ON ZERO MASS SOLUTIONS OF VISCOUS CONSERVATION LAWS

Grzegorz Karch[1] and Maria Elena Schonbek[2]

[1]Instytut Matematyczny, Uniwersytet Wrocławski
pl. Grunwaldzki 2/4, 50–384 Wrocław, Poland
karch@math.uni.wroc.pl
[2]Department of Mathematics, University of California
Santa Cruz, CA 95064, U.S.A.
schonbek@math.ucsc.edu

August 28, 2001

Abstract. In the paper, we consider the large time behavior of solutions to the convection-diffusion equation $u_t - \Delta u + \nabla \cdot f(u) = 0$ in $\mathbb{R}^n \times [0, \infty)$, where $f(u) \sim u^q$ as $u \to 0$. Under the assumption that $q \geq 1 + 1/(n+\beta)$ and the initial condition u_0 satisfies: $u_0 \in L^1(\mathbb{R}^n)$, $\int_{\mathbb{R}^n} u_0(x)\, dx = 0$, and $\|e^{t\Delta} u_0\|_{L^1(\mathbb{R}^n)} \leq Ct^{-\beta/2}$ for fixed $\beta \in (0,1)$, all $t > 0$, and a constant C, we show that the solution has the same decay in $L^1(\mathbb{R}^n)$ as its linear counterpart. Moreover, we prove that, for small initial conditions, the exponent $q^* = 1 + 1/(n+\beta)$ is critical in the following sense. For $q > q^*$ the large time behavior of solutions is weakly nonlinear (i.e. given by solutions to the linear heat equation) and for $q = q^*$ the behavior as $t \to \infty$ of solutions is described by a new class of self-similar solutions to a nonlinear convection-diffusion equation.

1 Introduction

In this paper, we study the large time behavior of solutions $u = u(x,t)$ $(x \in \mathbb{R}^n,\ t > 0)$ to the Cauchy problem for the nonlinear convection-diffusion

2000 Mathematics Subject Classification: 35B40, 35K55.
Key words and phrases: the Cauchy problem, the convection-diffusion equation, large time behavior of solutions, self-similar solutions.

equation

$$u_t - \Delta u + a \cdot \nabla(u|u|^{q-1}) = 0, \tag{1.1}$$

$$u(x,0) = u_0(x), \tag{1.2}$$

where $q > 1$ and the vector $a \in I\!R^n$ are fixed, under the assumption $u_0 \in L^1(I\!R^n)$ and $\int_{I\!R^n} u_0(x)\,dx = 0$.

The typical nonlinear term occurring in hydrodynamics in the one-dimensional case has the form $uu_x = (u^2/2)_x$ (as in the case of the viscous Burgers equation). The most obvious generalization of this nonlinearity consists in replacing the square by a power u^q where q is a positive integer. Here, however, we intend to observe a more subtle interaction of the nonlinearity with dissipation, consequently, we need to consider a continuous range of parameters q. The problem then appears with the definition of u^q for negative u and for non-integer q. In order to avoid this difficulty, we chose the nonlinear term of the from $a \cdot \nabla(u|u|^{q-1})$. This was done to shorten notation in this report, only. Note that, in fact, the following property of the nonlinearity will only be important throughout this work:

- the nonlinear term in (1.1) has the form $\nabla \cdot f(u)$ where the C^1-vector function f satisfies $|f(u)| \le C|u|^q$, $|f'(u)| \le C|u|^{q-1}$ for every $u \in I\!R$, $q > 1$, and a constant C. Moreover, if the balanced case is considered (i.e. $q = 1 + 1/(n+\beta)$), the limits

$$\lim_{u\to 0^-} f(u)/|u|^q, \quad \text{and} \quad \lim_{u\to 0^+} f(u)/|u|^q$$

  should exist and the both should be different from 0.

Recent publications developed versatile functional analytic tools to study the long time behavior of solutions of this initial value problem.

Concerning the decay of solutions of (1.1)-(1.2) and, more generally, of scalar parabolic conservation laws of the form $u_t - \Delta u + \nabla \cdot f(u) = 0$ with integrable initial conditions, Schonbek [25] was the first who proved that the L^2-norm tends to 0 as $t \to \infty$ with the rate $t^{-n/4}$. To deal with this problem, she introduced the so-called *Fourier splitting method.* The results from [25] were extended in the later work [26], where the decay of solutions in $L^p(I\!R^n)$, $(1 \le p \le \infty)$ was obtained, again, by a method based on the Fourier splitting technique. It was emphasized in [26] that the decay rates are the same as for the underlying linear equations.

Next, Escobedo and Zuazua [12] proved decay estimates of the L^p-norms of solutions by a different method under more general assumptions on nonlinearity and under less restrictive assumptions on initial data. Finally, by the use of the *logarithmic Sobolev inequality*, Carlen and Loss [7] showed that solutions of viscous conservation laws satisfy

$$\|u(\cdot,t)\|_p \le Ct^{-(n/2)(1/r-1/p)}\|u_0\|_r$$

for each $1 \le r \le p \le \infty$, all $t > 0$, and a numerical constant $C > 0$ depending on p and q, only. Here, we would also like to recall results on algebraic decay rates of solution to systems of parabolic conservation laws, obtained by Kawashima [21], Hopf and Zumbrun [15], Jeffrey and Zhao [17], and Schonbek and Süli [28]. Smallness assumptions on initial conditions were often imposed in those papers.

The first term of the asymptotic expansion was studied as the next step in analysis of the long time behavior of solutions to (1.1)-(1.2). Assuming that $u_0 \in L^1(\mathbb{R}^n)$, roughly speaking, these results, cf. e.g. [8, 21, 12, 13, 14, 10, 11, 1, 2, 3, 19, 20], fall into three cases:

- *Case I:* $q > 1 + 1/n$, when the asymptotics is linear, i.e.

  $$t^{(n/2)(1-1/p)}\|u(\cdot,t) - MG(\cdot,t)\|_p \to 0 \text{ as } t \to \infty, \tag{1.3}$$

  where $M = \int_{\mathbb{R}^n} u_0(x)\,dx$, $G(x,t) = (4\pi t)^{-n/2}\exp(-|x|^2/(4t))$ is the fundamental solution of the heat equation. Hence, this case can be classified as *weakly nonlinear*, since in this situation the linear diffusion prevails and the nonlinearity is asymptotically negligible.

- Case II: $q = 1 + 1/n$, when

  $$t^{(n/2)(1-1/p)}\|u(\cdot,t) - U_M(\cdot,t)\|_p \to 0 \text{ as } t \to \infty, \tag{1.4}$$

  where $U_M(x,t) = t^{-n/2}U_M(xt^{-1/2},1)$ is the self-similar solution of (1.1) with $u_0(x) = M\delta_0$. Here, diffusion and the convection are balanced, and the asymptotics is determined by a special solution of a nonlinear equation.

- Case III: $1 < q < 1 + 1/n$, when

  $$t^{(n+1)(1-1/p)/(2q)}\|u(\cdot,t) - U_M(\cdot,t)\|_p \to 0 \text{ as } t \to \infty, \tag{1.5}$$

  holds, where U_M is a particular self-similar solution of the partly viscous conservation law $U_t - \Delta_y U + \frac{\partial}{\partial x_n}(U|U|^{q-1}) = 0$ such that $u_0(x) = M\delta_0$ in the sense of measures. Here $x = (y, x_n)$, $y = (x_1, \ldots, x_{n-1})$, and $\Delta_y = \sum_{j=1}^{n-1}\frac{\partial^2}{\partial x_j^2}$. Hence, the asymptotics of solutions is determined by solutions of an equation with strong convection and partial dissipation.

Finally, we recall that, in the weakly nonlinear case, Zuazua [32] found, for solutions to (1.1)-(1.2), the second order term in the asymptotic expansion as $t \to \infty$. He observed that asymptotic behavior of the solution differs depending if q satisfies $1 + 1/n < q < 1 + 2/n$, $q = 1 + 2/n$, or $q > 1 + 2/n$. Analogous results for Lévy conservation laws were obtained in [1, 2] and for convection-diffusion equations with dispersive effects – in [19, 20].

Note now that if we assume that $M = \int_{I\!R^n} u_0(x)\,dx = \int_{I\!R^n} u(x,t)\,dx = 0$ the corresponding self-similar intermediate asymptotics in (1.3)-(1.5) equal to 0 for every $q > 1$. Moreover, for $p = 1$ the asymptotic formulae in (1.3)-(1.5) say nothing else but

$$\|u(\cdot,t)\|_1 \to 0 \quad \text{as} \quad t \to \infty.$$

The goal of this paper is to find the first term of the asymptotic expansion in $L^p(I\!R^n)$ of solutions to (1.1)-(1.2) with $M = 0$ imposing additional conditions on initial data. We assume that u_0 satisfies $\|e^{t\Delta}u_0\|_1 \leq Ct^{-\beta/2}$ for some $\beta \in (0,1)$, all $t > 0$, and C independent of t. Such a decay estimate of solutions to the linear heat equation is optimal for a large class of initial conditions (cf. Propositions 2.1 and 2.2, below). Under these assumptions, we improve the known algebraic decay rates of the solutions to (1.1)-(1.2) in the L^p-norms for every $1 \leq p \leq \infty$. In addition, if the initial data are sufficiently small, we discover the new critical exponent

$$q^* = 1 + \frac{1}{n+\beta}$$

such that

- for $q > q^*$ the asymptotics of solutions to (1.1)-(1.2) is linear and described by self-similar solutions to the heat equation (cf. Corollaries 2.1 and 2.2, below);
- $q = q^*$ corresponds to the balanced case, and the asymptotics is described by a new class of self-similar solutions to the nonlinear equation (1.1) (cf. Theorem 2.3 and the remark following it.).

In the next section of this report, we present and discuss our results. The proofs of all results corresponding to the weakly nonlinear case are contained in Section 3. Theorems 2.2 and 2.3 are proved in Section 4.

Notation.

The notation to be used is mostly standard. For $1 \leq p \leq \infty$, the L^p-norm of a Lebesgue measurable real-valued function defined on $I\!R^n$ is denoted by $\|v\|_p$.

We will always denote by $\|\cdot\|_{\mathcal{X}}$ the norm of any other Banach space $\mathcal{X}$ used in this paper.
If k is a nonnegative integer, $W^{k,p}(I\!R^n)$ will be the Sobolev space consisting of functions in $L^p(I\!R^n)$ whose generalized derivatives up to order k belong also to $L^p(I\!R^n)$.
The Fourier transform of v is defined as $\widehat{v}(\xi) \equiv (2\pi)^{-n/2} \int_{I\!R^n} e^{-ix\xi} v(x)\, dx$.
Given a multi-index $\gamma = (\gamma_1, ..., \gamma_n)$, we denote $\partial^\gamma = \partial^{|\gamma|}/\partial_{x_1}^{\gamma_1} ... \partial_{x_n}^{\gamma_n}$. On the other hand, for $\beta > 0$, the operator D^β is defined via the Fourier transform as $(\widehat{D^\beta w})(\xi) = |\xi|^\beta |\widehat{w}(\xi)$.
The letter C will denote generic positive constants, which do not depend on t but may vary from line to line during computations.

2 Results and comments

For every $u_0 \in L^1(I\!R^n)$, the Cauchy problem (1.1)-(1.2) has a unique solution in $C([0,\infty); L^1(I\!R^n))$ satisfying

$$u \in C((0,\infty); W^{2,p}(I\!R^n)) \cap C^1((0,\infty), L^p(I\!R^n))$$

for all $p \in (1,\infty)$. The proof is based on a standard iteration procedure involving the integral representation of solutions of (1.1)-(1.2)

$$u(t) = e^{t\Delta} u_0 - \int_0^t a \cdot \nabla e^{(t-\tau)\Delta} (u|u|^{q-1})(\tau)\, d\tau \tag{2.1}$$

(see, e.g. [12] for details). Here, $e^{t\Delta}u_0$ is the solution to the linear heat equation given by the convolution of the initial datum u_0 with the Gauss-Weierstrass kernel $G(x,t) = (4\pi t)^{-n/2} \exp(-|x|^2/(4t))$. Formula (2.1) will be one of the main tools used in the analysis of the long time behavior of solutions.
Let us also recall that sufficiently regular solutions of (1.1)-(1.2) satisfy the estimate

$$\|u(\cdot,t)\|_p \le C(p,r) t^{-(n/2)(1/r-1/p)} \|u_0\|_r \tag{2.2}$$

for all $1 \le r \le p \le \infty$, all $t > 0$, and a constant $C(p,r)$ depending on p and r, only. Inequalities (2.2) are due to Carlen and Loss [7, Theorem 1]. We also refer the reader to [1, 2] where counterparts of (2.2) were proved for more general equations: so-called Lévy conservation laws.

We begin our consideration by the analysis of the large time asymptotics of solutions to the linear heat equation. Easy calculations show that for every $u_0 \in L^1(I\!R^n)$ such that $\int_{I\!R^n} u_0(x)\, dx = 0$ we have $\|e^{t\Delta}u_0\|_1 \to 0$ as $t \to \infty$.

The following two propositions assert the existence of a large class of initial conditions for which the large time behavior of $e^{t\Delta}u_0$ is self-similar. Here, we need the notion of the Riesz potential I_β and the fractional derivative D^β defined in the Fourier variables as

$$(\widehat{I_\beta w})(\xi) = \frac{\widehat{w}(\xi)}{|\xi|^\beta} \quad \text{and} \quad (\widehat{D^\beta w})(\xi) = |\xi|^\beta \widehat{w}(\xi). \tag{2.3}$$

Proposition 2.1 *Let $\beta > 0$ and $\gamma = (\gamma_1, ..., \gamma_n)$ be a multi-index with $\gamma_i \geq 0$. Assume that $I_\beta u_0 \in L^1(I\!\!R^n)$. Denote*

$$A = \lim_{|\xi|\to 0} \frac{\widehat{u}_0(\xi)}{|\xi|^\beta} = \int_{I\!\!R^n} (I_\beta u_0)(x)\, dx. \tag{2.4}$$

Then

$$\|\partial^\gamma e^{t\Delta} u_0\|_1 \leq C t^{-\beta/2-|\gamma|/2} \|I_\beta u_0\|_1 \tag{2.5}$$

for all $t > 0$ and $C = C(\beta, \gamma)$ independent of t and u_0; moreover,

$$t^{\beta/2+|\gamma|/2} \|\partial^\gamma e^{t\Delta} u_0(\cdot) - A \partial^\gamma D^\beta G(\cdot, t)\|_1 \to 0 \tag{2.6}$$

as $t \to \infty$.

Remark 2.1. The L^1-decay of solutions to the linear heat equation formulated in (2.5) was proved by Miyakawa [23] under the assumptions

$$u_0 \in L^1(I\!\!R^n), \ \int_{I\!\!R^n} u_0(x)\, dx = 0, \ \int_{I\!\!R^n} |x|^\beta |u_0(x)|\, dx < \infty. \tag{2.7}$$

Our assumptions on initial conditions are weaker than those by Miyakawa in view of the inequality

$$\|I_\beta u_0\|_1 \leq C \int_{I\!\!R^n} |x|^\beta |u_0(x)|\, dx \tag{2.8}$$

valid for every u_0 satisfying (2.7) with $\beta \in (0, 1)$. Let us sketch the proof of (2.8), however, it does not play any role in our considerations, below. It is well known that $(I_\beta u_0)(x) = C(\beta, n) \int_{I\!\!R^n} |x-y|^{\beta-n} u_0(y)\, dy$ (in fact, this representation holds true for every $\beta \in (0, n)$). Hence, using the assumption $\int_{I\!\!R^n} u_0(y)\, dy = 0$ and changing the order of integration we obtain

$$\|I_\beta u_0\|_1 \leq C(\beta, n) \int_{I\!\!R^n} \left(\int_{I\!\!R^n} \left| \frac{1}{|x-y|^{n-\beta}} - \frac{1}{|x|^{n-\beta}} \right| dx \right) |u_0(y)|\, dy.$$

Next, note that the integral with respect to x in the inequality above is finite for every $y \in I\!\!R^n$, because its integrand $||x-y|^{\beta-n} - |x|^{\beta-n}|$ is locally integrable and behaves like $|x|^{\beta-1-n}$ as $|x| \to \infty$ (here, the assumption $\beta \in (0,1)$ is crucial). Hence, by the change of variables, we obtain

$$\int_{I\!\!R^n} \left| \frac{1}{|x-y|^{n-\beta}} - \frac{1}{|x|^{n-\beta}} \right| \, dx = |y|^\beta \int_{I\!\!R^n} \left| \frac{1}{|\omega - y/|y||^{n-\beta}} - \frac{1}{|\omega|^{n-\beta}} \right| \, d\omega.$$

Since $\sup_{y \in I\!\!R^n \setminus \{0\}} \int_{I\!\!R^n} ||\omega - y/|y||^{\beta-n} - |\omega|^{\beta-n}| \, d\omega < \infty$ (we skip the proof of this elementary fact), we obtain (2.8). □

We can derive the self-similar asymptotics of $e^{t\Delta}u_0$ in $L^p(I\!\!R^n)$ with $p \in [2,\infty]$ under weaker assumptions on u_0.

Proposition 2.2 *Let $\ell = \ell(\xi)$ denote a function homogeneous of degree $\beta > 0$. Assume that u_0 satisfies*

$$\sup_{\xi \in I\!\!R^n \setminus \{0\}} \frac{\widehat{u}_0(\xi)}{\ell(\xi)} < \infty \qquad \text{and} \qquad \lim_{|\xi| \to 0} \frac{\widehat{u}_0(\xi)}{\ell(\xi)} = A \tag{2.9}$$

for some $A \in I\!\!R$. Denote by $\mathcal{L}$ the Fourier multiplier operator defined via the formula $\widehat{\mathcal{L}v}(\xi) = \ell(\xi)\widehat{v}(\xi)$. Under these assumptions, for every $p \in [2,\infty]$ and for every multi-index γ

$$t^{n(1-1/p)/2+\beta/2+|\gamma|/2} \|\partial^\gamma e^{t\Delta}u_0 - A\partial^\gamma \mathcal{L}G(t)\|_p \to 0$$

as $t \to \infty$.

Propositions 2.1 and 2.2 are proved in the beginning of Section 3.

In our first theorem on the large time behavior of solutions to the nonlinear problem (1.1)-(1.2), we assume the decay of $\|e^{t\Delta}u_0\|_1$ with a given rate and we prove that the same decay estimate holds true for solutions to (1.1)-(1.2).

Theorem 2.1 *Fix $0 < \beta < 1$. Assume that $u_0 \in L^1(I\!\!R^n) \cap L^q(I\!\!R^n)$ satisfies the inequality*

$$\|e^{t\Delta}u_0\|_1 \le Ct^{-\beta/2} \tag{2.10}$$

for all $t > 0$ and a constant C independent of t. Let u be the solution to (1.1)-(1.2) with u_0 as the initial datum. If $q > 1 + 1/n$, then there exists a constant C such that

$$\|u(\cdot,t)\|_1 \le C(1+t)^{-\beta/2} \tag{2.11}$$

for all $t > 0$. The same conclusion holds true for $1 + 1/(n+\beta) \le q \le 1 + 1/n$ provided $u_0 \in L^1(I\!\!R^n) \cap L^\infty(I\!\!R^n)$ and $\sup_{t>0} t^{\beta/2}\|e^{t\Delta}u_0\|_1$ is sufficiently small.

Remark 2.2. The assumption (2.10) means that u_0 belongs to the homogeneous Besov space $\mathcal{B}_1^{-\beta,\infty}$ (cf. (2.17), below) which will play an important role in the analysis of the balanced case $q = 1 + 1/(n+\beta)$. □

The approach formulated in Theorem 2.1, saying that the decay estimates imposed on the heat semigroup lead to the analogous estimates of solutions to a nonlinear problem, appears in several recent papers. Here, we would like only to recall (the list is by no mean exhaustive) the works on the Navier-Stokes system by Schonbek [27] and Wiegner [31] where the L^2-decay of solutions was studied as well as by Miyakawa [22] where decay of the L^1-norm and $\mathcal{H}^p$-norms (the Hardy spaces) of weak solutions was shown. Moreover, our results extend essentially the recent paper by Schonbek and Süli [28] where general conservation laws were considered.

The decay of the L^1-norm in (2.11) is crucial in the proof of the following stronger result.

Corollary 2.1 *Under the assumptions of Theorem 2.1, for every $p \in [1,\infty]$ there exists $C = C(u_0, p)$ independent of t such that*

$$\|u(\cdot,t)\|_p \le C(1+t)^{-(n/2)(1-1/p)-\beta/2} \tag{2.12}$$

for all $t > 0$, and

$$\begin{aligned} &\|u(\cdot,t) - e^{t\Delta}u_0(\cdot)\|_p \\ &\le\ C\begin{cases} t^{-(n/2)(q-1/p)-(\beta q-1)/2} & \text{for} \quad q \in \left(1+\frac{1}{n+\beta}, \frac{n+2}{n+\beta}\right), \\ t^{-(n/2)(1-1/p)-1/2}\log(e+t) & \text{for} \quad q = \frac{n+2}{n+\beta}, \\ t^{-(n/2)(1-1/p)-1/2} & \text{for} \quad q > \frac{n+2}{n+\beta} \end{cases} \end{aligned} \tag{2.13}$$

for all $t \ge 1$.

As the immediate consequence of (2.13), we obtain that, under the assumptions of Theorem 2.1,

$$t^{(n/2)(1-1/p)+\beta/2}\|u(\cdot,t) - e^{t\Delta}u_0(\cdot)\|_p \to 0 \quad \text{as} \quad t \to \infty \tag{2.14}$$

for $q > 1 + 1/(n+\beta)$ and every $p \in [1,\infty]$. Moreover, the results from (2.14) combined with Propositions 2.1 and 2.2 may be summarized by saying that the large time behavior of solutions to (1.1)-(1.2) with $q > 1 + 1/n$ (or, if the data are sufficiently small, for $q > 1 + 1/(n+\beta)$) is *weakly nonlinear.* This is worth stating more precisely.

Corollary 2.2 *Under the assumptions of Theorem 2.1 and either Proposition 2.1 or Proposition 2.2 with* $\ell(\xi) = |\xi|^\beta$*, the solution to (1.1)-(1.2) with* $q > 1 + 1/(n+\beta)$ *satisfies*

$$t^{(n/2)(1-1/p)+\beta/2}\|u(\cdot,t) - AD^\beta G(\cdot,t)\|_p \to 0 \quad \textit{as} \quad t \to \infty.$$

Remark 2.3. If the nonlinear term in (1.1) has the form $\nabla \cdot f(u)$ and the function f is sufficiently regular at zero, it is possible to improve the conclusion of the last corollary to

$$t^{(n/2)(1-1/p)+(\beta+|\gamma|)/2}\|\partial^\gamma u(\cdot,t) - A\partial^\gamma D^\beta G(\cdot,t)\|_p \to 0 \quad \text{as} \quad t \to \infty$$

for the multi-index γ depending on the regularity of f. □

Remark 2.4. Let us look at Corollary 2.2 in the context of the viscous Burgers equation

$$u_t - u_{xx} + (u^2/2)_x = 0.$$

It is well known (cf. e.g. [16, 8, 12, 3, 10, 11]) that the large time behavior of solutions to this equation supplemented with the integrable initial condition is described by so-called *nonlinear diffusion waves*. If, however, we assume that u_0 satisfies (2.10) with some $\beta > 0$, the asymptotics for large t of solutions to the Burgers equation are *weakly nonlinear*. □

Remark 2.5. The conditions formulated in (2.9) appear in a natural way if the Hardy spaces are considered. Let us recall that a tempered distribution v belongs to the Hardy space $\mathcal{H}^p$ on $I\!R^n$ for some $0 < p < \infty$ whenever $v^+ = \sup_{t>0}|(\phi_t * v)| \in L^p(I\!R^n)$, where $\phi_t(x) = t^{-n}\phi(x/t)$ with $\phi \in \mathcal{S}(I\!R^n)$ such that $\int_{I\!R^n}\phi(x)\,\mathrm{d}x = 1$. We refer the reader to [30] where several properties of the Hardy spaces are derived. We recall that $\mathcal{H}^1$ is a Banach space strictly contained in $L^1(I\!R^n)$ and that $L^p(I\!R^n) = \mathcal{H}^p$ for $p > 1$ with the equivalent norms. Suppose now, that $p \le 1$ and $u_0 \in \mathcal{H}^p$. It is known (cf. [30, Chapter III, §5.4]) that the Fourier transform $\widehat{u}_0$ is continuous on $I\!R^n$ and $|\widehat{u}_0(\xi)| \le C|\xi|^{n(1/p-1)}\|u_0\|_{\mathcal{H}^p}$ for all $\xi \in I\!R^n$. Moreover, near the origin, this can be refined to $\lim_{\xi\to 0}\widehat{u}_0(\xi)|\xi|^{-n(1/p-1)} = 0$. Hence, assumptions (2.9) are satisfied with $\ell(\xi) = |\xi|^\beta$, $\beta \in (0,1)$, and $A = 0$, if e.g. $u_0 \in \mathcal{H}^{n/(n+\beta)}$. □

Remark 2.6. In this paper, we limit ourselves to the case $\beta \in (0,1)$ for the following reason. Suppose that

$$u_0 \in L^1(I\!R^n, (1+|x|)\ dx) \quad \text{and} \quad \int_{I\!R^n} u_0(x)\ dx = 0. \tag{2.15}$$

It is proved in [9] that $\|e^{t\Delta}u_0\|_1 \le Ct^{-1/2}\|u_0\|_{L^1(I\!R^n,|x|\,dx)}$ for all $t>0$ and a constant C; moreover,

$$t^{1/2}\left\|e^{t\Delta}u_0 - \int_{I\!R^n} xu_0(x)\,dx\cdot\nabla G(x,t)\right\|_1 \to 0 \quad \text{as} \quad t\to\infty.$$

Now, using the second order asymptotic expansion by Zuazua [32] (cf. also [2] for analogous results with more general diffusion operators and less regular initial conditions) of solutions to (1.1)-(1.2) with $q>1+2/n$, we obtain that the quantity

$$t^{1/2}\left\|u(\cdot,t) - \left(\int_{I\!R^n} xu_0(x)\,dx - a\int_0^\infty\int_{I\!R^n}(u|u|^{q-1})(x,\tau)\,dxd\tau\right)\cdot\nabla G(x,t)\right\|_1$$

tends to 0 as $t\to\infty$. This asymptotic result shows that the large time behavior of solutions with the initial data satisfying (2.15) cannot be classified as weakly nonlinear for every $q>1+2/n$, because the constant in front of ∇G depends on the nonlinearity in an essential way. Hence, assuming that $\|e^{t\Delta}u_0\|_1\le Ct^{-\beta/2}$ for some $\beta\ge 1$ one should expect asymptotic expansions of solutions completely different from that in Corollary 2.2. □

Our next results correspond to the balanced case

$$q = 1+\frac{1}{n+\beta}$$

for some fixed $0<\beta<1$, where following the ideas form [4, 5, 6, 18] we construct self-similar solutions to (1.1). Elementary calculations show that, for this exponent, if $u(x,t)$ is a solution to (1.1) then so is $\lambda^{n+\beta}u(\lambda x,\lambda^2 t)$ for every $\lambda>0$. Self-similar solutions should satisfy the equality $u(x,t)=\lambda^{n+\beta}u(\lambda x,\lambda^2 t)$, hence choosing $\lambda=\lambda(t)=1/\sqrt{t}$ we obtain its self-similar form

$$u(x,t) = t^{-\frac{n+\beta}{2}}U\left(\frac{x}{\sqrt{t}}\right) \tag{2.16}$$

where $U(x)=u(x,1)$, $x\in I\!R^n$ and $t>0$. These solutions will be proved to be asymptotically stable in the sense that they describe the asymptotic behavior, as $t\to\infty$, of a large class of solutions to (1.1)-(1.2).

We will work in the Besov space $\mathcal{B}_1^{-\beta,\infty}$ defined by

$$\mathcal{B}_1^{-\beta,\infty} = \{v\in\mathcal{S}'(I\!R^n)\;:\;\|v\|_{\mathcal{B}_1^{-\beta,\infty}}<\infty\},$$

where $\mathcal{S}'(I\!R^n)$ is the space of tempered distributions and the norm is defined by

$$\|v\|_{\mathcal{B}_1^{-\beta,\infty}} \equiv \sup_{s>0} s^{\beta/2}\|e^{s\Delta}v\|_1. \tag{2.17}$$

The standard way of defining norms in Besov spaces is based on the Paley-Littlewood dyadic decomposition. Here, the choice of the equivalent norm (2.17) allows us to simplify several calculations. Lemmata 4.1 and 4.2, and their elementary proofs show the usefulness of this definition.

The first theorem below constructs global-in-time solutions to (1.1)-(1.2) for suitably small initial data in the space $\mathcal{B}_1^{-\beta,\infty}$. The next one studies asymptotic stability of solutions.

Theorem 2.2 *Fix $\beta \in (0,1)$ and put $q = 1 + 1/(n+\beta)$. There is $\varepsilon > 0$ such that for each $u_0 \in \mathcal{B}_1^{-\beta,\infty}$ satisfying $\|u_0\|_{\mathcal{B}_1^{-\beta,\infty}} < \varepsilon$ there exists a solution of (1.1)-(1.2) for all $t \geq 0$ in the space*

$$\begin{aligned} \mathcal{X} \;\; &\equiv \;\; \mathcal{C}([0,\infty) : \mathcal{B}_1^{-\beta,\infty}) \\ &\qquad \cap \{u : (0,\infty) \to L^q(I\!\!R^n) \; : \; \sup_{t>0} t^{(n/2)(1-1/q)+\beta/2} \|u(t)\|_q < \infty\}. \end{aligned}$$

This is the unique solution satisfying the condition

$$\sup_{t>0} t^{(n/2)(1-1/q)+\beta/2} \|u(t)\|_q \leq 2\varepsilon.$$

Remark 2.7. Proposition 2.1 describes a large subset in $\mathcal{B}_1^{-\beta,\infty}$ of initial conditions u_0. Moreover, let us remark that $D^\beta \delta_0 \in \mathcal{B}_1^{-\beta,\infty}$ (the fractional derivative of order β of the Dirac delta δ_0). This is an easy consequence of the definitions of $e^{t\Delta}$ and δ_0, since $e^{t\Delta} D^\beta \delta_0 = D^\beta G(\cdot,t)$. Hence (3.1) below yields $\|e^{t\Delta} D^\beta \delta_0\|_1 = t^{-\beta/2} \|D^\beta G(\cdot,1)\|_1$. Note that the tempered distribution $D^\beta \delta_0$ is homogeneous of degree $-n-\beta$. Using this important property and applying the standard reasoning (cf. e.g. [5, Section 3]) based on the uniqueness result from Theorem 2.2, one can easily deduce that the solution $U(x,t)$ corresponding to $D^\beta \delta_0$ as the initial datum is self-similar, hence of the form (2.16).

□

Theorem 2.3 *Let the assumptions from Theorem 2.2 hold true. Assume that u and v are two solutions of (1.1)-(1.2) constructed in Theorem 2.2 corresponding to the initial data $u_0, v_0 \in \mathcal{B}_1^{-\beta,\infty}$, respectively. Suppose that*

$$\lim_{t\to\infty} t^{\beta/2} \|e^{t\Delta}(u_0 - v_0)\|_1 = 0. \tag{2.18}$$

Choosing $\varepsilon > 0$ in Theorem 2.2 sufficiently small, we have

$$\lim_{t\to\infty} t^{(n/2)(1-1/p)+\beta/2} \|u(\cdot,t) - v(\cdot,t)\|_p = 0 \tag{2.19}$$

for every $p \in [1,\infty]$.

Remark 2.8. Let $U_A(x,t)$ be the self-similar solution corresponding to the initial datum $u_0 = AD^\beta\delta_0$ for some $A \in I\!R$ (cf. Remark 2.7). Combining Proposition 2.1 with Theorem 2.3, we obtain a large class of initial data such that the asymptotic behavior in $L^p(I\!R^n)$ of corresponding solutions to (1.1)-(1.2) is described by $U_A(x,t)$. □

3 Weakly nonlinear asymptotics

Proof of Proposition 2.1. Let us note that the limit in (2.4) exists, because $\widehat{u}_0(\xi)/|\xi|^\beta$ is continuous as the Fourier transform of an integrable function.
First, we prove that $\partial^\gamma D^\beta G(\cdot,1) \in L^1(I\!R^n)$. Obviously, $\partial^\gamma D^\beta G(\cdot,1)$ is bounded and continuous because its Fourier transform $(i\xi)^\gamma|\xi|^\beta e^{-|\xi|^2}$ is integrable. Moreover, it follows from [29, Ch. 5, Lemma 2] that for every $\beta > 0$ there exists a finite measure μ_β on $I\!R^n$ such that

$$\widehat{\mu}_\beta(\xi) = \frac{|\xi|^\beta}{(1+|\xi|^2)^{\beta/2}}.$$

Hence, $\partial^\gamma D^\beta G(\cdot,1) = \mu_\beta * K_{\beta,\gamma}$ where the function $K_{\beta,\gamma}$ is defined via the Fourier transform as $\widehat{K}_{\beta,\gamma}(\xi) = (i\xi)^\gamma(1+|\xi|^2)^{\beta/2}e^{-|\xi|^2}$. It is easy to prove that $K_{\beta,\gamma} \in \mathcal{S}(I\!R^n)$ (the Schwartz class of rapidly decreasing smooth function), and this implies the integrabilty of $\partial^\gamma D^\beta G(\cdot,1)$ for every multi-index γ.
Now, by the change of variables, we obtain that $\partial^\gamma D^\beta G(x,t)$ has the self-similar form:

$$\partial^\gamma D^\beta G(x,t) = t^{-n/2-\beta/2-|\gamma|/2}(\partial^\gamma D^\beta G)(x/\sqrt{t},1) \tag{3.1}$$

for all $x \in I\!R^n$ and $t > 0$.
We are ready to prove (2.5). By the Young inequality for the convolution and by (3.1), we have

$$\begin{aligned}
\|\partial^\gamma e^{t\Delta}u_0\|_1 &= \|\partial^\gamma D^\beta G(t) * I_\beta u_0\|_1 \\
&\le \|\partial^\gamma D^\beta G(\cdot,t)\|_1 \|I_\beta u_0\|_1 \\
&\le t^{-\beta/2-|\gamma|/2}\|\partial^\gamma D^\beta G(\cdot,1)\|_1 \|I_\beta u_0\|_1
\end{aligned}$$

for all $t > 0$.
For the proof of (2.6), we first observe that the change of variables $z = x/\sqrt{t}$ combined with (3.1) lead to the following expression

$$\begin{aligned}
& t^{\beta/2+|\gamma|/2}\|\partial^\gamma e^{t\Delta}u_0(\cdot) - A\partial^\gamma D^\beta G(\cdot,t)\|_1 \\
&= t^{\beta/2+|\gamma|/2}\int_{I\!R^n}\left|\int_{I\!R^n}\left[\partial^\gamma D^\beta G(x-y,t) - \partial^\gamma D^\beta G(x,t)\right] I_\beta u_0(y)\, dy\right| dx \qquad (3.2)\\
&\le \iint_{I\!R^n\times I\!R^n} |I_\beta u_0(y)| \left|(\partial^\gamma D^\beta G)(z-y/\sqrt{t},1) - (\partial^\gamma D^\beta G)(z,1)\right|\, dydz
\end{aligned}$$

We already know from the first part of this proof that the function $\partial^\gamma D^\beta G(z,1)$ is continuous, hence the integrand on the right hand side of (3.2) tends to 0 as $t \to \infty$ for all $y, z \in I\!R^n$. Denote

$$\mathcal{A}(z,y,t) \equiv (\partial^\gamma D^\beta G)(z - y/\sqrt{t}, 1) - (\partial^\gamma D^\beta G)(z,1).$$

Now, to apply the Lebesgue Dominated Convergence Theorem to the integral on the right hand side of (3.2), we show that there exists $F \in L^1(I\!R^n)$ independent of $y \in I\!R^n$ and $t \geq 1$, such that

$$|\mathcal{A}(z,y,t)| \leq F(z) \tag{3.3}$$

for all $z, y \in I\!R^n$ and $t \geq 1$. For this reason, note first that

$$\mathcal{A}(z,y,,t) = \int_{R^n} |\xi|^\beta (i\xi)^\gamma \left[e^{-iy/\sqrt{t}} - 1\right] e^{-|\xi|^2} e^{iz\xi} d\xi.$$

Moreover, the symbol $b(\xi,y,t) \equiv (1+|\xi|^2)^{\beta/2}(i\xi)^\gamma \left[e^{-iy/\sqrt{t}} - 1\right] e^{-|\xi|^2}$ is a C^∞ function of $(\xi, y) \in I\!R^n \times I\!R^n$, and satisfies the differential inequalities

$$|\partial_\xi^\alpha \partial_y^\gamma b(\xi,y,t)| \leq C(\alpha,\gamma,N)(1+|\xi|)^{-N-\alpha}$$

for all multi-indices α and γ, all $N \in I\!N$, and $C(\alpha,\gamma,N)$ independent of $\xi, y \in I\!R^n$ and $t \geq 1$. By [30, Ch. VI, Sec. 4, Prop. 1], the (inverse) Fourier transform with respect to ξ of $b(\xi,y,t)$ satisfies the estimate

$$|\mathcal{F}_\xi^{-1} b(\cdot,y,t)(z)| \leq C(1+|z|)^{-N}$$

for all $N \in I\!N$, and a constant $C = C(N)$ independent of $z, y \in I\!R^n$ and $t \geq 1$. Finally, the use of the measure μ_β from the first part of this proof combined with standard properties of the Fourier transform and the convolution lead to the representation $\mathcal{A}(\cdot,y,t) = \mu_\beta * \mathcal{F}_\xi^{-1} b(\cdot,y,t)$. Hence, (3.3) holds true for the function $F(z) = C[\mu_\beta * (1+|\cdot|^{-N})](z)$ with any $N > n$.
This completes the proof of Proposition 2.1. □

Proof of Proposition 2.2. The tool here is the Hausdorff–Young inequality

$$\|\widehat{v}\|_p \leq C\|v\|_q, \tag{3.4}$$

valid for every $1 \leq q \leq 2 \leq p \leq \infty$ such that $1/p + 1/q = 1$. Hence, by (3.4), the change of variables $\xi t^{1/2} = \omega$, and the homogeneity of ℓ, we obtain

$$\begin{aligned}
&\|\partial^\gamma e^{t\Delta} u_0 - A\partial^\gamma \mathcal{L}G(t)\|_p^q \\
&\quad \leq C \int_{I\!R^n} \left| (i\xi)^\gamma e^{-t|\xi|^2} \ell(\xi) \left(\frac{\widehat{u}_0(\xi) - A\ell(\xi)}{\ell(\xi)} \right) \right|^q d\xi \\
&\quad = Ct^{-n/2-(\beta/2+|\gamma|/2)q} \int_{I\!R^n} \left| (i\omega)^\gamma e^{-|\omega|^2} \ell(\omega) \left(\frac{\widehat{u}_0(\omega/t^{1/2})}{\ell(\omega/t^{1/2})} - A \right) \right|^q d\xi.
\end{aligned}$$

Now, the assumptions on u_0 in (2.9) allow us to apply the Lebesgue Dominated Convergence Theorem in order to prove that the integral on the right hand side tends to 0 as $t \to \infty$. □

Proof of Theorem 2.1. We use systematically the integral equation (2.1) combined with inequality (2.2). First, note that since $u_0 \in L^1(I\!R^n) \cap L^q(I\!R^n)$, by (2.2), we obtain

$$\|u(\cdot,t)\|_q^q \le C(\|u_0\|_1, \|u_0\|_q)(1+t)^{-(n/2)(q-1)} \tag{3.5}$$

for all $t \ge 0$. Hence, computing the L^1-norm of (2.1), using the assumption on u_0, and (3.5) we obtain

$$\begin{aligned}
\|u(\cdot,t)\|_1 &\le \|e^{t\Delta}u_0\|_1 + \int_0^t \|a\cdot\nabla G(\cdot,t-\tau)\|_1 \|u(\cdot,t)\|_q^q\, d\tau \\
&\le Ct^{-\beta/2} + C\int_0^t (t-\tau)^{-1/2}(1+\tau)^{-(n/2)(q-1)}\, d\tau \\
&\le Ct^{-\beta/2} + C\begin{cases} t^{1/2-(n/2)(q-1)}, & \text{for} \quad q \in \left(1+\frac{1}{n}, 1+\frac{2}{n}\right); \\ t^{-1/2}\log(e+t), & \text{for} \qquad q = 1+\frac{2}{n}; \\ t^{-1/2}, & \text{for} \qquad q > 1+\frac{2}{n}. \end{cases}
\end{aligned} \tag{3.6}$$

Now, for $q \ge 1 + (\beta+1)/n$ we derive (2.11) immediately from (3.6), because $1/2 - (n/2)(q-1) \le -\beta/2$ in this range of q.
Next, we consider $1 + 1/n < q < 1 + (\beta+1)/n$. A simple calculation shows that $\alpha = -(1/2 - (n/2)(q-1))$ satisfies $0 < \alpha < \beta/2$. Moreover, it follows from (3.6) that

$$\|u(\cdot,t)\|_1 \le C(1+t)^{-\alpha}. \tag{3.7}$$

Combining inequality (2.2) with (3.7) we obtain the improved decay of the L^q-norm

$$\begin{aligned}
\|u(\cdot,t)\|_q &\le C(1+t/2)^{-(n/2)(1-1/q)}\|u(\cdot,t/2)\|_1 \\
&\le C(1+t)^{-(n/2)(1-1/q)-\alpha}.
\end{aligned} \tag{3.8}$$

Hence, repeating the calculations from (3.6), using (3.8) instead of (3.5), we obtain

$$\|u(\cdot,t)\|_1 \le Ct^{-\beta/2} + \int_0^t (t-\tau)^{-1/2}(1+\tau)^{-(n/2)(q-1)-q\alpha/2} d\tau. \tag{3.9}$$

If $-(n/2)(q-1) - q\alpha/2 \le -1$, the integral on the right hand side of (3.9) tends to 0 as $t \to \infty$ faster than $t^{-\beta/2}$ and this ends the proof. On the other hand, if $-(n/2)(q-1) - q\alpha/2 > -1$, by the definition of α, it follows from (3.9) that

$$\|u(\cdot,t)\|_1 \le Ct^{-\beta/2} + Ct^{-\alpha(q+1)}.$$

Hence, if $-\alpha(q+1) \leq -\beta/2$, the proof is finished. In the opposite case, we use the estimate

$$\|u(\cdot,t)\|_1 \leq C(1+t)^{-\alpha(q+1)}$$

in (3.8) (with α replaced by $\alpha(q+1)$) in order to get an improved decay of the L^q-norm. Consequently, after a finite number of repetition of the reasoning above, we obtain (2.11).

Finally, let us prove (2.11) for $1+1/(n+\beta) \leq q \leq 1+1/n$ under the assumption that $\sup_{t>0} \|e^{t\Delta}u_0\|_1$ is sufficiently small. For simplicity of notation, we put

$$q^* = 1 + \frac{1}{n+\beta},$$

and we use systematically the following inequality (obtained from the Hölder inequality and from (2.2))

$$\|u(\cdot,t)\|_q^q \leq \|u(\cdot,t)\|_{q^*}^{q^*} \|u(\cdot,t)\|_\infty^{q-q^*} \leq C(\|u_0\|_\infty)\|u(\cdot,t)\|_{q^*}^{q^*} \tag{3.10}$$

for all $t>0$. We also define a nonnegative continuous function

$$g(t) \equiv \sup_{0\leq\tau\leq t} \left(\tau^{\beta/2}\|u(\cdot,\tau)\|_1\right) + \sup_{0\leq\tau\leq t} \left(\tau^{(1/2+\beta/2)/q^*}\|u(\cdot,\tau)\|_{q^*}\right).$$

Now, computing the L^1-norm of the integral equation (2.1) and using (3.10) we obtain

$$\begin{aligned} t^{\beta/2}\|u(\cdot,t)\|_1 &\leq t^{\beta/2}\|e^{t\Delta}u_0\|_1 + Ct^{\beta/2}\int_0^t (t-\tau)^{-1/2}\|u(\cdot,\tau)\|_q^q \, d\tau \\ &\leq t^{\beta/2}\|e^{t\Delta}u_0\|_1 \\ &\quad + g^{q^*}(t)\, Ct^{\beta/2}\int_0^t (t-\tau)^{-1/2}\tau^{-1/2-\beta/2}\, d\tau \end{aligned} \tag{3.11}$$

for all $t>0$. An elementary calculation shows that the quantity

$$t^{\beta/2}\int_0^t (t-\tau)^{-1/2}\tau^{-1/2-\beta/2}\, d\tau$$

is finite for every $t>0$ (since $0<\beta<1$) and independent of t. A similar reasoning gives

$$\begin{aligned} \|u(\cdot,t)\|_{q^*} &\leq (t/2)^{-(n/2)(1-1/q^*)}\|e^{(t/2)\Delta}u_0\|_1 \\ &\quad + g^{q^*}(t)\, C\int_0^t (t-\tau)^{-(n/2)(1-1/q^*)-1/2}\tau^{-1/2-\beta/2}\, d\tau. \end{aligned} \tag{3.12}$$

Note now that $-((n/2)(1-1/q^*) - \beta/2 = -(1/2+\beta/2)/q^*$. Moreover, the quantity

$$t^{(1/2+\beta/2)/q^*}\int_0^t (t-\tau)^{-(n/2)(1-1/q^*)-1/2}\tau^{-1/2-\beta/2}\, d\tau$$

is finite (because $-(n/2)(1-1/q^*)-1/2>-1$) and independent of t (by the change of variables).
Combining inequalities (3.11) and (3.12) we obtain

$$g(t)\le C_1\sup_{0\le\tau}\tau^{\beta/2}\|e^{t\Delta}u_0\|_1+C_2g^{q^*}(t) \tag{3.13}$$

for all $t\ge 0$ and constants C_1 and C_2 independent of t.
Now, we consider the function

$$F(y)=A+C_2y^{q^*}-y \quad \text{where} \quad A=C_1\sup_{0\le\tau}\tau^{\beta/2}\|e^{t\Delta}u_0\|_1$$

and $q^*>1$. If $A>0$ is sufficiently small, there exists $y_0>0$ such that $F(y_0)=0$ and $F(y)>0$ if $y\in[0,y_0)$. Moreover, it follows from (3.13) that $F(g(t))\ge 0$. Since $g(t)$ is a nonnegative, continuous function such that $g(0)=0$, we deduce that $g(t)\in[0,y_0)$ for all $t\ge 0$. This completes the proof of Theorem 2.1. □

Proof of Corollary 2.1. We obtain (2.12) combining inequality (2.2) with (2.11) as in (3.8) replacing q by p and α by $\beta/2$.
To prove (2.13), in view of the integral equation (2.1), it suffices to estimate the L^p-norm of the second term on the right hand side of (2.1). Here, we split the integration range with respect to τ into $[0,t/2]\cup[t/2,t]$ and we study each term separately as follows. Using the Young inequality for the convolution and (2.12) we obtain

$$\begin{aligned}
&\int_0^{t/2}\|a\cdot\nabla e^{(t-\tau)\Delta}(u|u|^{q-1})(\tau)\|_p\,d\tau\\
&\le \int_0^{t/2}\|a\cdot\nabla G(\cdot,t-\tau)\|_p\|u(\cdot,\tau)\|_q^q\,d\tau\\
&\le C\int_0^{t/2}(t-\tau)^{-(n/2)(1-1/p)-1/2}(1+\tau)^{-(n/2)(q-1)-\beta q/2}\,d\tau\\
&\le C\begin{cases} t^{-(n/2)(q-1/p)-(\beta q-1)/2} & \text{for}\quad q\in\left(1+\frac{1}{n+\beta},\frac{n+2}{n+\beta}\right),\\ t^{-(n/2)(1-1/p)-1/2}\log(e+t) & \text{for}\quad q=\frac{n+2}{n+\beta},\\ t^{-(n/2)(1-1/p)-1/2} & \text{for}\quad q>\frac{n+2}{n+\beta}\end{cases}
\end{aligned} \tag{3.14}$$

for all $t>0$.
A similar calculation gives

$$\begin{aligned}
&\int_{t/2}^{t}\|a\cdot\nabla e^{(t-\tau)\Delta}(u|u|^{q-1})(\tau)\|_p\,d\tau\\
&\qquad\le \int_{t/2}^{t}(t-\tau)^{-1/2}\|u(\cdot,\tau)\|_{pq}^q\,d\tau
\end{aligned} \tag{3.15}$$

$$
\begin{aligned}
&\le C\int_{t/2}^{t}(t-\tau)^{-1/2}(1+\tau)^{-(n/2)(q-1/p)-\beta q/2}\,d\tau\\
&\le Ct^{-(n/2)(q-1/p)-(\beta q-1)/2}
\end{aligned}
$$

for all $t>0$.
Finally, combining (3.14) and (3.15) (note that $-(n/2(q-1/p)-(\beta q-1)/2 \le -(n/2)(1-1/p)-1/2$ for $q\ge (n+2)/(n+\beta)$) we obtain (2.13). $\square$

4 Nonlinear asymptotics

The following two lemmata give the crucial estimates for the integral equation (2.1) systematically used in the proof of Theorem 2.2.

Lemma 4.1 *Let $a\in I\!\!R^n$ be a fixed constant vector. There exists a constant $C>0$ such that for every $w\in L^1(I\!\!R^n)$ we have*

$$
\|a\cdot\nabla e^{t\Delta}w\|_{\mathcal{B}_1^{-\beta,\infty}} \le Ct^{(\beta-1)/2}\|w\|_1 \tag{4.1}
$$

for all $t>0$.

Proof. Here, we use the definition of the norm in $\mathcal{B}_1^{-\beta,\infty}$ and properties of the heat semigroup as follows

$$
\begin{aligned}
\|a\cdot\nabla e^{t\Delta}w\|_{\mathcal{B}_1^{-\beta,\infty}} &= \sup_{s>0} s^{\beta/2}\|e^{s\Delta}a\cdot\nabla e^{t\Delta}w\|_1\\
&= \sup_{s>0} s^{\beta/2}\|a\cdot\nabla e^{(t+s)\Delta}w\|_1\\
&\le C\|w\|_1\sup_{s>0} s^{\beta/2}(t+s)^{-1/2}
\end{aligned}
$$

for all $t>0$. Now, a direct calculation shows that $\sup_{s>0}s^{\beta/2}(t+s)^{-1/2} = C(\beta)t^{(\beta-1)/2}$ with $C(\beta)$ independent of t. $\square$

Lemma 4.2 *Assume that $v\in\mathcal{B}_1^{-\beta,\infty}$. Then for each $p\in[1,\infty]$ there exists a constant $C>0$ such that*

$$
\|e^{t\Delta}v\|_p \le Ct^{-(n/2)(1-1/p)-\beta/2}\|v\|_{\mathcal{B}_1^{-\beta,\infty}}
$$

for all $t>0$.

Proof. Standard properties of the heat semigroup $e^{t\Delta}$ and the definition of the norm in $\mathcal{B}_1^{-\beta,\infty}$ give

$$\|e^{t\Delta}v\|_p \le C(t/2)^{-(n/2)(1-1/p)}\|e^{(t/2)\Delta}v\|_1 \le Ct^{-(n/2)(1-1/p)-\beta/2}\|v\|_{\mathcal{B}_1^{-\beta,\infty}}.$$

for all $t>0$ and a constant C. □

Proof of Theorem 2.2. Our reasoning is similar to that in [4, 5, 6, 18]. Moreover, the calculations below resemble those in the proof of Theorem 2.1 with $1+1/(n+\beta)\le q\le 1+1/n$, thus we shall be brief in details. Recall that

$$q=q^*=1+\frac{1}{n+\beta}$$

which is equivalent to

$$\frac{n}{2}\left(1-\frac{1}{q}\right)+\frac{\beta}{2}=\frac{1}{q}\left(\frac{1}{2}+\frac{\beta}{2}\right).$$

We equip the space $\mathcal{X}$ with the norm

$$\|u\|_{\mathcal{X}}=\max\{\sup_{t>0}\|u(t)\|_{\mathcal{B}_1^{-\beta,\infty}},\ \sup_{t>0}t^{(n/2)(1-1/q)+\beta/2}\|u(t)\|_q\}$$

and we show that the nonlinear operator

$$\mathcal{N}(u)(t)\equiv e^{t\Delta}u_0-\int_0^t a\cdot\nabla e^{(t-\tau)\Delta}(u|u|^{q-1})(\tau)\,d\tau \tag{4.2}$$

is a contraction on the box

$$B_{R,\varepsilon}=\{u\in\mathcal{X}\ :\ \|u(t)\|_{\mathcal{B}_1^{-\beta,\infty}}\le R\ \text{ and }\ \sup_{t>0}t^{(n/2)(1-1/q)+\beta/2}\|u(t)\|_q\le 2\varepsilon\}$$

for sufficiently large $R>0$ and a suitably small $\varepsilon>0$. This will be guaranteed provided we shall prove the following estimates

$$\|\mathcal{N}(u)(t)\|_{\mathcal{B}_1^{-\beta,\infty}} \le \|u_0\|_{\mathcal{B}_1^{-\beta,\infty}}+C\varepsilon^q, \tag{4.3}$$

$$t^{(n/2)(1-1/q)+\beta/2}\|\mathcal{N}(u)(t)\|_q \le C\|u_0\|_{\mathcal{B}_1^{-\beta,\infty}}+C\varepsilon^q, \tag{4.4}$$

and

$$\begin{aligned}&\|\mathcal{N}(u)(t)-\mathcal{N}(v)(t)\|_{\mathcal{B}_1^{-\beta,\infty}}\\&\qquad\le C\varepsilon^{q-1}\sup_{t>0}t^{(n/2)(1-1/q)+\beta/2}\|u(\cdot,t)-v(\cdot,t)\|_q\end{aligned} \tag{4.5}$$

$$\begin{aligned}&t^{(n/2)(1-1/q)+\beta/2}\|\mathcal{N}(u)(t)-\mathcal{N}(v)(t)\|_q\\&\qquad\le C\varepsilon^{q-1}\sup_{t>0}t^{(n/2)(1-1/q)+\beta/2}\|u(\cdot,t)-v(\cdot,t)\|_q.\end{aligned} \tag{4.6}$$

with constants C independent of u and t.
For the proof of (4.3) observe that $\|e^{t\Delta}u_0\|_{\mathcal{B}_1^{-\beta,\infty}} \le \|u_0\|_{\mathcal{B}_1^{-\beta,\infty}}$. Hence computing the $\mathcal{B}_1^{-\beta,\infty}$-norm of (4.2) for $u \in B_{R,\varepsilon}$ and applying Lemma 4.1 we obtain

$$\begin{aligned}
\|\mathcal{N}(u)(t)\|_{\mathcal{B}_1^{-\beta,\infty}} &\le \|e^{t\Delta}u_0\|_{\mathcal{B}_1^{-\beta,\infty}} + \int_0^t \|a\cdot\nabla e^{(t-\tau)\Delta}(u|u|^{q-1})(\tau)\|_{\mathcal{B}_1^{-\beta,\infty}}\, d\tau \\
&\le \|u_0\|_{\mathcal{B}_1^{-\beta,\infty}} + C\int_0^t (t-\tau)^{(\beta-1)/2}\|u(\tau)\|_q^q\, d\tau \\
&\le \|u_0\|_{\mathcal{B}_1^{-\beta,\infty}} + C\varepsilon^q \int_0^t (t-\tau)^{(\beta-1)/2}\tau^{-(n/2)(q-1)-\beta q/2}\, d\tau.
\end{aligned}$$

Note now that the assumptions $\beta \in (0,1)$ and $q = 1 + 1/(n+\beta)$ guarantee that the integral on the right hand side is finite for any $t > 0$. Moreover, since $(\beta-1)/2 - n(q-1)/2 - \beta q/2 + 1 = 0$, it follows that this integral is independent of t. Hence, estimate (4.3) holds true.
The proof of (4.4) is similar. It involves Lemma 4.2 as follows

$$\begin{aligned}
\|\mathcal{N}(u)(t)\|_q &\le \|e^{t\Delta}u_0\|_q + \int_0^t \|a\cdot\nabla e^{(t-\tau)\Delta}(u|u|^{q-1})(\tau)\|_q\, d\tau \\
&\le Ct^{-(n/2)(1-1/q)-\beta/2}\|u_0\|_{\mathcal{B}_1^{-\beta,\infty}} \\
&\quad + C\varepsilon^q \int_0^t (t-\tau)^{-(n/2)(1-1/q)-1/2}\tau^{-(n/2)(q-1)-\beta q/2}\, d\tau.
\end{aligned} \tag{4.7}$$

In this case, the conditions on β, q imply again that the integral on the right hand side is finite for every $t > 0$. In fact, by a change of variables, it equals $Ct^{-(n/2)(1-1/p)-\beta/2}$ for a constant $C > 0$. Hence (4.4) is proved.
The proofs of (4.5) and (4.6) are completely analogous. The only difference consists in using elementary inequality

$$\left\| u|u|^{q-1} - v|v|^{q-1} \right\|_1 \le C\|u-v\|_q \left(\|u\|_q^{q-1} + \|v\|_q^{q-1}\right) \tag{4.8}$$

valid for all $u, v \in L^q(I\!R^n)$.
Finally, it follows from (4.3)–(4.6) that $\mathcal{N} : B_{R,\varepsilon} \to B_{R,\varepsilon}$ is a contraction for $R > 2\|u_0\|_{\mathcal{B}_1^{-\beta,\infty}}$ and a suitably small $\varepsilon > 0$. Hence the sequence defined as $u_0(t) = e^{t\Delta}u_0$ and $u_{n+1}(t) = \mathcal{N}(u_n(t))$ converges to a unique (in $B_{R,\varepsilon}$) global-in-time solution to (1.1)-(1.2) provided $u_0(t) \in B_{R,\varepsilon}$, i.e. $\|u_0\|_{\mathcal{B}_1^{-\beta,\infty}} < \varepsilon$.

□

The proof of Theorem 2.3 requires the following result from [18, Lemma 6.1].

Lemma 4.3 *Let $w \in L^1(0,1)$, $w \ge 0$, and $\int_0^1 w(x)\, dx < 1$. Assume that f and g are two nonnegative, bounded functions such that*

$$f(t) \le g(t) + \int_0^1 w(\tau) f(\tau t)\, d\tau. \tag{4.9}$$

Then $\lim_{t\to\infty} g(t) = 0$ *implies* $\lim_{t\to\infty} f(t) = 0.$ □

Proof of Theorem 2.3. The subtraction of equation (2.1) for v from the analogous expression for u leads to the following identity

$$
\begin{aligned}
u(t) - v(t) &= e^{t\Delta}(u_0 - v_0) \\
&\quad - \int_0^t a\cdot\nabla e^{(t-\tau)\Delta}\left(u|u|^{q-1} - v|v|^{q-1}\right)(\tau)\, d\tau.
\end{aligned} \tag{4.10}
$$

Next, repeating the reasoning from the proof of (4.4) involving inequality (4.8) we estimate

$$
\begin{aligned}
&\|u(\cdot,t) - v(\cdot,t)\|_q \\
&\quad\le Ct^{-(n/2)(1-1/q)-\beta/2}\left((t/2)^{\beta/2}\|e^{(t/2)\Delta}(u_0 - v_0)\|_1\right) \\
&\quad\quad +C\int_0^t (t-\tau)^{-(n/2)(1-1/q)-1/2}\|u(\cdot,\tau) - v(\cdot,\tau)\|_q \\
&\quad\quad\quad \times\left(\|u(\cdot,\tau)\|_q^{q-1} + \|v(\cdot,\tau)\|_q^{q-1}\right)\, d\tau.
\end{aligned} \tag{4.11}
$$

By Theorem 2.2, the both quantities

$$
\sup_{t>0} t^{(n/2)(1-1/q)+\beta/2}\|u(\cdot,t)\|_q \quad\text{and}\quad \sup_{t>0} t^{(n/2)(1-1/q)+\beta/2}\|v(\cdot,t)\|_q
$$

are bounded by 2ε. Hence, multiplying (4.11) by $t^{(n/2)(1-1/q)+\beta/2}$, putting

$$
f(t) = t^{(n/2)(1-1/q)+\beta/2}\|u(\cdot,t) - v(\cdot,t)\|_q, \tag{4.12}
$$

and changing variable $\tau = ts$, we get

$$
\begin{aligned}
f(t) &\le C(t/2)^{\beta/2}\|e^{(t/2)\Delta}(u_0 - v_0)\|_1 \\
&\quad +2C\varepsilon^{q-1}\int_0^1 (1-s)^{-(n/2)(1-1/q)-1/2}s^{-(n/2)(q-1)-\beta q/2}f(ts)\, ds.
\end{aligned} \tag{4.13}
$$

Since $(1-s)^{-(n/2)(1-1/q)-1/2}s^{-(n/2)(q-1)-\beta q/2} \in L^1(0,1)$ (cf. comments following inequalities (4.7)), we may apply Lemma 4.3 obtaining $f(t)\to 0$ as $t\to\infty$ for sufficiently small $\varepsilon > 0$. This proves (2.19) for $p = q$.
Next, we prove (2.19) for $p = 1$. Computing the L^1-norm of (4.10) and repeating the calculations from (4.11) and (4.13) yield

$$
\begin{aligned}
t^{\beta/2}\|u(\cdot,t) - v(\cdot,t)\|_1 &\le t^{\beta/2}\|e^{t\Delta}(u_0 - v_0)\|_1 \\
&\quad + C\int_0^1 (1-s)^{-1/2}s^{-(n/2)(q-1)-\beta q/2}f(ts)\, ds,
\end{aligned}
$$

where f, defined in (4.12), is a bounded function satisfying $\lim_{t\to\infty} f(t) = 0$, by the first part of this proof. Hence (2.18) and the Lebesgue Dominated Convergence Theorem give

$$\lim_{t\to\infty} t^{\beta/2}\|u(\cdot,t)-v(\cdot,t)\|_1 = 0. \tag{4.14}$$

The next stage of the proof deals with (2.19) for all $p \in (1,\infty)$. The calculations from (3.8) show that $\|u(\cdot,t)\|_\infty$ and $\|v(\cdot,t)\|_\infty$ can be both bounded by $Ct^{-n/2-\beta/2}$ for all $t > 0$ and a constant C independent of t. Hence, by the Hölder inequality and (4.14) it follows that

$$\begin{aligned} \|u(\cdot,t)-v(\cdot,t)\|_p &\le C\|u(\cdot,t)-v(\cdot,t)\|_1^{1/p} \\ &\quad \times \left(\|u(\cdot,t)\|_\infty^{1-1/p} + \|v(\cdot,t)\|_\infty^{1-1/p}\right) \\ &= o\left(t^{-(n/2)(1-1/p)-\beta/2}\right) \quad \text{as} \quad t\to\infty, \end{aligned}$$

where we used the following inequality

$$\left|g|g|^{q-1} - h|h|^{q-1}\right| \le C(q)|g-h|\left(|g|^{q-1}+|h|^{q-1}\right) \tag{4.15}$$

valid for all $g, h \in I\!R$, $q > 1$, and $C(q)$ depending on q, only.
Finally, the proof of (2.19) for $p = \infty$ involves equation (4.10) and (2.19) proved already for all $p \in [1,\infty)$. Standard $L^p - L^q$ estimates of the of the heat semigroup imply that

$$\begin{aligned} t^{n/2+\beta/2}\|e^{t\Delta}(u_0-v_0)\|_\infty &\le Ct^{n/2+\beta/2}(t/2)^{-n/2}\|e^{(t/2)\Delta}(u_0-v_0)\|_1 \\ &= C(t/2)^{\beta/2}\|e^{(t/2)\Delta}(u_0-v_0)\|_1 \to 0 \end{aligned}$$

as $t\to\infty$ by assumption (2.18).
To study the second term on the right hand side of (4.10), the integration range with respect to τ is decomposed into $[0,t] = [0,t/2]\cup[t/2,t]$.
Combining inequality (4.15) with estimates of the heat semi-group and the Hölder inequality yields

$$\begin{aligned} &\left\|a\cdot\nabla e^{(t-\tau)\Delta}\left(u|u|^{q-1}-v|v|^{q-1}\right)(\tau)\right\|_\infty \\ &\le C(t-\tau)^{-n/2-1/2}\|u(\tau)-v(\tau)\|_1\left(\|u(\tau)\|_\infty^{q-1}+\|v(\tau)\|_\infty^{q-1}\right) \\ &\le C(t-\tau)^{-n/2-1/2}\tau^{-\beta/2-(n+\beta)(q-1)/2}f_1(\tau), \end{aligned} \tag{4.16}$$

where C is independent of t and τ, and $f_1(\tau) = \tau^{\beta/2}\|u(\tau)-v(\tau)\|_1$ is the bounded function which tends to 0 as $t\to\infty$ by (2.19) for $p = 1$.

Moreover, choosing $1/r + 1/z = 1$, similar calculations lead to

$$\begin{aligned} &\left\| a \cdot \nabla e^{(t-\tau)\Delta} \left(u|u|^{q-1} - v|v|^{q-1} \right)(\tau) \right\|_\infty \\ &\qquad \le C(t-\tau)^{-(n/2)(1-1/z)-1/2} \tau^{-(n/2)(1-1/r)-\beta/2-(n+\beta)(q-1)/2} f_r(\tau) \end{aligned} \tag{4.17}$$

where $f_r(\tau) = \tau^{(n/2)(1-1/r)+\beta/2}\|u(\tau) - v(\tau)\|_r$ also tends to 0 as $t \to \infty$ by (2.19). Hence, by the change of variables $\tau = ts$, it follows from (4.16) that

$$\begin{aligned} &\int_0^{t/2} \left\| a \cdot \nabla e^{(t-\tau)\Delta} \left(u|u|^{q-1} - v|v|^{q-1} \right)(\tau) \right\|_\infty \, d\tau \\ \le \quad & Ct^{-n/2-\beta/2} \int_0^{1/2} (1-s)^{-n/2-1/2} s^{-\beta/2-(n+\beta)(q-1)/2} f_1(st) \, ds. \end{aligned}$$

The integral on the right hand side is finite (recall that $q = 1 + 1/(n + \beta)$), because

$$-\frac{\beta}{2} - \frac{(n+\beta)(q-1)}{2} = -\frac{\beta+1}{2} > -1 \quad \text{for } \beta \in (0,1).$$

This integral tends to 0 as $t \to \infty$ by the Lebesgue Dominated Convergence Theorem.
The case of the integral $\int_{t/2}^t \dots d\tau$ involves inequality (4.17) with $z > 1$ chosen such that $-(n/2)(1 - 1/z) - 1/2 > -1$. The proof here is analogous as in the last case and as such will be omitted. This completes the proof of Theorem 2.3. □

ACKNOWLEDGEMENTS

Part of this research was done while G.K. was invited to Department of Mathematics, University of California, Santa Cruz, in September 1999. During the preparation of the paper, G.K. was partially supported by the Foundation for Polish Science. Grant support from KBN 0050/P03/2000/18 is gratefully acknowledged.

References

[1] P. Biler, G. Karch, & W.A. Woyczyński, *Multifractal and Lévy conservation laws*, C. R. Acad. Sci. Paris, Ser. I, **330** (2000), 343-348.

[2] P. Biler, G. Karch, & W.A. Woyczyński, *Asymptotics for conservation laws involving Lévy diffusion generators,* to appear in Studia Math. `http://www.math.uni.wroc.pl/~karch`

[3] P. Biler, G. Karch, & W.A. Woyczyński, *Critical nonlinearity exponent and self-similar asymptotics for Lévy conservation laws,* to appear in Ann. Inst. H. Poincaré, Analyse non linéaire.
`http://www.math.uni.wroc.pl/~karch`

[4] M. Cannone, "Ondelettes, paraproduits et Navier–Stokes", Diderot Editeur, Arts et Sciences, Paris, New York, Amsterdam, 1995.

[5] M. Cannone, *A generalization of a theorem by Kato on Navier-Stokes equations,* Rev. Mat. Iberoamericana **13** (1997), 515–541.

[6] M. Cannone & F. Planchon, *Self-similar solutions for Navier-Stokes equations in* $\mathbb{R}^3$*,* Comm. Partial Differential Equations **21** (1996), 179–194.

[7] E.A. Carlen & M. Loss, *Optimal smoothing and decay estimates for viscously damped conservation laws, with applications to the 2-D Navier–Stokes equation*, Duke Math. J. **86** (1996), 135–157.

[8] I.-Liang Chern & Tai-Ping Liu, *Convergence to diffusion waves of solutions for viscous conservation laws,* Comm. Math. Phys. **110** (1987), 503–517.

[9] J. Duoandikoetxea & E. Zuazua, *Moments, masses de Dirac et décomposition de fonctions,* C. R. Acad. Sci. Paris, Sér. I **315** (1992), 693–698.

[10] G. Duro & E. Zuazua, *Large time behavior for convection-diffusion equations in* $\mathbb{R}^N$ *with asymptotically constant diffusion,* Comm. Partial Differential Equations **24** (1999), 1283–1340.

[11] G. Duro & E. Zuazua, *Large time behavior for convection-diffusion equations in* $\mathbb{R}^N$ *with periodic coefficients,* J. Differential Equations **167** (2000), 275–315.

[12] M. Escobedo & E. Zuazua, *Large time behavior for convection-diffusion equations in* $\mathbb{R}^N$*,* J. Funct. Anal. **100** (1991), 119–161.

[13] M. Escobedo, J.L. Vázquez, & E. Zuazua, *Asymptotic behavior and source-type solutions for a diffusion-convection equation*, Arch. Rat. Mech. Anal. **124** (1993), 43–65.

[14] M. Escobedo, J.L. Vázquez, & E. Zuazua, *A diffusion-convection equation in several space dimensions*, Indiana Univ. Math. J. **42** (1993), 1413–1440.

[15] D. Hoff & K. Zumbrun, *Multidimensional diffusion waves for the Navier-Stokes equations of compressible flow,* Indiana Univ. Math. J., **44** (1995), 603–676.

[16] E. Hopf, *The partial differential equation* $u_t + uu_x = \mu u_{xx}$*,* Comm. Pure Appl. Math. 3 (1950), 201–230.

[17] A. Jeffrey & H. Zhao, *Global existence and optimal temporal decay estimates for systems of parabolic conservation laws. II. The multidimensional case,* J. Math. Anal. Appl. **217** (1998), 597–623.

[18] G. Karch, *Scaling in nonlinear parabolic equations,* J. Math. Anal. Appl. **234** (1999), 534–558.

[19] G. Karch, *Self-similar large time behavior of solutions to Korteweg-de Vries-Burgers equation,* Nonlinear Analysis 35 (1999), 199–219.

[20] G. Karch, *Large-time behavior of solutions to non-linear wave equations: higher-order asymptotics,* Math. Methods Appl. Sci. **22** (1999), 1671–1697.

[21] S. Kawashima, *Large-time behavior of solutions to hyperbolic-parabolic systems of conservation laws and applications,* Proc. Royal Soc. Edinburgh **106A** (1987), 169–194.

[22] T. Miyakawa, *Application of Hardy space techniques to the time-decay problem for incompressible Navier-Stokes flows in* $I\!R^n$*,* Funkc. Ekv. **41** (1998) 383–434.

[23] T. Miyakawa, *Hardy spaces of solenoidal vector fields, with applications to the Navier-Stokes equations*, Kyushu J. Math., **50** (1996), 1–64.

[24] F. Planchon, *Global strong solutions in Sobolev or Lebesgue spaces for the incompressible Navier-Stokes equations in* $I\!R^n$*,* Ann. Inst. H. Poincaré, Analyse non linéaire **13** (1996), 319–336.

[25] M.E. Schonbek, *Decay of solutions to parabolic conservation laws,* Comm. Partial Differential Equations **5** (1980), 449–473.

[26] M.E. Schonbek, *Uniform decay rates for parabolic conservation laws,* Nonlinear Analysis, Theory, Methods & Applications **10** (1986), 943-956.

[27] M.E. Schonbek, *Large time behavior of solutions to the Navier-Stokes equations*, Comm. Partial Differential Equations **11** (1986), 733-763.

[28] M.E. Schonbek & E. Süli, *Decay of total variation and Hardy norms of solutions to parabolic conservation laws,* to appear in Nonlinear Analysis.

[29] E.M. Stein, "Singular Integrals and Differentiability Properties of Functions", Princeton University Press,Princeton, New Jersey, 1970.

[30] E.M. Stein, "Harmonic Analysis: Real-Variable Methods, Orthogonality, and Oscillatory Integrals", Princeton University Press, Princeton, New Jersey, 1993.

[31] M. Wiegner, *Decay results for weak solutions of the Navier-Stokes equations on* $\mathbb{R}^n$*,* J. London Math. Soc. **35** (1987), 303-313.

[32] E. Zuazua, *Weakly nonlinear large time behavior in scalar convection-diffusion equations*, Differential Integral Equations **6** (1993), 1481–1491.