}y \\ & - \int_{t/2}^{t} \int_{\mathbb{R}^n} g_i(u)_{x_i} \frac{\mathrm{e}^{-(x+c-y)^2/4(t-s)}}{(4(t-s)\pi)^{n/2}}\,\mathrm{d}y\,\mathrm{d}s. \end{aligned} \tag{41}$$

Since we are working with smooth solutions, the derivative can be passed inside the integrand. Thus,

$$\begin{aligned} \partial_j u(x,t) = & - \int_{\mathbb{R}^n} u(y,t/2)\, \frac{x_j + c_j - y_j}{2t}\, \frac{\mathrm{e}^{-(x+c-y)^2/2t}}{(2t\pi)^{n/2}}\,\mathrm{d}y \\ & - \int_{t/2}^{t} \int_{\mathbb{R}^n} g_i(u)_{x_i} \frac{x_j - y_j}{4(t-s)}\, \frac{\mathrm{e}^{-(x+c-y)^2/4(t-s)}}{(4(t-s)\pi)^{n/2}}\,\mathrm{d}y\,\mathrm{d}s = I_1(t) + I_2(t). \end{aligned} \tag{42}$$

Let us define, as before,

$$\mathscr{K}(x,t) = \frac{\mathrm{e}^{-|x-ct|^2/4t}}{(4t\pi)^{n/2}}. \tag{43}$$

Now we proceed to bound the terms I_1 and I_2:

$$I_1 \leq \frac{C}{t^{1/2}} \int_{\mathbb{R}^n} |u(y,t/2)| \mathscr{K}(x-y,t/2)\,\mathrm{d}y \tag{44}$$

and

$$I_2 \le \int_{t/2}^{t} \frac{C}{(t-s)^{1/2}} \int_{\mathbb{R}^n} |g_i(u)_{x_i}| \mathscr{K}(x-y,t-s)\,\mathrm{d}y\,\mathrm{d}s. \tag{45}$$

Thus, integrating $|\partial_j u|$ yields

$$\begin{aligned}\int_{\mathbb{R}^n} |\partial_j u|\,\mathrm{d}x \le{}& \frac{C}{t^{1/2}} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} u(y,t/2)\mathscr{K}(x-y,t)\,\mathrm{d}y\,\mathrm{d}x \\ &+ \int_{\mathbb{R}^n}\int_{t/2}^{t} \frac{C}{(t-s)^{1/2}} \int_{\mathbb{R}^n} |g_i(u)_{x_i}|\mathscr{K}(x-y,t-s)\,\mathrm{d}y\,\mathrm{d}s\,\mathrm{d}x \\ ={}& A_1 + A_2.\end{aligned}$$

Next we bound A_1 and A_2. Since the heat kernel integrates to one,

$$A_1 \le \frac{C}{t^{1/2}} \int_{\mathbb{R}^n} |u(y,t/2)|\,\mathrm{d}y \le \frac{C}{t^{1/2}} \int_{\mathbb{R}^n} |u_0(y)|\,\mathrm{d}y. \tag{46}$$

The last inequality is a consequence of the well-known fact that the L^1 norm of a solution to a scalar parabolic conservation laws is bounded provided that the L^1 norm of the initial datum is bounded. To bound A_2, we change the order of integration and integrate the heat kernel to one. Hölder's inequality yields

$$A_2 \le \int_{t/2}^{t} \frac{C}{(t-s)^{1/2}} \int_{\mathbb{R}^n} |g_i(u)_{x_i}|\,\mathrm{d}y \int_{\mathbb{R}^n} \mathscr{K}(x-y,t-s)\,\mathrm{d}x\,\mathrm{d}s.$$

Thus,

$$A_2 \le \int_{t/2}^{t} \frac{C}{(t-s)^{1/2}} \left(\int_{\mathbb{R}^n} |u|^{2k}\,\mathrm{d}y\right)^{1/2} \left(\int_{\mathbb{R}^n} |\nabla u|^2\,\mathrm{d}y\right)^{1/2} \mathrm{d}s.$$

Now, we use the decay results on the L^p norm and the L^2 norm of the gradient to complete the bound on A_2:

$$A_2 \le C(t/2+1)^{-\alpha} \int_{t/2}^{t} \frac{1}{(t-s)^{1/2}}\,\mathrm{d}s \le Ct^{1/2}(t/2+1)^{-\alpha}$$

with $\alpha = kn/2 + 1/2$. Hence A_2 decays at the rate of $t^{-nk/2}$, since $p > 1$ here. □

Note, finally, that for systems satisfying the conditions of Theorem 4 the same result holds as in the scalar case. Thus, we have the following theorem.

Theorem 16. *Let u_0 satisfy the conditions of Theorem* 4 *and let $u_0 \in \mathscr{H}^1 \cap H^m(\mathbb{R}^n)$ with $m > [n/2]$. Let $|\nabla F_j(u)| \le C|u|^k,\ j = 1,\dots,n$, where C is a positive constant and $k > 1$. Suppose that $u(x,t)$ is a solution to* (3) *with data u_0. Then*

$$\int_{\mathbb{R}^n} |\nabla u(x,t)|\,\mathrm{d}x \le C_0 t^{-1/2},$$

where C_0 depends only on norms of the data.

Proof. The proof is the same as for scalar parabolic conservation laws, replacing the necessary results from Theorems 1 and 2 by those in Theorem 4. The only step which is different from the scalar case is to note that, with our hypothesis, $u(T)$ is in $\mathscr{H}^1$ and thus in L^1.

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