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Nonexistence of singular pseudo-self-similar solutions of the Navier–Stokes system

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Abstract. We show that there are no singular pseudo-self-similar solutions of the Navier-Stokes system with finite energy.

1 Introduction

In his 1934 pioneering paper, Jean Leray [1] asked whether it is possible to construct a self-similar solution to the Navier-Stokes system in $\mathbf{R}^3$

$$\frac{\partial \mathbf{u}}{\partial t} - \triangle \mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p = 0, \tag{1}$$

$$\operatorname{div} \mathbf{u} = 0 \tag{2}$$

of the form

$$\mathbf{u}(x,t) = \frac{1}{\sqrt{T-t}}\mathbf{U}\left(\frac{x}{\sqrt{T-t}}\right), \tag{3}$$

$$p(x,t) = \frac{1}{T-t}P\left(\frac{x}{\sqrt{T-t}}\right). \tag{4}$$

The motivation for studying such of solutions is that they would possess a singularity when $t = T$; indeed $\|\nabla \mathbf{u}(\cdot,t)\|_{L_2(\mathbf{R}^3)} = \frac{1}{\sqrt{T-t}}\|\nabla \mathbf{U}\|_{L_2(\mathbf{R}^3)}$. This question was first answered in 1996 by Nečas, Růžička, and Šverák in the negative. Specifically, in [3], they showed that the only self-similar solution with $\mathbf{U} \in L_3(\mathbf{R}^3) \cap W^1_{2,loc}(\mathbf{R}^3)$ is the trivial solution. Later, Málek, Nečas, Pokorný, and Schonbek [2] showed that any self-similar solution with $\mathbf{U} \in W^1_2(\mathbf{R}^3)$ was

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trivial, and this was extended to solutions that merely have locally finite energy by Tsai in [5,6].

In [2], Nečas posed an extension of the original problem of Leray, namely could we construct pseudo-self-similar solutions of the Navier-Stokes system of the form

$$\mathbf{u}(x,t) = \mu(t)\mathbf{U}(\lambda(t)x), \tag{5}$$

$$p(x,t) = \mu^2(t)P(\lambda(t)x), \tag{6}$$

for all $t < T$ and some $T > 0$ where $\lambda, \mu \in C^1[0,T)$. Like the self-similar solutions, it was hoped that pseudo-self-similar solutions would provide an example of a singular solution to the Navier-Stokes system. In that paper [2] Málek, Nečas, Pokorný, and Schonbek were only able to give a partial answer to this problem. They showed that if there was a constant c so that $\lambda = c\mu$, then the problem could be reduced to the self-similar case, and hence $\mathbf{u} = 0$. Further, possibly singular solutions for which

$$\lambda(t) = (T-t)^{-\gamma_1} \qquad \mu(t) = (T-t)^{-\gamma_2} \tag{7}$$

were also shown to be of the Leray type, so that $\gamma_1 = \gamma_2 = 1/2$, and hence were trivial. On the other hand, for general λ and μ it was only shown that if such solutions were to exist, then they had a very specific form in frequency space, namely that

$$\hat{\mathbf{U}}(\xi) = |\xi|^{-\frac{\beta}{c_2}} \mathrm{e}^{-\frac{|\xi|^2}{2c_2}} \mathbf{S}\left(\frac{\xi}{|\xi|}\right) \tag{8}$$

for some function $\mathbf{S}$ and some constants β and c_2.

In this paper we close the question by showing that there are no singular pseudo-self-similar solutions of the Navier-Stokes system with finite energy. In particular, we shall prove the following.

Theorem 1 *There are no pseudo-self-similar solutions of the Navier-Stokes system that satisfy*

$$\operatorname*{ess\,sup}_{0<t<T} \|\mathbf{u}(\cdot,t)\|_{L_2(\mathbf{R}^3)} < \infty, \tag{9}$$

$$\|\nabla\mathbf{u}\|_{L_2(\mathbf{R}^3\times(0,T))} < \infty, \tag{10}$$

$$\lim_{t\uparrow T} \|\nabla\mathbf{u}(\cdot,t)\|_{L_2(\mathbf{R}^3)} = \infty \tag{11}$$

for any T.

2 Proof

Following [2], we can substitute (5) and (6) into (1) and (2) to obtain

$$\frac{\mu'}{\mu^2\lambda}\mathbf{U} + \frac{\lambda'}{\mu\lambda^2}(y\cdot\nabla)\mathbf{U} - \frac{\lambda}{\mu}\triangle\mathbf{U} + (\mathbf{U}\cdot\nabla)\mathbf{U} + \nabla P = 0, \tag{12}$$

$$\operatorname{div}\mathbf{U} = 0. \tag{13}$$

The conditions (9)–(10) then imply that $\mathbf{U} \in W_2^1(\mathbf{R}^3)$. An ordinary differential equation for λ and μ can be found by multiplying (12) by $\mathbf{U}$ and integrating to obtain

$$\frac{\mu'}{\mu^2\lambda} - \frac{3}{2}\frac{\lambda'}{\mu\lambda^2} = -\frac{\lambda}{\mu}K_3 \tag{14}$$

where $K_3 = \|\nabla\mathbf{U}\|_2^2/\|\mathbf{U}\|_2^2 > 0$. (The notation here and elsewhere is chosen to be consistent with [2].)

Further it was shown in [2, Lemmas 2.1 & 2.2] that $\mathbf{U} \in W_2^2(\mathbf{R}^3) \cap L_\infty(\mathbf{R}^3)$ and $P \in W_2^2(\mathbf{R}^3) \cap L_\infty(\mathbf{R}^3)$. It was also shown that the requirement $\lambda \neq c\mu$ implies that

$$\frac{\lambda}{\mu} + \frac{\lambda'}{\mu\lambda^2}\frac{1}{c_2} = K_2 \tag{15}$$

for some $c_2 > 0$ and some K_2. It was already noted in [2] that if $K_2 = 0$ then the solution is nonsingular, so we shall reserve our primary attention for the case $K_2 \neq 0$.

Next we shall take advantage of the symmetry of the problem. Indeed, note that if $\mathbf{U}$, P, λ and μ satisfy (12), (13), (14) and (15), then so does

$$\tilde{\mu} = -\mu, \quad \tilde{K}_2 = -K_2, \quad \tilde{\mathbf{U}} = -\mathbf{U}, \quad \tilde{P} = -P, \tag{16}$$

$$\tilde{\lambda} = \lambda, \qquad \tilde{K}_3 = K_3, \qquad \tilde{c}_2 = c_2. \tag{17}$$

As a consequence, we can assume without loss of generality that $K_2 > 0$.

We can then substitute (14) into (12) to obtain

$$\frac{\lambda'}{\mu\lambda^2}\left[(y\cdot\nabla)\mathbf{U} + \frac{3}{2}\mathbf{U}\right] - \frac{\lambda}{\mu}\left[\triangle\mathbf{U} + K_3\mathbf{U}\right] + (\mathbf{U}\cdot\nabla)\mathbf{U} + \nabla P = 0. \tag{18}$$

Next, use (15) to substitute for the λ/μ factor to obtain

$$\frac{\lambda'}{\mu\lambda^2}\left[(y\cdot\nabla)\mathbf{U} + \frac{3}{2}\mathbf{U}\right] - \left(K_2 - \frac{1}{c_2}\frac{\lambda'}{\mu\lambda^2}\right)\left[\triangle\mathbf{U} + K_3\mathbf{U}\right] + (\mathbf{U}\cdot\nabla)\mathbf{U} + \nabla P = 0. \tag{19}$$

Combining like terms, we find that

$$\begin{aligned}
&-K_2(\triangle\mathbf{U}+K_3\mathbf{U})+(\mathbf{U}\cdot\nabla)\mathbf{U}+\nabla P\\
&\qquad=\left[-\frac{1}{c_2}(\triangle\mathbf{U}+K_3\mathbf{U})-(y\cdot\nabla)\mathbf{U}-\frac{3}{2}\mathbf{U}\right]\frac{\lambda'}{\mu\lambda^2}
\end{aligned}\tag{20}$$

Since the left side is independent of t, we know that the right side must be constant in t; thus either the first factor is zero or $\lambda'/\mu\lambda^2$ is a constant in time. The latter case is disallowed because (15) would imply that λ/μ is constant. Since the first factor is zero, the whole right side is zero and we have the equations

$$-K_2(\triangle\mathbf{U}+K_3\mathbf{U})+(\mathbf{U}\cdot\nabla)\mathbf{U}+\nabla P=0\tag{21}$$

and

$$-\frac{1}{c_2}(\triangle\mathbf{U}+K_3\mathbf{U})-(y\cdot\nabla)\mathbf{U}-\frac{3}{2}\mathbf{U}=0.\tag{22}$$

We remark that if we make the substitution $K_3=-\beta+(3/2)c_2$ and then take the Fourier transform of the second equation, we obtain

$$-|\xi|^2\hat{\mathbf{U}}+(3c_2-\beta)\hat{\mathbf{U}}+c_2\left(-|\xi|\frac{\partial}{\partial|\xi|}\hat{\mathbf{U}}-3\hat{\mathbf{U}}\right)=0.\tag{23}$$

If we solve the resulting ordinary differential equation for the radial part of $\hat{\mathbf{U}}$, we obtain (8).

Let $a\in\mathbf{R}$ be determined later and set

$$\tilde{\mathbf{U}}=\mathbf{U}+ay,\tag{24}$$

$$\tilde{P}=P-\tfrac{1}{2}a^2|y|^2.\tag{25}$$

Substitute this into (21) to obtain the equation

$$-K_2\triangle\tilde{\mathbf{U}}+(\tilde{\mathbf{U}}\cdot\nabla)\tilde{\mathbf{U}}-a(y\cdot\nabla)\mathbf{U}+\nabla\tilde{P}=K_2K_3\mathbf{U}+a\mathbf{U}.\tag{26}$$

Then use (22) to substitute for $(y\cdot\nabla)\mathbf{U}$, giving us

$$-\left(K_2-\frac{a}{c_2}\right)\triangle\tilde{\mathbf{U}}+(\tilde{\mathbf{U}}\cdot\nabla)\tilde{\mathbf{U}}+\nabla\tilde{P}=\left[K_2K_3-a\left(\frac{1}{2}+\frac{K_3}{c_2}\right)\right]\mathbf{U}.\tag{27}$$

Set

$$a=\frac{K_2K_3}{\frac{1}{2}+\frac{K_3}{c_2}}=K_2c_2\frac{2K_3}{c_2+2K_3}\tag{28}$$

and

$$\nu=K_2-\frac{a}{c_2}=K_2\left(1-\frac{2K_3}{c_2+2K_3}\right)=\frac{c_2K_2}{2K_3+c_2}.\tag{29}$$

Our restrictions on c_2, K_2, and K_3 imply that $\nu > 0$; hence

$$-\nu\triangle\tilde{\mathbf{U}} + (\tilde{\mathbf{U}}\cdot\nabla)\tilde{\mathbf{U}} + \nabla\tilde{P} = 0, \tag{30}$$

$$\operatorname{div}\tilde{\mathbf{U}} = 3a. \tag{31}$$

We can multiply (30) by $\tilde{\mathbf{U}}$ to obtain

$$-\nu\triangle(\tfrac{1}{2}|\tilde{\mathbf{U}}|^2) + (\tilde{\mathbf{U}}\cdot\nabla)(\tfrac{1}{2}|\tilde{\mathbf{U}}|^2) + (\tilde{\mathbf{U}}\cdot\nabla)\tilde{P} + \nu|\nabla\tilde{\mathbf{U}}|^2 = 0. \tag{32}$$

On the other hand, if we take the divergence of (30), we find

$$-\nu\triangle(\operatorname{div}\tilde{\mathbf{U}}) + \frac{\partial\tilde{U}_i}{\partial y_j}\frac{\partial\tilde{U}_j}{\partial y_i} + \tilde{U}_j\frac{\partial}{\partial y_j}(\operatorname{div}\tilde{\mathbf{U}}) + \triangle\tilde{P} = 0; \tag{33}$$

then since $\operatorname{div}\tilde{\mathbf{U}} = 3a$ is constant,

$$\triangle\tilde{P} = -\frac{\partial\tilde{U}_i}{\partial y_j}\frac{\partial\tilde{U}_j}{\partial y_i}. \tag{34}$$

Substitute this into (32) to obtain

$$-\nu\triangle(\tfrac{1}{2}|\tilde{\mathbf{U}}|^2 + \tilde{P}) + (\tilde{\mathbf{U}}\cdot\nabla)(\tfrac{1}{2}|\tilde{\mathbf{U}}|^2 + \tilde{P}) + \nu\left(|\nabla\tilde{\mathbf{U}}|^2 - \frac{\partial\tilde{U}_i}{\partial y_j}\frac{\partial\tilde{U}_j}{\partial y_i}\right) = 0. \tag{35}$$

If we define

$$\begin{aligned} X &= \tfrac{1}{2}|\tilde{\mathbf{U}}|^2 + \tilde{P} \\ &= \tfrac{1}{2}(\mathbf{U} + ay)\cdot(\mathbf{U} + ay) + P - \tfrac{1}{2}a^2|y|^2 \\ &= \tfrac{1}{2}|\mathbf{U}|^2 + P + a(\mathbf{U}\cdot y) \end{aligned} \tag{36}$$

then we find

$$-\nu\triangle X + (\tilde{\mathbf{U}}\cdot\nabla)X + \nu\left(|\nabla\tilde{\mathbf{U}}|^2 - \frac{\partial\tilde{U}_i}{\partial y_j}\frac{\partial\tilde{U}_j}{\partial y_i}\right) = 0. \tag{37}$$

Next we would like to replace $\tilde{\mathbf{U}}$ by $\mathbf{U}$. We note that

$$|\nabla\tilde{\mathbf{U}}|^2 = |\nabla\mathbf{U}|^2 + 2a\operatorname{div}\mathbf{U} + a^2\delta_{ij}\delta_{ij} = |\nabla\mathbf{U}|^2 + 3a^2 \tag{38}$$

while

$$\frac{\partial\tilde{U}_i}{\partial y_j}\frac{\partial\tilde{U}_j}{\partial y_i} = \frac{\partial U_i}{\partial y_j}\frac{\partial U_j}{\partial y_i} + 2a\operatorname{div}\mathbf{U} + 3a^2 = \frac{\partial U_i}{\partial y_j}\frac{\partial U_j}{\partial y_i} + 3a^2. \tag{39}$$

Making the substitutions, we find that

$$-\nu\triangle X + (\mathbf{U}\cdot\nabla)X + a(y\cdot\nabla)X + \nu\left(|\nabla\mathbf{U}|^2 - \frac{\partial U_i}{\partial y_j}\frac{\partial U_j}{\partial y_i}\right) = 0. \tag{40}$$

In [2, Lemma 3.2] the following was proven.

Lemma 2 *Let $a > 0$, $\nu > 0$, and suppose that*

$$-\nu\triangle X + (\mathbf{U}\cdot\nabla)X + a(y\cdot\nabla)X \leq 0. \tag{41}$$

Then either $X \leq 0$ or X is a positive constant.

For the reader's convenience, we shall sketch the proof. For $\beta > 0$, define $X_\beta = X\mathrm{e}^{-\beta|y|^2}$. Then

$$-\nu\triangle X_\beta + b_j(y)\frac{\partial X_\beta}{\partial y_j} + b(y)X_\beta \leq 0 \tag{42}$$

where $b_j(y) = U_j(y) + (a - 4\beta\nu)y_j$ and

$$b(y) = 2\beta(a|y|^2 - 2\beta\nu|y|^2 + \mathbf{U}\cdot y - 3\nu). \tag{43}$$

We can find $\beta_o > 0$ so that $b(y) > 0$ if $0 < \beta < \beta_o$ and $|y| \geq R$; choose such a pair. Let $M = \max_{|y|=R} X$ and let us first suppose that $M > 0$. Because $\mathbf{U}$ and P are bounded, there exists some $R_\beta > R$ so that $X_\beta(y) < M/2$ for all $|y| > R_\beta$. Applying the maximum principle to (42) on annuli, we conclude that $X_\beta \leq M\mathrm{e}^{-\beta R^2}$ if $|y| \geq R$. Letting $\beta \downarrow 0$ we see that $X \leq M$ if $|y| \geq R$. Apply the strong maximum principle for (41) on B_ρ for $\rho > R$; since the maximum is achieved in B_ρ on $|y| = R$, we conclude that X is constant in B_ρ for all $\rho > R$.

Suppose that $M \leq 0$. The boundedness of $\mathbf{U}$ and P imply that for all $\epsilon > 0$ there is some $R_\epsilon > R$ so that $X_\beta(y) < \epsilon$ if $|y| > R_\epsilon$. Applying the maximum principle for (42) on annuli we conclude that $X_\beta \leq \epsilon$ if $|y| > R$ and since ϵ is arbitrary, that $X \leq 0$ if $|y| > R$. Apply the maximum principle once more on B_ρ for $\rho > R$ to conclude that $X \leq 0$. This proves the lemma.

We can strengthen Lemma 1 as follows. If we set

$$X^* = X + c \tag{44}$$

for some constant c, we see that X^* also satisfies (41). Repeating the previous argument for X^* we find that either $X + c \leq 0$ for all constants c, or X is constant; we conclude that X is constant.

Because the last term in (40) is nonnegative we can apply this result to conclude that X is constant. As a consequence, (40) reduces to the equation

$$|\nabla\mathbf{U}|^2 = \frac{\partial U_i}{\partial y_j}\frac{\partial U_j}{\partial y_i}. \tag{45}$$

Integrate this over $\mathbf{R}^3$ to see that

$$\int_{\mathbf{R}^3} |\nabla\mathbf{U}|^2 = -\int_{\mathbf{R}^3} U_i\frac{\partial}{\partial y_j}\frac{\partial U_i}{\partial y_j} = -\int_{\mathbf{R}^3} U_i\frac{\partial}{\partial y_i}\operatorname{div}\mathbf{U} = 0; \tag{46}$$

thus $\mathbf{U}$ is a constant. Since $\mathbf{U} \in L_2(\mathbf{R}^3)$, we conclude that $\mathbf{U} = 0$.

3 Nontrivial pseudo-self-similar solutions

The preceding argument did more than show that there are no singular pseudo-self-similar solutions of the Navier-Stokes system. In fact, it shows that every pseudo-self-similar solution with finite energy is trivial, at least in the case where $K_2 \neq 0$. It was already shown in [2] that if $K_2 = 0$ then the solution is nonsingular; we shall now present some additional remarks about what occurs in this case.

If $K_2 = 0$ we can solve (15) directly for $\lambda(t)$ to determine that

$$\lambda(t) = \frac{\lambda_o}{\sqrt{1 + 2\lambda_o^2 c_2 t}} \tag{47}$$

for some arbitrary constant λ_o. We can then use (14) to see that

$$\mu(t) = \frac{\mu_o}{(1 + 2\lambda_o^2 c_2 t)^{\frac{3}{2} + \frac{K_3}{c_2}}} \tag{48}$$

where μ_o is also arbitrary.

The question of whether or not there exist nontrivial pseudo-self-similar solutions with finite energy in this form is still open. We remark that if $\mathbf{u}(x, t)$ and $p(x, t)$ are pseudo-self-similar solutions in this form, then so is $k^\alpha \mathbf{u}(kx, k^2 t)$ and $k^{2\alpha} p(kx, k^2 t)$ for any α and k. Using this scaling in the Navier-Stokes system then implies that $\mathbf{u}$ and p must satisfy

$$(\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p = 0, \tag{49}$$

and

$$\mathbf{u}_t - \triangle \mathbf{u} = 0. \tag{50}$$

Note that these are equivalent to (21) and (22) respectively with $K_2 = 0$. Finally, we remark that it is known that (49) and (50) have nontirival solutions in an even number of spatial dimensions; see [4, Theorem 5.1].

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