On optimal decay rates for weak solutions to the Navier-Stokes equations in $\mathbb{R}^n$

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Dedicated to Professor Jindrich Necas on his 70th birthday

1. Introduction and the results

Consider the Navier-Stokes equations in $\mathbb{R}^n$, $n \geq 2$, which will be treated in this paper in the form of the integral equation

$$u(t) = e^{-tA}a - \int_0^t \nabla \cdot e^{-(t-s)A} P(u \otimes u)(s)ds, \tag{NS}$$

for prescribed initial velocity $a(x) = (a_1(x), \cdots, a_n(x))$, $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$, and unknown velocity $u(x,t) = (u_1(x,t), \cdots, u_n(x,t))$. Here, $A = -\Delta$ is the Laplacian on $\mathbb{R}^n$; $\{e^{-tA}\}_{t\geq 0}$ is the heat semigroup ; $P = (P_{jk})$ is the bounded projection onto divergence-free vector fields ; $u \otimes v$ is the matrix with entries $(u \otimes v)_{jk} = u_j v_k$; $\nabla = (\partial_1, \cdots, \partial_n)$ with $\partial_j = \partial/\partial x_j$; and

$$(\nabla \cdot e^{-tA} P(u \otimes u))_j = \sum_{k,\ell=1}^{n} \partial_\ell e^{-tA} P_{jk}(u_\ell u_k), \qquad j = 1, \cdots, n.$$

It is well known that for each $a \in \boldsymbol{L}^2$ with $\nabla \cdot a = 0$, (NS) has a weak solution u defined for all $t \geq 0$, satisfying the energy inequality

$$\|u(t)\|_2^2 + 2\int_0^t \|\nabla u\|_2^2 ds \leq \|a\|_2^2 \qquad \text{for all } t \geq 0.$$

Hereafter $\|\cdot\|_r$ denotes the L^r-norm.

As shown in [10], there exists a weak solution u such that

$$\|u(t)\|_2 \leq C(1+t)^{-\frac{n+2}{4}}, \tag{1.1}$$

whenever

$$a \in \boldsymbol{L}^2, \qquad \nabla \cdot a = 0 \quad \text{and} \quad \int (1+|y|)|a(y)|dy < \infty. \tag{1.2}$$

Assumption (1.2) implies $a \in \boldsymbol{L}^1$; so the divergence-free condition gives (see [4])

$$\int a(y)dy = 0. \tag{1.3}$$

Furthermore, it is shown in [2] that in this case the solution u satisfies

$$(1.4) \qquad \lim_{t\to\infty} t^{\frac{n+2}{4}} \left\| u_j(t) + (\partial_k E_t)(\cdot)\int y_k a_j(y)dy + F_{\ell,jk}(\cdot,t)\int_0^\infty\!\!\int (u_\ell u_k)(y,s)dyds \right\|_2 = 0$$

for $j = 1, \cdots, n$, where

$$E_t(x) = (4\pi t)^{-n/2} e^{-|x|^2/4t}, \qquad F_{\ell,jk}(x,t) = \partial_\ell E_t(x)\delta_{jk} + \int_t^\infty \partial_\ell\partial_j\partial_k E_s(x)ds.$$

(Hereafter, we use the summation convention.) Equation (NS) is then written in the form

$$u_j(x,t) = \int E_t(x-y)a_j(y)dy - \int_0^t\!\!\int F_{\ell,jk}(x-y,t-s)(u_\ell u_k)(y,s)dyds, \qquad j = 1,\cdots,n,$$

as proved in [2]; and the integrals in (1.4) are finite, due to (1.1) and (1.2). Assertion (1.4) was first proved in [1] for smooth solutions when $n = 3$, and then extended in [2] to the case of weak solutions in all space dimensions by applying the spectral method as given in [3,5].

The argument of [10] suggests that the decay property (1.1) will be optimal in general. So we are interested in finding a class of weak solutions u satisfying the reverse estimate

$$\|u(t)\|_2 \geq Ct^{-\frac{n+2}{4}} \qquad \text{at least for large } t.$$

In this paper we discuss this kind of *lower bound problem.*

Theorem A. *Under the assumption* (1.2), *let*

$$b_{k\ell} = \int y_\ell a_k(y)dy, \qquad c_{k\ell} = \int_0^\infty\!\!\int (u_\ell u_k)(y,s)dyds.$$

(i) *We have*

$$(1.5) \qquad \lim_{t\to\infty} t^{\frac{n+2}{4}}\|u(t)\|_2 = 0$$

if and only if $(b_{k\ell}) = 0$ *and* $(c_{k\ell}) = (c\delta_{k\ell})$ *for some constant* $c \geq 0$.

(ii) *There exists* $c' > 0$ *such that*

$$(1.6) \qquad \|u(t)\|_2 \geq c't^{-\frac{n+2}{4}} \qquad \text{for large } t > 0,$$

if and only if $(b_{k\ell}) \neq 0$ *or* $(c_{k\ell}) \neq (c\delta_{k\ell})$. *In particular,* u *satisfies* (1.6) *whenever* $(b_{k\ell}) \neq 0$.

Remark. Theorem A (i) implies only that

$$(1.5') \qquad \limsup_{t\to\infty} t^{\frac{n+2}{4}}\|u(t)\|_2 > 0$$

if and only if $(b_{k\ell}) \neq 0$ or $(c_{k\ell}) \neq (c\delta_{k\ell})$. Note, however, that our second assertion (1.6) is more stringent than (1.5′). Moreover, (1.6) holds for *all* large $t > 0$ and for *all* space dimensions, although $\|u(t)\|_2$ is only known to be lower semicontinuous when $n \geq 5$. We know nothing about the characterization of solutions satisfying $(c_{k\ell}) = (c\delta_{k\ell})$.

We next consider weak solutions u satisfying

$$\|u(t)\|_2 \le C(1+t)^{-\frac{n}{4}}. \tag{1.7}$$

As shown in [3,6,10], such solutions exist for all $a \in \boldsymbol{L}^2$ satisfying

$$\nabla \cdot a = 0, \qquad \|e^{-tA}a\|_2 \le C(1+t)^{-\frac{n}{4}}. \tag{1.8}$$

Theorem B. *Suppose a satisfies* (1.8) *and let u be a weak solution satisfying* (1.7). *Then*

$$\|u(t)\|_2 \ge ct^{-\frac{n}{4}} \qquad \text{for large } t > 0, \tag{1.9}$$

if and only if

$$\|e^{-tA}a\|_2 \ge ct^{-\frac{n}{4}} \qquad \text{for large } t > 0. \tag{1.10}$$

The lemma below gives simple examples of a satisfying (1.10).

Lemma. *Let $a \in \boldsymbol{L}^2$, $\nabla \cdot a = 0$, and suppose that*

$$\int_{S^{n-1}} |\widehat{a}(r,\omega)|^2 d\omega \in L^\infty(\mathbb{R}_+), \qquad \liminf_{r\to 0} \int_{S^{n-1}} |\widehat{a}(r,\omega)|^2 d\omega > 0, \tag{1.11}$$

where the Fourier transform $\widehat{a}$ is defined by

$$\widehat{a}(\xi) = \int e^{-ix\cdot\xi} a(x)dx, \qquad i = \sqrt{-1},$$

S^{n-1} is the unit sphere of $\mathbb{R}^n$, and $\xi = (r,\omega)$ in polar coordinates. Then,

$$\|e^{-tA}a\|_2 \le C(1+t)^{-\frac{n}{4}} \quad \text{for all } t>0\,; \qquad \|e^{-tA}a\|_2 \ge c't^{-\frac{n}{4}} \quad \text{for large } t > 0, \tag{1.12}$$

with constants $C > 0$ and $c' > 0$ independent of t.

Proof. Parseval's relation gives

$$\|e^{-tA}a\|_2^2 = (2\pi)^{-n} \int e^{-2t|\xi|^2} |\widehat{a}(\xi)|^2 d\xi = (8\pi^2 t)^{-\frac{n}{2}} \int e^{-|\eta|^2} |\widehat{a}(\eta(2t)^{-\frac{1}{2}})|^2 d\eta$$

so that

$$(8\pi^2 t)^{\frac{n}{2}} \|e^{-tA}a\|_2^2 = \int e^{-|\eta|^2} |\widehat{a}(\eta(2t)^{-\frac{1}{2}})|^2 d\eta.$$

The assumption and Fatou's lemma together imply

$$\begin{aligned} \liminf_{t\to\infty} (8\pi^2 t)^{\frac{n}{2}} \|e^{-tA}a\|_2^2 &= \liminf_{t\to\infty} \int e^{-|\eta|^2} |\widehat{a}(\eta(2t)^{-\frac{1}{2}})|^2 d\eta \\ &\ge \int_0^\infty e^{-r^2} \left(\liminf_{t\to\infty} \int_{S^{n-1}} |\widehat{a}(r(2t)^{-\frac{1}{2}},\omega)|^2 d\omega \right) r^{n-1} dr > 0. \end{aligned}$$

This proves the second estimate of (1.12). The first estimate follows from $\|e^{-tA}a\|_2 \le \|a\|_2$ and

$$\|e^{-tA}a\|_2^2 = (8\pi^2 t)^{-\frac{n}{2}} \int e^{-|\eta|^2} |\widehat{a}(\eta(2t)^{-\frac{1}{2}})|^2 d\eta \le Ct^{-\frac{n}{2}} \left\| \int_{S^{n-1}} |\widehat{a}(\cdot,\omega)|^2 d\omega \right\|_\infty \int_0^\infty e^{-r^2} r^{n-1} dr.$$

The proof is complete.

Remarks. (i) Condition (1.11) implies that $\widehat{a}$ is discontinuous at $\xi = 0$. Indeed, since $\nabla \cdot a = 0$, we have $\xi \cdot \widehat{a}(\xi) = 0$; so if $\widehat{a}$ is continuous at $\xi = 0$, we get $\omega \cdot \widehat{a}(0) = 0$ for all unit vectors ω, and $\widehat{a}(0) = 0$. (For this reason, $a \in \boldsymbol{L}^1$ implies (1.3).)

(ii) The assumption of Lemma is not vacuous. Indeed, suppose $\widehat{a}$ is written in the form

$$\widehat{a}(\xi) = f(|\xi|)g(\xi/|\xi|),$$

in terms of functions f and g such that

$$g \in \boldsymbol{L}^2(S^{n-1}), \qquad g \not\equiv 0, \qquad \omega \cdot g(\omega) \equiv 0 \quad (\omega \in S^{n-1})$$

and

$$f \in BC([0,\infty)), \qquad \int_0^\infty |f(r)|^2 r^{n-1} dr < \infty, \qquad f(0) \neq 0.$$

Then, $\widehat{a}$ satisfies condition (1.11).

(iii) In this connection, we note that under condition (1.2) we have

$$\|e^{-tA}a\|_2 \geq ct^{-\frac{n+2}{4}} \qquad \text{for large } t > 0 \tag{1.10$'$}$$

if and only if $(b_{k\ell}) \neq 0$. Indeed, using (1.2) and (1.3), we have (see Section 4)

$$\lim_{t\to\infty} t^{\frac{n+2}{4}} \|e^{-tA}a_k + \partial_\ell E_t b_{k\ell}\|_2 = 0, \qquad k = 1, \cdots, n. \tag{1.4$'$}$$

Suppose $(b_{k\ell}) \neq 0$. Then $(\sum_k \|\partial_\ell E_t b_{k\ell}\|_2^2)^{1/2} = Ct^{-\frac{n+2}{4}}$ with $C > 0$; so we get

$$\|e^{-tA}a\|_2 \geq (\textstyle\sum_k \|\partial_\ell E_t b_{k\ell}\|_2^2)^{1/2} - \left(\textstyle\sum_k \|e^{-tA}a_k + \partial_\ell E_t b_{k\ell}\|_2^2\right)^{1/2} \geq ct^{-\frac{n+2}{4}}$$

for large $t > 0$. Conversely, if we assume (1.10$'$), then by (1.4$'$) we get

$$(\textstyle\sum_k \|\partial_\ell E_t b_{k\ell}\|_2^2)^{1/2} \geq \|e^{-tA}a\|_2 - \left(\textstyle\sum_k \|e^{-tA}a_k + \partial_\ell E_t b_{k\ell}\|_2^2\right)^{1/2} \geq ct^{-\frac{n+2}{4}}$$

for large $t > 0$. Hence $\sum_k \|\partial_\ell E_t b_{k\ell}\|_2^2 > 0$ for large $t > 0$, which implies $(b_{k\ell}) \neq 0$.

The L^2 decay problem for weak solutions of the Navier-Stokes equations was successfully studied for the first time by [5] and the result was then systematically developed by [3,10]. Estimates (1.6) and (1.9) are studied in [6,7,8] in case $n = 2, 3$, and some sufficient conditions are obtained. Our Theorems A and B provide *necessary and sufficient conditions* for those estimates to hold. We further note that our lower bound estimates (1.6) and (1.9) hold in all space dimensions $n \geq 2$, although the function $\|u(t)\|_2$ is known only to be lower semicontinuous when $n \geq 3$. As will be seen in the proof below, this is due to (1.4) and the fact that the functions $\partial_\ell E_t(x)$ and $F_{\ell,jk}(x,t)$ are written in the form $t^{-\frac{n+1}{2}}K(xt^{-\frac{1}{2}})$ in terms of some bounded, integrable and uniformly continuous functions K.

We finally consider an example of two-dimensional flows u with $(b_{k\ell}) = 0$, $(c_{k\ell}) = (c\delta_{k\ell})$, which was first treated by [7].

Theorem C. *When $n = 2$, there is a smooth weak solution u such that $(b_{k\ell}) = 0$, $(c_{k\ell}) = (c\delta_{k\ell})$, and, with some constant $\gamma > 0$,*

$$\|u(t)\|_q \le C_q e^{-\gamma t} \quad \text{and} \qquad |u(x,t)| \le C_m e^{-\gamma t}(1+|x|)^{-m} \tag{1.13}$$

for all $1 \le q \le \infty$ and all integers $m \ge 0$.

The above example was studied by [7,8], in which is given the exponential decay of $\|u(t)\|_q$ for $2 \le q \le \infty$. Our estimates (1.13) include the case $1 \le q < 2$ as well as the decay estimates in the spatial direction.

In what follows we prove Theorems A and B. A proof of Theorem C is given in [2] and so omitted here. We conclude the paper with the proof of (1.4) which was given also in [2].

2. Proof of Theorem A

We begin with the following

Proposition 2.1. *Let $(b_{k\ell})$ and $(c_{k\ell})$ be real $n \times n$ matrices and let $(c_{k\ell})$ be symmetric. Then*

$$b_{k\ell}\partial_\ell E_t(x)\delta_{jk} + c_{k\ell}F_{\ell,jk}(x,t) = 0, \qquad j = 1,\cdots,n, \tag{2.1}$$

for all $x \in \mathbb{R}^n$ and for some $t > 0$, if and only if

$$(b_{k\ell}) = 0 \quad \text{and} \quad (c_{k\ell}) = (c\delta_{k\ell}) \qquad \text{for some } c \in \mathbb{R}. \tag{2.2}$$

Furthermore, (2.2) implies that (2.1) holds for all x and for all $t > 0$.

Proof. Assumption (2.1) implies

$$b_{k\ell}\xi_\ell e^{-t|\xi|^2}\delta_{jk} = -c_{k\ell}\xi_\ell\left(e^{-t|\xi|^2}\delta_{jk} - \xi_j\xi_k\int_t^\infty e^{-s|\xi|^2}ds\right) = -(c_{j\ell} - |\xi|^{-2}c_{k\ell}\xi_j\xi_k)\xi_\ell e^{-t|\xi|^2}$$

for some $t > 0$, and we get $|\xi|^2(b_{j\ell} + c_{j\ell})\xi_\ell = \xi_j c_{k\ell}\xi_k\xi_\ell$. Taking $\xi_j = 0$ for any fixed j, $\xi_\ell = 1$ for any fixed $\ell \ne j$, and $\xi_k = 0$ for all k such that $k \ne j$ and $k \ne \ell$, we easily obtain $b_{j\ell} + c_{j\ell} = 0$ whenever $j \ne \ell$, and so

$$|\xi|^2(b_{jj} + c_{jj})\xi_j = \xi_j c_{k\ell}\xi_k\xi_\ell, \qquad j = 1,\cdots,n.$$

We let $\xi_j = 1$ and $\xi_k = 0$ for $k \ne j$, to get $b_{jj} + c_{jj} = c_{jj}$; so $b_{jj} = 0$. This implies

$$|\xi|^2 c_{jj}\xi_j = \xi_j c_{k\ell}\xi_k\xi_\ell, \qquad j = 1,\cdots,n. \tag{2.3}$$

Hence, $c_{11} = \cdots = c_{nn} = c_{k\ell}\xi_k\xi_\ell|\xi|^{-2}$. We then set $j = 1$, $\xi_1 = \xi_2 = 1$ and $\xi_k = 0$ for $k \ge 3$ in (2.3), to get $2c_{11} = c_{11}+c_{22}+c_{12}+c_{21} = 2(c_{11}+c_{12})$ since $c_{k\ell} = c_{\ell k}$ by assumption. Therefore, $c_{12} = 0$. We thus obtain $c_{j\ell} = 0 = -b_{j\ell}$ whenever $j \ne \ell$; so $(b_{k\ell}) = 0$ and $(c_{k\ell}) = (c\delta_{k\ell})$. That (2.2) implies (2.1) for *all* $t > 0$ is easily seen from

$$F_{k,jk} = \partial_j E_t + \int_t^\infty \partial_j \Delta E_s ds = \partial_j E_t + \int_t^\infty \partial_j\partial_s E_s ds = \partial_j E_t - \partial_j E_t = 0,$$

where $\partial_s = \partial/\partial s$. The proof of Proposition 2.1 is complete.

To establish Theorem A, it suffices in view of (1.4) to prove the following

Proposition 2.2. *Let a satisfy* (1.2) *and define*

$$b_{k\ell} = \int y_\ell a_k(y)dy, \qquad c_{k\ell} = \int_0^\infty \int (u_\ell u_k)(y,s)dyds.$$

Then we have

$$\text{either} \qquad (b_{k\ell}) \neq 0 \qquad \text{or} \qquad (c_{k\ell}) \neq (c\delta_{k\ell}), \tag{2.4}$$

if and only if a corresponding weak solution u satisfies

$$\|u(t)\|_2 \geq c' t^{-\frac{n+2}{4}} \qquad \text{for large } t > 0 \tag{2.5}$$

with a constant $c' > 0$ indenpendent of t.

Proof. In what follows we write

$$\boldsymbol{b}_\ell = (b_{1\ell}, \cdots, b_{n\ell}), \qquad \boldsymbol{F}_{\ell,k} = (F_{\ell,1k}, \cdots, F_{\ell,nk}).$$

Assume first (2.4). By Proposition 2.1, we have $\|\partial_\ell E_t \boldsymbol{b}_\ell + \boldsymbol{F}_{\ell,k} c_{k\ell}\|_2 = Ct^{-\frac{n+2}{4}}$ for all $t > 0$ with some $C > 0$, and so (1.4) implies

$$\begin{aligned} \|u(t)\|_2 &\geq \|\partial_\ell E_t \boldsymbol{b}_\ell + \boldsymbol{F}_{\ell,k} c_{k\ell}\|_2 - \|u(t) + \partial_\ell E_t \boldsymbol{b}_\ell + \boldsymbol{F}_{\ell,k} c_{k\ell}\|_2 \\ &= Ct^{-\frac{n+2}{4}} - o(t^{-\frac{n+2}{4}}) \geq c' t^{-\frac{n+2}{4}} \end{aligned}$$

for large $t > 0$. Assume next (2.5). By (1.4) we have

$$\|\partial_\ell E_t \boldsymbol{b}_\ell + \boldsymbol{F}_{\ell,k} c_{k\ell}\|_2 \geq \|u(t)\|_2 - \|u(t) + \partial_\ell E_t \boldsymbol{b}_\ell + \boldsymbol{F}_{\ell,k} c_{k\ell}\|_2 \geq c' t^{-\frac{n+2}{4}} - o(t^{-\frac{n+2}{4}}),$$

and so

$$\|\partial_\ell E_t \boldsymbol{b}_\ell + \boldsymbol{F}_{\ell,k} c_{k\ell}\|_2 > 0 \qquad \text{for large } t > 0.$$

We thus obtain (2.4) by Proposition 2.1. This proves Proposition 2.2.

3. Proof of Theorem B

Suppose that $n \geq 3$. We have

$$c_{k\ell} = \int_0^\infty \int (u_\ell u_k)(y,s)dyds < \infty\,;$$

so the argument given in [2,Sect. 5] applies to our present situation, implying

$$\lim_{t\to\infty} t^{\frac{n+2}{4}} \left\| u(t) - e^{-tA} a + \boldsymbol{F}_{\ell,k} c_{k\ell} \right\|_2 = 0. \tag{3.1}$$

Suppose (1.9) holds. Since $\|\boldsymbol{F}_{\ell,k}c_{k\ell}\|_2 = Ct^{-\frac{n+2}{4}}$, it follows from (3.1) that

$$\begin{aligned}\|e^{-tA}a\|_2 &\geq \|u(t)\|_2 - \|-u(t) + e^{-tA}a - \boldsymbol{F}_{\ell,k}c_{k\ell} + \boldsymbol{F}_{\ell,k}c_{k\ell}\|_2 \\ &\geq \|u(t)\|_2 - \|u(t) - e^{-tA}a + \boldsymbol{F}_{\ell,k}c_{k\ell}\|_2 - \|\boldsymbol{F}_{\ell,k}c_{k\ell}\|_2 \\ &\geq ct^{-\frac{n}{4}} - Ct^{-\frac{n+2}{4}} \geq c't^{-\frac{n}{4}}\end{aligned}$$

for large $t > 0$. This proves (1.10). Conversely, if (1.10) holds, then (3.1) implies

$$\begin{aligned}\|u(t)\|_2 &\geq \|e^{-tA}a\|_2 - \|\boldsymbol{F}_{\ell,k}c_{k\ell}\|_2 - \|u(t) - e^{-tA}a + \boldsymbol{F}_{\ell,k}c_{k\ell}\|_2 \\ &\geq ct^{-\frac{n}{4}} - Ct^{-\frac{n+2}{4}} \geq c't^{-\frac{n}{4}}\end{aligned}$$

for large $t > 0$. This proves (1.9) in case $n \geq 3$.

When $n = 2$, we introduce

$$c_{k\ell}(t) = \int_0^{t/2}\int (u_\ell u_k)(y,s)\,dy\,ds$$

instead of $c_{k\ell}$. The argument of [2,Sect. 5] is then modified to yield

$$\|u(t) - e^{-tA}a + \boldsymbol{F}_{\ell,k}c_{k\ell}(t)\|_2 \leq Ct^{-1}\log(1+t). \tag{3.1$'$}$$

Since

$$\|\boldsymbol{F}_{\ell,k}c_{k\ell}(t)\|_2 \leq Ct^{-1}\int_0^{t/2}\|u(s)\|_2^2\,ds \leq Ct^{-1}\log(1+t),$$

this implies $\|u(t) - e^{-tA}a\|_2 \leq Ct^{-1}\log(1+t)$. Now we can prove the result in the same way as in the case $n \geq 3$. Indeed, (1.10) implies

$$\|u(t)\|_2 \geq \|e^{-tA}a\|_2 - \|u(t) - e^{-tA}a\|_2 \geq ct^{-\frac{1}{2}} - Ct^{-1}\log(1+t) \geq c't^{-\frac{1}{2}}$$

for large $t > 0$, while (1.9) yields

$$\|e^{-tA}a\|_2 \geq \|u(t)\|_2 - \|u(t) - e^{-tA}\|_2 \geq ct^{-\frac{1}{2}} - Ct^{-1}\log(1+t) \geq c't^{-\frac{1}{2}}$$

for large $t > 0$. The proof of Theorem B is complete.

4. Proof of (1.4)

Here we present the proof of (1.4) given in [2]. The same method can be applied to the proof of (3.1) and (3.1$'$). Let a satisfy (1.2) and so (1.3). We first prove

$$\lim_{t\to\infty} t^{\frac{n+2}{4}}\left\|e^{-tA}a + (\partial_k E_t)(\cdot)\int y_k a(y)\,dy\right\|_2 = 0. \tag{4.1}$$

Direct calculation gives

$$\begin{aligned}e^{-tA}a &= \int [E_t(x-y) - E_t(x)]a(y)\,dy = -\int\int_0^1 (\partial_k E_t)(x - y\theta)y_k a(y)\,d\theta\,dy \\ &= -(\partial_k E_t)(x)\int y_k a(y)\,dy - \int\int_0^1 [(\partial_k E_t)(x-y\theta) - (\partial_k E_t)(x)]y_k a(y)\,d\theta\,dy,\end{aligned}$$

so

$$e^{-tA}a + (\partial_k E_t)(x) \int y_k a(y) dy = -\iint_0^1 [(\partial_k E_t)(x - y\theta) - (\partial_k E_t)(x)] y_k a(y) d\theta dy.$$

We can write $(\partial_k E_t)(x) = t^{-\frac{n+1}{2}}(\partial_k E_1)(xt^{-\frac{1}{2}})$, to obtain

$$\left\| e^{-tA}a + (\partial_k E_t)(\cdot) \int y_k a(y) dy \right\|_2 \le Ct^{-\frac{n+2}{4}} \iint_0^1 \varphi_t(y,\theta)|y||a(y)| d\theta dy.$$

Here $\varphi_t(y,\theta) = \|(\nabla E_1)(\cdot - y\theta t^{-\frac{1}{2}}) - (\nabla E_1)(\cdot)\|_2$ is bounded and $\lim_{t\to\infty} \varphi_t(y,\theta) = 0$ for any fixed (y,θ). Since $|y||a(y)|$ is integrable by (1.2), the dominated convergence theorem yields

$$\lim_{t\to\infty} \iint_0^1 \varphi_t(y,\theta)|y||a(y)| d\theta dy = 0.$$

This proves (4.1). Now let u satisfy (1.1). We next show that the function

$$w(t) = u(t) - e^{-tA}a = -\int_0^t \int \boldsymbol{F}_{\ell,k}(x - y, t - s)(u_\ell u_k)(y,s) dy ds$$

satisfies

$$\lim_{t\to\infty} t^{\frac{n+2}{4}} \left\| w(t) + \boldsymbol{F}_{\ell,k}(\cdot,t) \int_0^\infty \int (u_\ell u_k)(y,s) dy ds \right\|_2 = 0. \tag{4.2}$$

Indeed, we have

$$\begin{aligned}
w(t) &+ \boldsymbol{F}_{\ell,k}(x,t) \int_0^\infty \int (u_\ell u_k)(y,s) dy ds \\
&= \boldsymbol{F}_{\ell,k}(x,t) \int_{t/2}^\infty \int (u_\ell u_k)(y,s) dy ds \\
&\quad - \int_0^{t/2} \int [\boldsymbol{F}_{\ell,k}(x-y,t-s) - \boldsymbol{F}_{\ell,k}(x,t-s)](u_\ell u_k)(y,s) dy ds \\
&\quad - \int_0^{t/2} \int [\boldsymbol{F}_{\ell,k}(x,t-s) - \boldsymbol{F}_{\ell,k}(x,t)](u_\ell u_k)(y,s) dy ds \\
&\quad - \int_{t/2}^t \int \boldsymbol{F}_{\ell,k}(x-y,t-s)(u_\ell u_k)(y,s) dy ds \\
&\equiv I_1 + I_2 + I_3 + I_4.
\end{aligned}$$

It is easy to see that

$$t^{\frac{n+2}{4}} \|I_1\|_2 \le C \int_{t/2}^\infty (1+s)^{-1-\frac{n}{2}} ds \to 0 \qquad \text{as } t \to \infty. \tag{4.3}$$

We write I_3 in the form

$$I_3 = \int_0^{t/2} \iint_0^1 s(\partial_t \boldsymbol{F}_{\ell,k})(x, t - s\theta)(u_\ell u_k)(y,s) d\theta dy ds$$

to get

$$\|I_3\|_2 \le C \int_0^{t/2} \iint_0^1 s(t - s\theta)^{-1-\frac{n+2}{4}} |u(y,s)|^2 d\theta dy ds \le Ct^{-1-\frac{n+2}{4}} \int_0^{t/2} s\|u(s)\|_2^2 ds$$

and so

$$t^{\frac{n+2}{4}}\|I_3\|_2 \le Ct^{-1}\int_0^t (1+s)^{-\frac{n}{2}}ds \to 0 \qquad \text{as } t\to\infty. \tag{4.4}$$

To estimate I_2, note that we can write $\boldsymbol{F}_{\ell,k}(x,t)=t^{-\frac{n+1}{2}}K(xt^{-\frac{1}{2}})$, to get

$$\begin{aligned}\|I_2\|_2 \;&\le\; Ct^{-\frac{n+2}{4}}\int_0^{t/2}\!\!\int \|K(\cdot - y(t-s)^{-\frac{1}{2}}) - K(\cdot)\|_2 |u(y,s)|^2 dyds \\ &\equiv\; Ct^{-\frac{n+2}{4}}\int_0^{t/2}\!\!\int \varphi_t(y,s)|u(y,s)|^2 dyds \equiv Ct^{-\frac{n+2}{4}}\int_0^{t/2}\psi_t(s)ds.\end{aligned}$$

Since $\psi_t(s)\le C\|u(s)\|_2^2$, the dominated convergence theorem implies

$$\lim_{t\to\infty}\int_0^M \psi_t(s)ds = 0 \qquad \text{for any fixed } M>0.$$

Given $\varepsilon>0$, choose $M>0$ so that $\int_M^\infty \|u(s)\|_2^2 ds<\varepsilon$. Then for $t>2M$,

$$\int_0^{t/2}\psi_t(s)ds \le \int_0^M \psi_t(s)ds + C\int_M^\infty \|u(s)\|_2^2 ds \le \int_0^M \psi_t(s)ds + C\varepsilon.$$

This implies that

$$\lim_{t\to\infty} t^{\frac{n+2}{4}}\|I_2\|_2 = 0. \tag{4.5}$$

It remains to prove

$$\lim_{t\to\infty} t^{\frac{n+2}{4}}\|I_4\|_2 = 0. \tag{4.6}$$

To do so, we follow the arguments of [3,5]. The function

$$v(t) = -\int_\tau^t\!\!\int \boldsymbol{F}_{\ell,k}(x-y,t-s)(u_\ell u_k)(y,s)dyds = u(t) - e^{-(t-\tau)A}u(\tau)$$

defined for $t\ge\tau>0$ satisfies

$$\partial_t v + Av = -P(u\cdot\nabla u) \quad (t>\tau), \qquad v(\tau)=0.$$

(We may assume v is smooth, replacing u by the approximate solutions u_N given in [3].) Since $(P(u\cdot\nabla v),v)=(u\cdot\nabla v,v)=0$, the standard energy integral method gives

$$\partial_t\|v\|_2^2 + 2\|A^{1/2}v\|_2^2 = -2(u\cdot\nabla u,v) = 2(u\cdot\nabla v,u) = 2(u\cdot\nabla v,u_0)$$

and

$$\begin{aligned}2|(u\cdot\nabla v,u_0)| \;&\le\; 2\|u\|_2\|A^{1/2}v\|_2\|u_0\|_\infty \le C\|u\|_2\|A^{1/2}v\|_2(t-\tau)^{-\frac{n}{4}}\tau^{-\frac{n+2}{4}} \\ &\le\; C\|A^{1/2}v\|_2(t-\tau)^{-\frac{n+1}{2}}\tau^{-\frac{n+2}{4}} \le \|A^{1/2}v\|_2^2 + C(t-\tau)^{-n-1}\tau^{-1-\frac{n}{2}},\end{aligned}$$

where $u_0(t) = e^{-(t-\tau)A}u(\tau)$. We thus obtain

$$\partial_t \|v\|_2^2 + \|A^{1/2}v\|_2^2 \le C(t-\tau)^{-n-1}\tau^{-1-\frac{n}{2}}.$$

Let $\{E_\lambda\}_{\lambda\ge 0}$ be the spectral measure associated to A. Since $\|A^{1/2}v\|_2^2 \ge \varrho(\|v\|_2^2 - \|E_\varrho v\|_2^2)$ for any $\varrho > 0$, the above estimate yields

$$\partial_t \|v\|_2^2 + \varrho\|v\|_2^2 \le \varrho\|E_\varrho v\|_2^2 + C(t-\tau)^{-n-1}\tau^{-1-\frac{n}{2}}.$$

But, $\|E_\varrho v\|_2^2 \le C\varrho^{\frac{n+2}{2}}\left(\int_\tau^t \|u\|_2^2 ds\right)^2$ as shown in [3,5] ; so

$$\partial_t \|v\|_2^2 + \varrho\|v\|_2^2 \le \varrho^{\frac{n+4}{2}}\left(\int_\tau^t \|u\|_2^2 ds\right)^2 + C(t-\tau)^{-n-1}\tau^{-1-\frac{n}{2}}.$$

Here we set $\varrho = m/(t-\tau)$, $m > 0$, and multiply both sides by $(t-\tau)^m$, to obtain

$$\partial_t((t-\tau)^m\|v\|_2^2) \le C_m(t-\tau)^{m-\frac{n}{2}-2}\left(\int_\tau^t \|u\|_2^2 ds\right)^2 + C(t-\tau)^{m-n-1}\tau^{-1-\frac{n}{2}}.$$

Now fix m so that $m > n/2+2$ and $m > n+1$, and integrate the above inequality, to get

$$\|v(t)\|_2^2 \le C(t-\tau)^{-2-\frac{n}{2}}\int_\tau^t \left(\int_\tau^s \|u\|_2^2 d\sigma\right)^2 ds + C(t-\tau)^{-n}\tau^{-1-\frac{n}{2}}.$$

Inserting $\tau = t/2$ yields $v(t) = I_4$, so

$$t^{n+\frac{n}{2}}\|I_4\|_2^2 \le Ct^{n-1}\left(\int_{t/2}^\infty \|u\|_2^2 ds\right)^2 + Ct^{-1} \le Ct^{-1} \to 0$$

as $t \to \infty$. This proves (4.6).

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References

[1] A. Carpio, Large-time behavior in incompressible Navier-Stokes equations. *SIAM J. Math. Anal.* **27** (1996), 449–475.

[2] Y. Fujigaki and T. Miyakawa, Asymptotic profiles of nonstationary incompressible Navier-Stokes flows in $\mathbb{R}^n$. Preprint, Kobe University, 2000.

[3] R. Kajikiya and T. Miyakawa, On L^2 decay of weak solutions of the Navier-Stokes equations in $\mathbb{R}^n$. *Math. Z.* **192** (1986), 135–148.

[4] T. Miyakawa, Application of Hardy space techniques to the time-decay problem for incompressible Navier-Stokes flows in $\mathbb{R}^n$. *Funkcial. Ekvac.* **41** (1998), 383–434.

[5] M. E. Schonbek, L^2 decay for weak solutions of the Navier-Stokes equations. *Arch. Rational Mech. Anal.* **88** (1985), 209–222.

[6] M. E. Schonbek, Large time behaviour of solutions to the Navier-Stokes equations. *Commun. Partial Diff. Eq.* **11** (1986), 733–763.

[7] M. E. Schonbek, Lower bounds of rates of decay for solutions to the Navier-Stokes equations. *J. Amer. Math. Soc.* **4** (1991), 423–449.

[8] M. E. Schonbek, Asymptotic behavior of solutions to the three-dimensional Navier-Stokes equations. *Indiana Univ. Math. J.* **41** (1992), 809–823.

[9] E. M. Stein, *Harmonic Analysis*. Princeton University Press, Princeton, 1993.

[10] M. Wiegner, Decay results for weak solutions of the Navier-Stokes equations in $\mathbb{R}^n$. *J. London Math. Soc.* **35** (1987), 303–313.

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