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MOMENTS AND LOWER BOUNDS IN THE FAR-FIELD OF SOLUTIONS TO QUASI-GEOSTROPHIC FLOWS

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Abstract. We consider the long time behavior of moments of solutions and of the solutions itself to dissipative Quasi-Geostrophic flow (QG) with sub-critical powers. The flow under consideration is described by the nonlinear scalar equation

$$\frac{\partial \theta}{\partial t} + u \cdot \nabla \theta + \kappa(-\triangle)^{\alpha}\theta = f,$$
$$\theta|_{t=0} = \theta_0$$

Rates of decay are obtained for moments of the solutions, and lower bounds of decay rates of the solutions are established.

1. Introduction.

We consider the solutions to the surface 2D dissipative Quasi-Geostrophic flows (DQG) with sub-critical powers α

$$\frac{\partial \theta}{\partial t} + u \cdot \nabla \theta + \kappa(-\triangle)^{\alpha}\theta = 0,$$
$$\theta|_{t=0} = \theta_0$$

Here $\alpha \in (0, 1]$, $\kappa > 0$, $\theta(t)$ is a real function of two space variables $x \in \mathbb{R}^2$ and a time variable t. The function $\theta(t) = \theta(x, t)$ represents the potential temperature. The fluid velocity u is determined from θ by a stream function ψ

$$(u_1, u_2) = (-\frac{\partial \psi}{\partial x_2}, \frac{\partial \psi}{\partial x_1}) \tag{1}$$

where the function ψ satisfies

$$(-\triangle)^{\frac{1}{2}}\psi = -\theta$$

Equation (1) is obtained when dissipative mechanisms are incorporated into the inviscid 2D-Quasi-Geostrophic equation (2DQG). The 2DQG is derived from the General Quasi Geostrophic (GQG) equations by reduction to the special case of solutions with constant potential vorticity in the interior and constant buoyancy

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frequency [3]. For information on the GQG equations we refer the reader to [9]. The fractional power $\alpha = 1/2$ is perhaps the most interesting one since it corresponds to a fundamental model of quasi-geostrophic equations, see [4] and [9]. As pointed out in [4] "Dimensionally the 2DQG equation with $\alpha = 1/2$ is the analogue of the 3D Navier-Stokes equations."

Two main questions will be addressed in this paper, provided $1/2 \leq \alpha \leq 1$; decay of specific moments of the solutions of (1) and lower bounds of rates of decay of the solutions in L^2 with data with zero or non zero total mass. The results will be presented in the frame work of $1/2 < \alpha \leq 1$. Due to the bounds obtained in [2] we believe that all the results presented here can be easily extended to the case $\alpha = 1/2$, provided the data is sufficiently small

We consider the moments of the solutions with data in appropriate weighted spaces satisfying $|\widehat{\theta}_0(\xi)| \leq C|\xi|^{\mu}$, where $0 \leq \mu \leq 1$. It will be shown that the moments of order one of the temperature with such data, decay in norm L^r, with $2 \leq r < \frac{1}{\alpha-1}$ like

$$\|x_j\theta(t)\|_r \leq C_0(1+t)^{-\lambda_r}, \text{ where } \lambda_r = \frac{1}{\alpha r'} + \frac{\mu-1}{2\alpha}$$

and the corresponding velocity moments for $2 < r < \frac{1}{1-\alpha}$ decay like

$$\|x_j u(t)\|_r \ \leq C_0(1+t)^{-\tau}, \text{ where } \ \tau = \min\{\lambda_r, (\frac{1}{\alpha}+\mu)(\frac{1}{r'}-\frac{1}{2})\}.$$

The decay of the moments of the velocity will be improved if in addition it holds that $I_\beta\theta_0 \in L^1(\mathbb{R}^2)$ for some $\beta > 0$, where I_β is the Riesz potential.

Once this decay is established it will be used to obtain lower bounds of rates of decay of the solutions to (1). The techniques to establish lower bounds are based on the ones used for the lower bounds of rates of decay for solutions to the Navier-Stokes equations ([6, 8]). In this direction the main result established is

$$\lim_{t\to\infty} t^{1/\alpha}\|\theta(t) - \Theta_0(t)\|_2 = 0. \tag{2}$$

where $\Theta_0(t)$ is the solution of the "linear diffusive" part of the equation:

$$\frac{\partial\theta}{\partial t} + \kappa(-\triangle)^{\alpha}\theta = 0,$$
$$\theta|_{t=0} = \theta_0$$

In what follows we will refer to the above equation as the "linear" one. An easy corollary from this last result is that solutions to the geostrophic equation have similar lower bounds than solutions to the "linear diffusive" equation. In particular one can show that

$$\|\theta(t)\|_2 \geq C_0(1+t)^{-1/\alpha},$$

where the constant C_0 will depend on norms of the datum. The class of solutions we consider, will include solutions such that $\widehat{\theta}(0) = 0$ as well as solutions with data such that $\widehat{\theta}(0) \neq 0$. The case when

$$\widehat{\theta}(\xi) \geq \lambda > 0, \text{ for } |\xi| \leq \gamma \tag{3}$$

was already considered by Constantin and Wu in [4]. This case is an easy corollary of our results.

The paper consists of an introduction, four sections and two appendices. In Section 2 notation is described and several preliminary results are mentioned. Section

3 considers the existence and decay of the moments. Section 4 has the main result describing the asymptotics of the solutions compared with the "linear" equation. As a corollary we obtain lower bounds of the rates of decay. The appendices contain several results, mostly with simple proofs, included to make the paper self-contained.

We note that under appropriate conditions for a forcing term f, many of the results obtained in this paper can be extended to treat the solutions of

$$\frac{\partial \theta}{\partial t} + u \cdot \nabla \theta + \kappa (-\triangle)^{\alpha} \theta = f,$$
$$\theta|_{t=0} = \theta_0$$

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2. **Notation and Preliminaries.** The Fourier transform of $v \in \mathcal{S}(\mathbb{R}^2)$ is defined by

$$\widehat{v}(\xi) = (2\pi)^{-1} \int_{\mathbb{R}^2} e^{-ix\cdot\xi} v(x)\, dx,$$

extended as usual to $\mathcal{S}'$. For a function $v : \mathbb{R}^2 \to \mathbb{C}$ and a multi-index $\gamma = (\gamma_1, \gamma_2)$, $D^\gamma v$ denotes derivation of order γ with respect to the two (space) variables. If v also depends on time, the symbol D_t^j is used to denote j derivatives with respect to t. We denote by $x^\gamma v$ (with some abuse of language) the function whose value at x is $x^\gamma v(x)$. Similarly, $x_j f$ is the function whose value at x is $x_j f(x)$, $j = 1, 2$.

If k is a nonnegative integer, $W^{k,p}(\mathbb{R}^2)$ will be, as is standard, the Sobolev space consisting of functions in $L^p(\mathbb{R}^2)$ whose generalized derivatives up to order k belong to $L^p(\mathbb{R}^2)$. As usual, when $p = 2$, then $W^{k,2}(\mathbb{R}^2) = H^k(\mathbb{R}^2)$ where the space H^s is defined for all $s \in \mathbb{R}$ as the space of all $f \in \mathcal{S}'$ such that $(1 + |\xi|^2)^{s/2} \hat{f}(\xi) \in L^2$.

Let $1 \leq r < \infty$. The spaces L^r_j, $j = 1, 2$, will denote the weighted spaces $L^r(\mathbb{R}^2, |x_j|^r\, dx)$; i.e., the spaces of all measurable functions f defined on $\mathbb{R}^2$ such that

$$\|f\|^r_{r,j} = \int_{\mathbb{R}^2} |x_j|^r |f(x)|^r\, dx < \infty.$$

We define the space L^r_w by

$$L^r_w = L^r(\mathbb{R}^2) \cap L^r_1 \cap L^r_2 = \{f \,:\, \int_{\mathbb{R}^2} (1 + |x|^r) |f(x)|^r\, dx < \infty\}.$$

Following Constantin and Wu [4], we denote by

$$\Lambda = (-\triangle)^{\frac{1}{2}}$$

the operator defined by $\widehat{\Lambda f}(\xi) = |\xi| \hat{f}(\xi)$. More generally, if $s \geq 0$, we define Λ^s by

$$\widehat{\Lambda^s f}(\xi) = |\xi|^s \hat{f}(\xi).$$

Clearly $\Lambda^s f$ is well defined (and in L^2) if $f \in H^s$. More generally, the domain of Λ^s will consist of all elements $f \in \mathcal{S}'$ such that $\hat{f}$ is a function (i.e., locally integrable); it is then clear that the definition given above defines $\Lambda^s f$ as a tempered distribution.

We denote by $\mathcal{R}_1, \mathcal{R}_2$ the Riesz-transforms in $\mathbb{R}^2$; i.e., $\widehat{\mathcal{R}_j f}(\xi) = -i(\xi_j/|\xi|)\hat{f}(\xi)$. The operator $\mathcal{R}^\perp$ taking scalar valued functions to vector valued functions is defined by

$$\mathcal{R}^\perp f = (-\partial_{x_2} \Lambda^{-1} f, \partial_{x_1} \Lambda^{-1} f) = (-\mathcal{R}_2 f, \mathcal{R}_1 f). \tag{4}$$

The relation between u and θ in (1) can then briefly be stated as $u = \mathcal{R}^\perp \theta$.

If F is a function defined on $\mathbb{R}^2 \times [0,\infty)$, we define for $t \geq 0$ the function $F(t)$ on $\mathbb{R}^2$ by $F(t)(x) = F(x,t)$. For such F, the Fourier transform (and inverse Fourier transform) is always with respect to the space variables; thus

$$\hat{F}(\xi,t) = \widehat{F(t)}(\xi)$$

for all $t \geq 0$. The letters C, C_0, C_1, etc., will denote generic positive constants, which may vary from expression to expression during computations.

Let $0 < \alpha \leq 1$. We collect here a few formulas and results concerning the operator $(-\Delta)^\alpha$ and the semi-group it generates. Most of the proofs are omitted and will be presented in the Appendix.

The following notation is used throughout. We let $K_\alpha : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{C}$ be given by

$$K_\alpha(x,t) = (2\pi)^{-2} \int_{\mathbb{R}^2} e^{ix\cdot\xi} e^{-t|\xi|^{2\alpha}}\, d\xi;$$

i.e., by

$$\hat{K}_\alpha(\xi,t) = (2\pi)^{-1} e^{-t|\xi|^{2\alpha}}.$$

Then

Lemma 1. *Let β, γ be multi-indices, $|\gamma| < |\beta| + 2\alpha \max(j,1)$, $j = 0,1,2,\ldots$, $1 \leq p \leq \infty$. Then*

$$\|x^\gamma D_t^j D^\beta K_\alpha(t)\|_p = C t^{\frac{|\gamma|-|\beta|}{2\alpha} - j - \frac{p-1}{\alpha p}}$$

for some constant C depending only on α, β, γ, j, p.

Proof. This is the case $n = 2$ of Lemma 6 proved in the Appendix. □

In particular, we see that $K_\alpha(t) \in L^1(\mathbb{R}^2)$ for all $t > 0$ and

$$\|K_\alpha(t)\|_1 \leq C$$

for some constant C depending only on α. It is also easy to see that $K_\alpha(t) * K_\alpha(s) = K_\alpha(s+t)$ for all $s, t > 0$ and that

$$\lim_{t\to 0} K_\alpha(t) * f = f$$

uniformly for every continuous function of compact support $f : \mathbb{R}^2 \to \mathbb{C}$. It follows that if,for $f \in L^p(\mathbb{R}^2)$, we define

$$e^{-t(-\Delta)^\alpha} f = K_\alpha(t) * f$$

then $\{e^{-t(-\Delta)^\alpha}\}$ is a one-parameter, strongly continuous, semi-group of uniformly bounded operators in $L^p(\mathbb{R}^2)$ for $1 \leq p < \infty$. We can then define $(-\Delta)^\alpha$ as the generator of this semi-group.

The next estimate will be used several times. It is stated as a lemma for easy reference.

Lemma 2. *Let $1 \leq p \leq q \leq \infty$. Assume $K \in L^p(\mathbb{R}^2)$, $u \in L^q(\mathbb{R}^2)$, $v \in L^{p'}(\mathbb{R}^2)$. Then*

$$\|K * (uv)\|_q \leq \|K\|_p \|u\|_q \|v\|_{p'}. \tag{5}$$

Proof. By Hausdorff-Young

$$\|K * (uv)\|_q \leq \|K\|_p \|uv\|_r$$

where $1/r = 1/q + 1/p'$; the condition $q \geq p$ guarantees that $1/r \leq 1$. The estimate now follows by Hölder. □

Notice that if $p \leq 2$, then $p' \geq 2 \geq p$ and (5) is valid with $q = p'$. The estimate will be applied when $1 < p < 2$ and K will be one of $K_\alpha(t)$, $\partial K_\alpha(t)/\partial x_j$, $x_j K_\alpha(t)$, or $x_i \partial K_\alpha(t)/\partial x_j$.

3. **The Moments.** This section is focused on mild solutions of the geostrophic equations (1); that is, solutions of the integral equation

$$\begin{aligned} \theta(t) &= \Theta_0(t) - \int_0^t K_\alpha(t-s) * [u(s)\cdot\nabla\theta(s)]\,ds \qquad (6)\\ &= \Theta_0(t) - \int_0^t \int_{\mathcal{R}^2} \nabla K_\alpha(t-s,\cdot - y)\mathcal{R}(\theta(y,s))\cdot\theta(y,s)\,dy\,ds, \end{aligned}$$

where $\Theta_0(t) = K_\alpha(t) * \theta_0$ and $\mathcal{R}(\theta(y,s) = (-\mathcal{R}_2\theta(y,s), \mathcal{R}_1\theta(y,s)) = u(y,s)$, and $\widehat{\mathcal{R}_j\theta}(y,s) = i\frac{\xi_j}{|\xi|}\hat{\theta}$ is the j-th Riesz Transform of the function θ. It will be supposed that $\alpha \in (1/2, 1]$. As mentioned before we expect that our results can be easily extended to the case $\alpha = 1/2$, provided the data is in H^2 and $\|\theta\|_\infty$ is sufficiently small, since then the solutions are bounded in H^2 (see [2]), and the techniques applied here and in [12] will hold. In the case that our datum is in a sufficiently high Sobolev space, the results of [4] or [12] yield that we are working with regular solutions.

The following decay rates for the moments of order one are the main results of this section. Let $2 \leq r < \frac{1}{1-\alpha}$ then the moments decay like

$$\|x_j\theta(t)\|_r\,dx \leq C_0(1+t)^{-\lambda_r}, \text{ where } \lambda_r = \min\{\frac{1}{\alpha}(\frac{1}{2}-\frac{1}{r}),\ \frac{\mu-1}{2\alpha}+\frac{1}{\alpha r'}\} \qquad (7)$$

and the corresponding velocity moment for $2 < r < \frac{1}{1-\alpha}$, $0 \leq \mu \leq 1$ decays like

$$\|x_j u(t)\|_r \ \leq C_0(1+t)^{-\tau}, \text{ where } \ \tau = \min\{\lambda_r, (\frac{1}{\alpha}+\mu)(\frac{1}{r'}-\frac{1}{2})\}. \qquad (8)$$

The decay of the moments of the velocity will be improved if in addition for $\beta > 0$, it holds that $I_\beta\theta_0 \in L^1(\mathbb{R}^2)$.

The first step in obtaining the decay (7) is to prove that (6) has a solution in $C([0,T], L^2_w)$ for some $T > 0$, then proving that this solution has to coincide with a standard solution of the integral equation which exists for all times $t \geq 0$. The following version of a fixed point theorem will be very useful in establishing local existence.

Lemma 3. *Let X be a Banach space and let $B : X \times X \to X$ be a bilinear mapping of norm η; i.e., such that*

$$\|B(x_1,x_2)\| \leq \eta\|x_1\|\|x_2\|. \qquad (9)$$

for all $x_1, x_2 \in X$. Then, for all $y \in X$ satisfying

$$4\eta\|y\| < 1,$$

there exists a unique $x \in X$ satisfying the equation

$$x = y + B(x,x) \qquad (10)$$

and such that

$$\|x\| \leq 2\|y\|. \qquad (11)$$

Proof. For a proof see [1] □

We apply this lemma with the bilinear operator B defined (at least formally) as follows. The space X of Lemma 3 will be a space of the form $C([0,T],Y)$, where Y is a Banach space to be specified. If $\omega_1, \omega_2 \in C([0,T],Y)$ we define

$$B(\omega_1,\omega_2)(t) = \int_0^t \nabla K_\alpha(t-s) * [\omega_2(s)\mathcal{R}(\omega_1(s))]\, ds. \tag{12}$$

For the rest of this section we fix p such that $1 < p < 2/(3-2\alpha)$ and set

$$\delta_p = \frac{3p-2}{2\alpha p}.$$

Remark 1. Notice that $2/(3-2\alpha) > 1$ (since we are assuming $\alpha > 1/2$) and that $0 < \delta_p < 1$. The significance of δ_p is due to Lemma 1 (with $n = 2$, $j = 0$, $|\beta| = 1$) according to which

$$\|\frac{\partial K_\alpha}{\partial x_j}\|_p = Ct^{-\delta_p}, j = 1, 2, \tag{13}$$

for all $t > 0$ and some constant C depending only on p.

Moreover, it is well known that if $\theta_0 \in L^{p'}(\mathbb{R}^2)$ there exists a unique mild solution $\theta \in C([0,\infty), L^{p'}(\mathbb{R}^2))$ such that $\theta(0) = \theta_0$. We include this result in the statement of our next theorem, which also establishes the boundedness of the moments of a mild solution with datum in L^2_w.

Theorem 1. *Let $1 \le p \le q \le \infty$, with p' the conjugate of p. Let $\theta_0 \in L^{p'}(\mathbb{R}^2)$. There exists a unique mild solution $\theta \in C\left([0,\infty), L^{p'}(\mathbb{R}^2)\right)$ of (1) with $\theta(0) = \theta_0$. If, in addition,*

i.: *$\theta_0 \in L^q(\mathbb{R}^2)$ for some $q \in [p,\infty]$, then $\theta \in C\left([0,\infty), L^q(\mathbb{R}^2)\right)$.*
ii.: *$\theta_0 \in L^2_w(\mathbb{R}^2)$, then $\theta \in C\left([0,\infty), L^2_w\right)$.*

Proof. If $T > 0$, $q \ge 1$, let $X_{q,T} = C([0,T], L^q(\mathbb{R}^2))$, a Banach space with the norm

$$\|f\|_{X_{q,T}} = \sup_{0\le t\le T} \|f(t)\|_q.$$

We begin sketching a proof of the existence and uniqueness of a mild solution θ with datum $\theta_0 \in L^{p'}(\mathbb{R}^2)$; i.e., a solution $\theta \in C\left([0,\infty), L^{p'}(\mathbb{R}^2)\right)$ of the integral equation (6). Using the bilinear operator B we can write the integral equation in the form

$$\theta = g + B(\theta,\theta), \tag{14}$$

where $g(t) = \Theta_0(t) = K_\alpha(t) * \theta_0$.

By Lemma 2, with K replaced by $\nabla K_\alpha(t-s)$, whose L^p-norm is given by (13),

$$\begin{aligned}
\|B(\omega_1,\omega_2)(t)\|_q &\le C\int_0^t (t-s)^{-\delta_p}\|\omega_2(s)\|_q\|\mathcal{R}(\omega_1(s))\|_{p'}\, ds \\
&\le C\int_0^t (t-s)^{-\delta_p}\|\omega_2(s)\|_q\|\omega_1(s)\|_{p'}\, ds \qquad (15)\\
&\le CT^{1-\delta_p}\|\omega_2\|_{X_{q,T}}\|\omega_1\|_{X_{p'T}}. \qquad (16)
\end{aligned}$$

Inequality (15) is due to the boundedness of the Riesz transforms in $L^{p'}$ (because $1 < p' < \infty$); the constant C depends only on p, q and varies from line to line.

Consider first the case $q = p'$. Notice that if $\theta, \tilde\theta$ are two solutions in $X_{p',T}$ of the integral equation (6) with initial datum $\theta_0, \tilde\theta_0$, respectively, then by (14)

$$\begin{aligned}
\|\theta(t) - \tilde\theta(t)\|_{p'} &\leq \|K_\alpha(t) * (\theta_0 - \tilde\theta_0\|_{p'} + \|B(\theta,\theta)(t) - B(\tilde\theta,\tilde\theta)(t)\|_{p'} \\
&\leq c\|\theta_0 - \tilde\theta_0\|_{p'} + \|B(\theta, \theta - \tilde\theta)(t)\|_{p'} + \|B(\theta - \tilde\theta, \tilde\theta)(t)\|_{p'} \\
&\leq c\|\theta_0 - \tilde\theta_0\|_{p'} + C\int_0^t (t-s)^{-\delta_p}\left(\|\theta(s)\|_{p'} + \|\tilde\theta(s)\|_{p'}\right)\|\theta(s) - \tilde\theta(s)\|_{p'}\,ds \\
&\leq c\|\theta_0 - \tilde\theta_0\|_{p'} + C\left(\|\theta\|_{X_{p',T}} + \|\tilde\theta\|_{X_{p',T}}\right)\int_0^t (t-s)^{-\delta_p}\|\theta(s) - \tilde\theta(s)\|_{p'}\,ds,
\end{aligned}$$

where $c \equiv \|K_\alpha(t)\|_1$ and C is the same constant as in (16). It follows by a modified Gronwall inequality argument* that

$$\|\theta(t) - \tilde\theta(t)\|_{p'} \leq \Phi_{\delta_p}\left(T, \|\theta\|_{X_{p',T}} + \|\tilde\theta\|_{X_{p',T}}\right)\|\theta_0 - \tilde\theta_0\|_{p'} \tag{17}$$

for some (continuous) function Φ_{δ_p} on $[0,\infty)\times[0,\infty)$. In particular $\|\theta(t) - \tilde\theta(t)\|_{p'} = 0$ for all $t \in [0,T]$ if $\theta_0 = \tilde\theta_0$, proving the uniqueness of the solution of the integral equation in $C([0,T], L^{p'})$, in any interval $[0,T]$ in which it is defined. Moreover, the short term existence of such a solution is also immediate. In fact, fix $T_0 > 0$ and let T be such that $T \leq T_0$ and $4C_0T^{1-\delta_p} < 1/(c\|\theta_0\|_{p'})$, where the solution to the"linear part" satisfies $\|\Theta_0\|_{p'} \leq c\|\theta_0\|_{p'}$. Lemma 3 applies (with $\eta = C_0T^{1-\delta_p}$) to prove the existence of $\theta \in X_{p',T}$ solving (6).

To see that this short term solution can be extended to a global solution in $C([0,\infty), L^{p'}(\mathbb{R}^2))$, as is usual, one only needs to see that if θ solves the integral equation in an interval $[0,T)$, $T < \infty$, then $\limsup_{t\to T-}\|\theta(t)\|_{p'} < \infty$. Thanks to (17) it suffices to prove this assuming θ_0 smooth. In this case the so called "maximum principle" (cf. [2]) implies that

$$\|\theta(t)\|_q \leq \|\theta_0\|_q \tag{18}$$

for t in any interval $[0,T)$ in which θ solves (1) with initial datum in L^q; $1 < q < \infty$. Applying this result with $q = p'$ completes the proof of the existence of a global solution $\theta \in C\left([0,\infty), L^{p'}(\mathbb{R}^2)\right)$ of the integral equation (6).

Assume now that, in addition, $\theta_0 \in C([0,\infty), Y)$ where Y is either $L^q(\mathbb{R}^2)$ with $q \in [p,\infty]$ or $L^2_w(\mathbb{R}^2)$. We need to prove that if $\theta_0 \in Y$, then $\Theta_0(t) = K_\alpha(t) * \theta_0$ stays bounded in Y and B (as defined above) is a bounded bilinear map

$$(X_{T,p'} \cap C([0,\infty), Y)) \times (X_{T,p'} \cap C([0,\infty), Y)) \to X_{T,p'} \cap C([0,\infty), Y)$$

of norm of order $o(1)$ as $T \to 0$. Once this is done, Lemma 3 establishes the short term existence of a solution $\theta \in X_{T,p'} \cap C([0,\infty), Y)$. Next we need to prove that this solution remains bounded in Y over bounded intervals; it can then be extended to a solution valid for all values of $t \geq 0$ that, by the uniqueness of the solution in $X_{T',p}$ (for all $T > 0$), must coincide with the previous one.

The case $Y = L^q(\mathbb{R}^2)$ is particularly simple. We have $\|\Theta_0(t)\|_q \leq c\|\theta_0\|_q$ for all $t \geq 0$, ($c \equiv \|K_\alpha(t)\|_1$) and (16) proves that the norm of B as a bilinear map from $(X_{T,p'} \cap X_{T,q}) \times (X_{T,p'} \cap X_{T,q})$ to $X_{T,p'} \cap X_{T,q}$ is of order $T^{1-\delta_p} \to 0$ as $T \to 0$. By the maximum principle (18) this solution cannot blow up in finite time, hence can be extended to a global solution.

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*see Appendix 2

Finally, consider the case $\theta_0 \in L^2_w(\mathbb{R}^2)$. By Lemma 1,

$$\|x_j K_\alpha(t)\|_1 \equiv Ct^{1/2\alpha}.$$

Using this and once again the fact that $\|K_a(t)\|_1$ is a constant, we get from

$$x_j\Theta_0(t) = x_j(K_\alpha(t) * \theta_0) = (x_j K_\alpha(t)) * \theta_0 + K_\alpha(t) * (x_j\theta_0)$$

that $\|x_j\Theta_0(t)\|_2 \le C(1+t)^{1/2\alpha}$ $(j = 1, 2)$. Thus $\|\Theta_0(t)\|_{L^2_w}$ remains bounded in bounded intervals, hence the same is true of its $L^{p'} \cap L^2_w$ norm.

Since B has already been proved bounded from $(X_{T,p'} \times X_{T,2}) \times (X_{T,p'} \times X_{T,2})$ to $X_{T,p'} \times X_{T,2}$, with norm going to 0 as $T \to 0$. It remains to be proved that it is similarly bounded from $C([0,T], L^2_j) \times C([0,T], L^2_j)$ to $C([0,T], \cap L^2_w)$ for $j = 1, 2$.

We have

$$\begin{aligned} x_j B(\omega_1, \omega_2)(t) \quad = \\ \int_0^t (x_j\nabla K_\alpha) * [\omega_2(s)\mathcal{R}(\omega_1(s))\, ds + \int_0^t \nabla K_\alpha(t-s) * [(x_j\omega_2(s))\mathcal{R}(\omega_1(s))]\, ds = I_1 + I_2. \end{aligned}$$

By Lemma 1 (with $|\gamma| = |\beta| = 1$, $j = 0$),

$$\|x_j \frac{\partial K_\alpha}{\partial x_k}\|_p = Ct^{-\rho},$$

with $\rho = 1/(\alpha p') < 1$ (since $\alpha > 1/2$, $p' \ge 2$) for $j, k = 1, 2$, $t > 0$. From this, and (5) (with $q = 2$),

$$\|I_1\|_2 \le C\int_0^t (t-s)^{-\rho}\|\omega_1(s)\|_{p'}\|\omega_2(s)\|_2\, ds.$$

In I_2 we estimate $\|\nabla K_\alpha(t-s)\|_p$ as before and get

$$\|I_2\|_2 \le C\int_0^t (t-s)^{-\delta_p}\|\omega_1(s)\|_{p'}\|x_j\omega_2(s)\|_2\, ds.$$

It follows that

$$\|x_j B(\omega_1, \omega_2)(t)\|_2 \le C(T^{1-\delta_p} + T^{1-\rho})\|\omega_1\|_{X_{T,p'}}\|\omega_2\|_{C([0,T],L^2_w)}$$

for $0 \le t \le T$, $j = 1, 2$. The desired boundedness of B has been proved, hence the short term existence of a solution taking values in $L^2_w \cap L^{p'}$; by uniqueness of the $L^{p'}$ solution this solution coincides with the $L^{p'}$ solution. To see that the solution remains in L^2_j for all $t \ge 0$ $(j = 1, 2)$, we observe that $x_j\theta$ solves the integral equation

$$(x_j\theta)(t) = g_j(t) - \int_0^t \nabla K_\alpha(t-s) * [(x_j\theta(s))u(s)]\, ds$$

where $u(s) = \mathcal{R}(\theta(s))$ thus $u \in C\left([0,\infty), L^{p'}(\mathbb{R}^2)\right)^2$ and

$$g_j(t) = x_j\Theta_0(t) - \int_0^t [x_j\nabla K_\alpha(t-s)] * [\theta(s)\mathcal{R}(\theta(s))]\, ds.$$

From what we proved, using the fact that $\theta \in C([0,\infty), L^{p'} \cap L^2)$, we see that $g_j \in C([0,\infty), L^{p'} \cap L^2)$. Thus

$$\|x_j\theta(t)\|_2 \le \|g_j(t)\|_2 + C\int_0^t (t-s)^{-\rho}\|u(s)\|_{p'}\|x_j\theta(s)\|_2\, ds$$

and a (modified) Gronwall inequality argument proves that $\|x_j\theta(t)\|_2$ cannot blow up in finite time. □

From now on, θ will always denote the solution of the geostrophic equations with initial datum $\theta_0 \in L^{p'} \cap L^2$ given by Theorem 1 and u will always denote the corresponding velocity, $u = \mathcal{R}^{\perp}\theta$. Thus $\theta \in C\left([0,\infty), L^2(\mathbb{R}^2) \cap L^{p'}(\mathbb{R}^2)\right)$ and (since The Riesz transforms are bounded in L^2, $L^{p'}$, it follows that $u \in C\left([0,\infty), L^2(\mathbb{R}^2) \cap L^{p'}(\mathbb{R}^2)\right)$. In the sequel we shall need the following properties of this solution.

1.: Let m be a non-negative integer and assume $\theta_0 \in L^1 \cap H^m \cap L^{p'}$. There exists a constant $C \geq 0$, depending only on $L^1 \cap H^m$-norm of θ_0 such that

$$\|\Lambda^m \theta(t)\|_2 \leq C(1+t)^{-\frac{m+1}{2\alpha}} \tag{19}$$

for all $t \geq 0$. See [4, Theorem 3.1] for the case $m = 0$, [12, Theorem 3.2] for the generalization to the case $m \geq 0$. We observe that since the space dimension is 2, the condition $\theta_0 \in L^{p'}$ (which assures uniqueness of our solution) is implied by $\theta_0 \in H^m$ if $m \geq 1$.

2.: With the same hypotheses as in **1.**,

$$\|\Lambda^m u(t)\|_2 \leq C(1+t)^{-\frac{m+1}{2\alpha}}, \tag{20}$$

same C as in (19). This follows at once from (19) because the Riesz transforms are bounded in L^2 and commute with Λ^m.

3.: Let $1 < q < \infty$ and assume that $\theta_0 \in L^q \cap L^{p'}$. Then

$$\|\theta(t)\|_q \leq \|\theta_0\|_q \tag{21}$$

for all $t \geq 0$. This is the so called maximum principle (see [2]).

The next theorem is a simple extension of the decay rate (19), which was obtained in [4] and [12]. The main tool used in the proof is Fourier splitting.

Theorem 2. *Assume θ is a solution of (1) with data $\theta_0 \in L^1 \cap H^m$, $m \geq 0$. Suppose additionally that $|\theta_0(\xi)| \leq C_0|\xi|^\mu$ for ξ in some neighborhood of the origins and for some constants C_0, μ; $C \geq 0$, $0 \leq \mu \leq 1$. Moreover, if $\alpha = 1$ assume that $\mu < 1$. Then*

$$\|\Lambda^m \theta(t)\|_{L^2} \leq C(t+1)^{-\frac{m+1+\mu}{2\alpha}}, \tag{22}$$

where C is a constant which depends only on C_0 and the norms of the initial datum.

Proof. Suppose first that $m = 0$. Because $\theta_0 \in L^1$ implies $\hat{\theta}_0 \in L^\infty$, we may assume that the inequality $|\hat{\theta}_0(\xi)| \leq C_0|\xi|^\mu$ is valid for all $\xi \in \mathbb{R}^2$. Claim that

$$|\hat{\theta}(\xi,t)| \leq C|\xi|^\mu \tag{23}$$

for $\xi \in \mathbb{R}^2$, where C does not depend on t. In fact, taking the Fourier Transform in (6) one sees that

$$\widehat{\theta} = \widehat{\theta}_0 e^{-|\xi|^{2\alpha}t} + \sum_{k=1}^{2} \int_0^t e^{-|\xi|^{2\alpha}(t-s)} \xi_k \widehat{u_k\theta}\, ds.$$

By (19), (20) (case $m = 0$),

$$|\widehat{u_k\theta}(s)| \leq \|u_k(s)\theta(s)\|_1 \leq \|u_k(s)\|_2 \|\theta(s)\|_2 \leq C(1+s)^{-1/\alpha},$$

so that
$$|\widehat{\theta}(\xi)| \leq C|\xi|^{\mu} + C|\xi| \int_0^t e^{-|\xi|^{2\alpha}(t-s)}(1+s)^{-1/\alpha}\, ds.$$
To establish the claim, it suffices to see that
$$\int_0^t e^{-|\xi|^{2\alpha}(t-s)}(1+s)^{-1/\alpha}\, ds \leq C|\xi|^{-\epsilon}$$
where C does not depend on t and $1-\epsilon \geq \mu$. This is obvious (with $\epsilon = 0$) if $\alpha < 1$, so assume $\alpha = 1$, in which case we assume $\mu < 1$. Then, if $\sigma > 1$, by Hölder,
$$\begin{aligned}
\int_0^t e^{-|\xi|^{2\alpha}(t-s)}(1+s)^{-1}\, ds &\leq \left(\int_0^t e^{-\sigma'|\xi|^{2\alpha}(t-s)}\, ds\right)^{1/\sigma'} \left(\int_0^t (1+s)^{-\sigma}\, ds\right)^{1/\sigma} \\
&\leq \left(\int_0^\infty e^{-\sigma'|\xi|^{2\alpha}s}\, ds\right)^{1/\sigma'} \left(\int_0^\infty (1+s)^{-\sigma}\, ds\right)^{1/\sigma} = C|\xi|^{-\frac{2\alpha}{\sigma'}}.
\end{aligned}$$
Taking σ' large enough, we get $\epsilon = 2\alpha/\sigma' \leq 1-\mu$. The claim is established.

We are ready for the Fourier Splitting argument. For this, multiply (1) by θ and integrate in space. Using Parseval and integration by parts it follows in frequency space that
$$\frac{d}{dt}\int_{\mathbb{R}^2} |\widehat{\theta}(\xi)|^2\, d\xi = -\int_{\mathbb{R}^2} |\xi|^{2\alpha}|\widehat{\theta}(\xi)|^2\, d\xi$$
Now split the domain of integration of the integral on the right hand side of the last equation yields into $\mathcal{S} \cup \mathcal{S}^c$ where $\mathcal{S}$ is defined by
$$\mathcal{S} = \{\xi : |\xi| \leq \left(\frac{k}{t+1}\right)^{1/2\alpha}\};$$
k a constant to be determined below. Noting that for $\xi \in \mathcal{S}^c$ one has $-|\xi|^{2\alpha} \leq -\frac{k}{t+1}$, it follows that
$$\frac{d}{dt}\int_{\mathbb{R}^2} |\widehat{\theta}(\xi)|^2\, d\xi \leq -\frac{k}{t+1}\int_{\mathcal{S}^c} |\widehat{\theta}(\xi)|^2 = -\frac{k}{t+1}\int_{\mathbb{R}^2} |\widehat{\theta}(\xi)|^2\, d\xi + \frac{k}{t+1}\int_{\mathcal{S}} |\widehat{\theta}(\xi)|^2\, d\xi.$$
By (23) the last integral can be estimated by
$$\int_{\mathcal{S}} |\widehat{\theta}(\xi)|^2\, d\xi \leq C\int_{\mathcal{S}} |\xi|^{2\mu}\, d\xi = C(t+1)^{-(1+\mu)/\alpha}$$
so that multiplying by $(t+1)^k$ one gets
$$\frac{d}{dt}(1+t)^k \int_{\mathbb{R}^2} |\widehat{\theta}(\xi)|^2\, d\xi \leq C(t+1)^{k-1-(1+\mu)/\alpha}.$$
We choose $k > (1+\mu)/\alpha$. Integrating from 0 to t and then dividing out $(t+1)^k$ yields
$$\begin{aligned}
\int_{\mathbb{R}^2} |\widehat{\theta}(\xi)|^2\, d\xi &\leq (t+1)^{-k}\int_{\mathbb{R}^2} |\widehat{\theta}_0(\xi)|^2\, d\xi + C(t+1)^{-k} + C(t+1)^{-(1+\mu)/\alpha} \\
&\leq C(t+1)^{-(1+\mu)/\alpha}
\end{aligned}$$
as desired. To establish the decay for the higher derivatives; i.e., cases with $m > 0$, follow the steps in the proof of [12, Theorem 3.4] where $\mu = 0$, and replace the decay for the L^2 norm with the new decay obtained for the case $\mu \geq 0$. The faster decay in L^2 will yield the desired faster decay for the L^2 norm of Λ^β. □

Remark 2. It is possible to include the case $\alpha = \mu = 1$ in the considerations of Theorem 2, except that the estimates need to be modified by a factor of $\log(t+1)$.

We can also extend (20).

Corollary 1. *Under the conditions of the last theorem we have the same decay rates for u;*

$$\|\Lambda^{\beta} u\|_{L^2} \leq C(t+1)^{-\frac{1+\beta+\mu}{2\alpha}}, \tag{24}$$

where C is a constant that depends only on the initial datum.

Proof. Follows immediately since the Riesz transforms are bounded in L^2 and commute with Λ. □

The estimates of Theorem 2 and Corollary 1 can be used to improve the L^p decay of the temperature and the velocity.

Lemma 4. *Assume θ is a solution of (1) with datum $\theta_0 \in L^1 \cap H^1$ and that $|\hat{\theta}_0(\xi)| \leq |\xi|^{\mu}$ for some $C \geq 0$, $\mu \in [0,1]$, $\mu < 1$ if $\alpha = 1$. Then, for $q \in [2,\infty)$, $t \geq 0$,*

$$\|\theta(t)\|_q \quad \leq \quad C(t+1)^{-\frac{1}{\alpha}(\frac{\mu}{2}+\frac{1}{q'})}, \tag{25}$$

$$\|u(t)\|_q \quad \leq \quad C(t+1)^{-\frac{1}{\alpha}(\frac{\mu}{2}+\frac{1}{q'})}. \tag{26}$$

Proof. By the Gagliardo-Nirenberg inequality (see [5, Chapter 1,Theorem 9.3]),

$$\|\theta\|_q \leq \|\theta\|_2^{1-a}\|\Lambda\theta\|_2^a$$

if $0 \leq a \leq 1$ and

$$\frac{1}{q} = a\left(\frac{1}{2}-\frac{1}{2}\right) + (1-a)\frac{1}{2} = (1-a)\frac{1}{2};$$

i.e., $a = (q-2)/q$, $1-a = 2/q$. Inequality (25) now follows from the estimates in Theorem 2. Since the Riesz transforms are bounded in L^q ($q \neq 1,\infty$), (26) is an immediate consequence of (25). See also [12, Corollary 3.1]. □

Corollary 2. *Let $\theta_0 \in L^1 \cap H^1$ then the solution θ with data θ_0 belongs to L^1 for all time.*

Proof. Write the solution in integral form then it follows that

$$\|\theta(t)\|_1 \leq \|\theta_0\|_1 + \int_0^t \|u\|_2\|\nabla\theta\|_2 \, ds$$

where we used Young's and Hölder inequalities to get the last term. From here, the decay of the L^2 norm of the velocity and the gradient of the temperature yields

$$\|\theta(t)\|_1 \leq \|\theta_0\|_1 + \int_0^t (1+t)^{-\frac{3}{2\alpha}} \, ds$$

and we conclude that the L^1norm of the solution is bounded. □

Remark 3. The next two theorems give the bounds for the moments of θ and of u. In the proof of the first one we will have occasion to use the following estimate. Assume $0 \leq \rho < 1$, $\tau \geq 0$. Then there exists a constant $C \geq 0$ (depending on ρ, τ) such that

$$\int_0^t (t-s)^{-\rho}(1+s)^{-\tau} \, ds \leq \begin{cases} Ct^{-\rho} & \text{if} \quad \tau > 1, \\ Ct^{-\rho}(1+\log(1+t)) & \text{if} \quad \tau = 1, \\ Ct^{-\rho}(1+t)^{1-\tau} & \text{if} \quad \tau < 1. \end{cases} \tag{27}$$

Theorem 3. *Let $\theta_0 \in L^1 \cap L^2_w \cap L^r_w \cap H^1$ where $2 \leq r < 1/(1-\alpha)$. Let $0 \leq \mu \leq 1$ be such that*

$$|\hat{\theta}_0(\xi)| \leq C|\xi|^\mu \tag{28}$$

for all ξ in a neighborhood of the origin, some $C \geq 0$;$\mu < 1$ if $\alpha = 1$. Let

$$\lambda_r = \frac{1}{\alpha r'} + \frac{\mu - 1}{2\alpha}.$$

Then

$$\|x_j\theta(t)\|_r \leq C(1+t)^{-\lambda_r}$$

for some constant C (depending on θ_0 and r), all $t \geq 0$.

Proof. Note first that since $\theta_0 \in L_1$ hence $\hat{\theta}_0 \in L^\infty$, and inequality (28) always holds for all $\xi \in \mathbb{R}^2$ with $\mu = 0$. The following auxiliary estimate for the L^2 norm of the temperature moment will be needed

$$\|x_j\theta(t)\|_2 \leq C(1 + \log(1+t)) \tag{29}$$

for all $t \geq 0$, where the constant $C \geq 0$, depends on θ_0. Here we work with no information on $\widehat{\theta}(\xi)$ near zero; that is, we assume (as we may) that $\mu = 0$. The first step is to rewrite equation (1) in the form

$$(x_j\theta)_t + (-\Delta)^\alpha(x_j\theta) - u \cdot \nabla(x_j\theta) = f_j \tag{30}$$

where $f_j = u_j\theta + h_j$ and

$$\hat{h}_j(\xi) = -2\alpha i|\xi|^{2\alpha-2}\xi_j\hat{\theta}(\xi).$$

We need to estimate $\|f_j(t)\|_2$. We suppose that $|\hat{\theta}_0(\xi)| \leq C|\xi|^\mu$, with $\mu \geq 0$ to derive some preliminary estimates. Due to the boundedness of the Riesz transforms in L^4,

$$\|u_j(t)\theta(t)\|_2 \leq \|u_j(t)\|_4\|\theta(t)\|_4 \leq C\|\theta(t)\|_4^2;$$

thus by (25), (26),

$$\|u_j(t)\theta(t)\|_2 \leq C(1+t)^{-\frac{3}{2\alpha}}.$$

Turning to the other term in f_j,

$$\|h_j(t)\|_2 = \|\hat{h}_j(t)\|_2 \leq 2\alpha\left\||\xi|^{2\alpha-1}\hat{\theta}(t)\right\|_2 = \|(-\Delta)^{\alpha-\frac{1}{2}}\theta(t)\|_2.$$

and by Theorem 2 (or [12, Theorem 3.4]) it follows that

$$\|h_j(t)\|_2 \leq C(1+t)^{-1}.$$

Since$3/(2\alpha) > 1$ we see that there exists a constant C such that

$$\|f_j(t)\|_2 \leq C(1+t)^{-1} \tag{31}$$

for $t \geq 0$. Let

$$E(t) = \int_{\mathbb{R}^2} x_j^2|\theta(t)|^2\,dx.$$

Then, by (30), with $(\cdot,\cdot)$ denoting the inner product of L^2, and considering that $(x_j\theta(t), u\cdot\nabla(x_j\theta)) = 0$ because div $u = 0$,

$$\begin{aligned} E'(t) &= 2(x_j\theta(t), (x_j\theta)_t(t)) = -2(x_j\theta(t), (-\Delta)^\alpha(x_j\theta(t)) + (x_j\theta, f_j) \\ &= -2\int_{R^2}|\xi|^{2\alpha}|[x_j\theta(t)]\hat{}(\xi)|^2\,d\xi + (x_j\theta, f_j). \end{aligned}$$

Thus

$$E'(t) \leq -2\int_{R^2}|\xi|^{2\alpha}|[x_j\theta(t)]\hat{}(\xi)|^2\,d\xi + \|f_j(t)\|_2 E(t)^{1/2}. \tag{32}$$

Dropping the first negative term on the right hand side, using (31), we get

$$E'(t) \leq C(1+t)^{-1}E(t)^{1/2} \tag{33}$$

hence integration yields

$$E(t) \leq (E(0)^{1/2} + C\log(1+t))^2 = C(1+\log(1+t))^2,$$

proving (29). Assume now $2 \leq r < 1/(1-\alpha)$, $\theta_0 \in L^r_w \cap L^2_w \cap H^1$, and $\hat{\theta}_0(\xi) = O(|\xi|^\mu)$ near 0; $0 \leq \mu < 1$, $\mu < 1$ if $\alpha = 1$. We can write

$$x_j\theta(t) = x_j\Theta_0(t) - \int_0^t (x_j\nabla K_\alpha)(t-s)*[\theta(s)u(s)]\,ds - \int_0^t \nabla K_\alpha(t-s)*[(x_j\theta(s))u(s)]\,ds. \tag{34}$$

From the assumptions on r, $1/2 - 1/r < \alpha - 1/2$ and we can select p such that $1/2 - 1/r < 1/p' < \alpha - 1/2$. Then $1/2 < 1/r + 1/p' < 1$ so that if we set

$$\frac{1}{q} = \frac{1}{r} + \frac{1}{p'} = \frac{1}{r} + 1 - \frac{1}{p}$$

we get $1 < q < 2$. Moreover, $1/r = 1/p + 1/q - 1$, hence $\|f * g\|_r \leq \|f\|_p\|g\|_q$, as well as $\|f * g\|_r \leq \|f\|_r\|g\|_1$. We use both Young estimates in (34) to get

$$\begin{aligned}
\|x_j\theta(t)\|_r &\leq \|x_j\Theta_0(t)\|_r + \int_0^t \|x_j\nabla K_\alpha)(t-s)\|_r\|\theta(s)u(s)\|_1\,ds \\
&\quad + \int_0^t \|\nabla K_\alpha(t-s)\|_p\|x_j\theta(s)u(s)\|_q\,ds \\
&\leq \|x_j\Theta_0(t)\|_r + \int_0^t \|x_j\nabla K_\alpha)(t-s)\|_r\|\theta(s)\|_2\|u(s)\|_2\,ds \\
&\quad + \int_0^t \|\nabla K_\alpha(t-s)\|_p\|x_j\theta(s)\|_2\|u(s)\|_\nu\,ds,
\end{aligned}$$

where ν is such that $1/2 + 1/\nu = 1/q$, thus $\nu > 2$ and the last estimates being due to Hölder's inequality. By Corollary 4 of the Appendix (case $n = 2$),

$$\|x_j\Theta_0(t)\|_r \leq C(1+t)^{-\frac{2+(\mu-1)r'}{2\alpha r'}} = C(1+t)^{-\lambda_r}$$

for all $t \geq 0$; by Theorem 2 $\|\theta(s)\|_2$, $\|u(s)\|_2$ are bounded by $C(s+1)^{-(1+\mu)/(2\alpha)}$, by (26), $\|u(s)\|_\nu$ is bounded by $C(s+1)^{-(1/2\alpha)(\mu/2+1/\nu')}$, while

$$\begin{aligned}
\|x_j\nabla K_\alpha)(t-s)\|_r &\leq C(t-s)^{-1/\alpha r'} \\
\|\nabla K_\alpha(t-s)\|_p &\leq C(t-s)^{-\frac{1}{2\alpha}-\frac{1}{\alpha p'}}
\end{aligned}$$

by Lemma 1. Using these estimates as well as the logarithmic bound (29) for $\|x_j\theta(t)\|_2$ in the last estimate of $\|x_j\theta(t)\|_r$, we get

$$\begin{aligned}
\|x_j\theta(t)\|_r &\leq C(1+t)^{-\lambda_r} + C\int_0^t (t-s)^{-1/\alpha r'}(1+s)^{-\frac{1+\mu}{\alpha}}\,ds \\
&\quad + \int_0^t (t-s)^{-\frac{1}{2\alpha}-\frac{1}{\alpha p'}}(1+\log(1+s))(1+s)^{-\frac{1}{\alpha}(\frac{\mu}{2}+\frac{1}{\nu'})}\,ds = I + II.
\end{aligned} \tag{35}$$

(Notice $1/(\alpha r') < 1$ by $r < 1/(1-\alpha)$ and $\frac{1}{2\alpha} + \frac{1}{\alpha p'} < 1$ since $\frac{1}{2} + \frac{1}{p'} < \alpha$.) To complete the proof, it suffices to show that the two integrals on the right hand side of (35) are bounded by $C(1+t)^{-\lambda_r}$. The first integral is smallest for $\mu = 0$, thus

$$\begin{aligned} I \le \int_0^t (t-s)^{-1/\alpha r'}(1+s)^{-1/\alpha}\,ds &= \int_0^{t/2} + \int_{t/2}^t \\ &\le Ct^{-\frac{1}{\alpha r'}}\int_0^{t/2}(1+s)^{-\frac{1}{\alpha}}\,ds + (1+t)^{-\frac{1}{\alpha}}\int_{t/2}^t (t-s)^{-\frac{1}{\alpha r'}}\,ds \\ &\le \begin{cases} Ct^{-1/\alpha r'}[1+(1+t)^{-1/\alpha}t], & \text{if } \alpha < 1, \\ Ct^{-1/r'}[(1+\log(1+t)) + (1+t)^{-1}t], & \text{if } \alpha = 1. \end{cases} \end{aligned}$$

This takes care of the first integral in (35). In fact, if $\alpha < 1$ then $1/\alpha r' \ge \lambda_r$ while if $\alpha = 1$ then $1/r' > \lambda_r$ because then $\mu < 1$.

To bound the second integral in (35) we notice first that

$$II \le (1+\log(1+t))\int_0^t (t-s)^{-\frac{1}{2\alpha}-\frac{1}{\alpha p'}}(1+s)^{-\frac{1}{\alpha}(\frac{\mu}{2}+\frac{1}{\nu'})}\,ds$$

and we can apply Remark 3 with

$$\rho = \frac{1}{2\alpha} + \frac{1}{\alpha p'}, \quad \tau = \frac{1}{\alpha}(\frac{\mu}{2} + \frac{1}{\nu'}).$$

In applying this remark, we can assume $t \ge 1$ and replace t by $t+1$ on the right hand sides. If $\tau \ge 1$, then

$$II \le C(1+t)^{-\rho}\,(1+\max(0,1-\tau)\log(t+1))\,(1+\log(t+1).$$

By the choice of p,

$$\rho = \frac{1}{\alpha}\left(\frac{1}{2} + \frac{1}{p'}\right) > \frac{1}{\alpha r'} \ge \lambda_r$$

Hence

$$II \le C(1+t)^{-\rho}(1+\log(t+1)^2 \le C(t+1)^{-\lambda_r}.$$

On the other hand, if $\tau \le 1$ then the estimate in Remark 3 implies

$$II \le C(t+1)^{-(\rho+\tau-1)}(1+\log(t+1).$$

An easy calculation shows that with ρ, τ as given above,

$$\rho + \tau - 1 = \frac{1}{\alpha}\left(\frac{1}{r'}\frac{\mu}{2} + 1 - \alpha\right) > \frac{1}{\alpha}\left(\frac{1}{r'}\frac{\mu}{2} - \frac{1}{2}\right) = \lambda_r.$$

Thus we also get $II \le C(t+1)^{-\lambda_r}$ in this case. This concludes the proof of the theorem. □

The case $r = 2$, $\mu = 0$ is an important case of the previous Theorem. We state it as a corollary.

Corollary 3. *Assume $\theta_0 \in L^2_w \cap L^1 \cap H^1$. Then there exists a constant $C \ge 0$ depending only on the initial datum θ_0 such that*

$$\|x_j\theta(t)\|_2 \le C$$

for all $t \ge 0$.

We also want to estimate the moments of u. The following result is auxiliary and gives a uniform bound for the first moments of the velocity.

Theorem 4. *Assume* $\theta_0 \in L^1 \cap L^2_w \cap L^r_w \cap H^1$ *where* $2 < r < 1/(1-\alpha)$. *Then there exists a constant* $C \geq 0$ *such that* $\|x_j u(t)\|_r \leq C$ *for all* $t \geq 0$. *The same result holds for* $r = 2$ *if* $\theta_0 \in L^1_w \cap L^2_w \cap H^1$ *and there exist constants* $c \geq 0$, $\mu > 0$ *such that* $|\hat{\theta}_0(\xi)| \leq C|\xi|^\mu$ *in a neighborhood of the origin.*

Proof. Let $j, k \in \{1, 2\}$; then $x_j \mathcal{R}_k = \mathcal{R}_k x_j + L_{kj}$, where (up to constant factor)

$$L_{kj} f(x) = \int_{\mathbb{R}^2} \frac{y_j y_k}{|y|^3} f(x-y)\, dy,$$

hence

$$|L_{kj} f(x)| \leq \int_{\mathbb{R}^2} \frac{1}{|y|} |f(x-y)|\, dy = I_1|f|(x),$$

where I_1 is the Riesz potential. In the case $r > 2$ we use the fact that I_1 is bounded from L^q to L^r, where $1/r = 1/q - 1/2$, thus

$$\|L_{kj}\theta(t)\|_r \leq \|\theta(t)\|_q \leq C\|\theta_0\|_q = C_0$$

the last inequality being due to the maximum principle (21); notice that $q > 1$ since $r > 2$. By Theorem 3 $\|x_j\theta(t)\|_r$ is (at least) bounded uniformly in t, thus

$$\begin{aligned} \|x_j u_\ell(t)\|_r &= \|x_j \mathcal{R}_k \theta(t)\|_r \leq \|\mathcal{R}_k(x_j\theta(t))\|_r + \|L_{kj}\theta(t)\|_r \\ &\leq C\|x_j\theta(t)\|_r + C_0 \leq C. \end{aligned}$$

Assume now $r = 2$ and $|\hat{\theta}_0(\xi)| \leq C|\xi|^\mu$ near $\xi = 0$. We may assume $\mu < 1$. Then, as shown in the proof of Theorem 2, estimate (23) is valid hence, by Parseval, if $R > 0$,

$$\begin{aligned} \|I_1(|\theta(t)|)\|_2^2 &= C\int_{\mathbb{R}^2} \frac{|\hat{\theta}(\xi,t)|^2}{|\xi|^2}\, d\xi \leq C\int_{|\xi|<R} |\xi|^{\mu-2}|\hat{\theta}(\xi,t)|^2\, d\xi + C\int_{|\xi|\geq R} \hat{\theta}(\xi,t)|^2\, d\xi \\ &\leq C\|\theta(t)\|_{\sigma'}^2 + \|\theta(t)\|_2^2 \end{aligned}$$

where σ satisfies $\sigma(2-\mu) < 2$. By the maximum principle (21),

$$\|I_1(|\theta(t)|)\|_2 \leq C(\|\theta_0\|_{\sigma'} + \|\theta_0\|_2)$$

(Since $\theta_0 \in H^1$, we have $\theta_0 \in L^{\sigma'}$ and the maximum principle applies). Thus, proceeding as for the case $r > 2$, we obtain

$$\begin{aligned} \|x_j u_\ell(t)\|_2 &= \|x_j \mathcal{R}_k \theta(t)\|_2 \leq \|\mathcal{R}_k(x_j\theta(t))\|_2 + \|L_{kj}\theta(t)\|_2 \\ &\leq C\|x_j\theta(t)\|_2 + \|I_1(|\theta(t)|)\|_2 \leq C. \end{aligned}$$

□

The following result can be found in [12]. Combined with the results of the last theorem one can use it to improve the algebraic decay of the velocity moments.

Theorem 5. *Let* $\beta > 0$, *assume that* $I_\beta\theta_0 \in L^1(\mathbb{R}^2)$, *and let* θ *be the solution of the homogeneous DQG with initial datum* θ_0.

i: *Assume* $\frac{1}{2} \leq \alpha < 1$. *Then*

$$\|\theta(t)\|_1 \leq Ct^{-\nu}$$

for all $t > 0$, *some constant* C, *where*

$$\nu = \begin{cases} \min(\beta, \frac{1}{2}) & \text{if} \quad \alpha = \frac{1}{2}, \\ \min(\frac{\beta}{2\alpha}, \frac{1}{2\alpha}) & \text{if} \quad \frac{1}{2} < \alpha < 1. \end{cases}$$

ii: *Assume $\alpha = 1$. Then*

$$\|\theta(t)\|_1 \leq \begin{cases} Ct^{-\frac{\beta}{2}} & \text{if} \quad \beta < 1, \\ Ct^{-\frac{1}{2}} \log(t+1) & \text{if} \quad \beta \geq 1, \end{cases}$$

for some constant C.

Proof. See [12] □

The next theorem combines Theorems 4, 5to give an improved decay of the moments of the velocity.

Theorem 6.

- *Assume $\theta_0 \in L^1 \cap L^2_w \cap L^r_w \cap H^1$ where $2 < r < 1/(1-\alpha)$. Then there exists a constant $C \geq 0$ such that*

$$\|x_j u(t)\|_r \leq C(t+1)^{-\tau}, \quad \text{for all} \ \ t \geq 0,$$

where $\tau = \min\{\lambda_r, [\frac{1}{\alpha}[\frac{1}{r'} - \frac{1}{2}]\}$, and λ_r was defined in (7) in the last theorem.
- *If θ_0 as before, $2 \leq r < 1/(1-\alpha)$ (for $r = 2$, we only require $\theta_0 \in L^1 \cap L^2_w \cap H^1$), and there exist constants $c \geq 0$, $0 \leq \mu \leq 1$ such that $|\hat{\theta}_0(\xi)| \leq C|\xi|^\mu$ in a neighborhood of the origin (as before $\mu < 1$ if $\alpha = 1$), then*

$$\|x_j u(t)\|_r \leq C(t+1)^{-\tau}$$

where $\tau = \min\{\lambda_r, [\frac{1}{\alpha} + \mu][\frac{1}{r'} - \frac{1}{2}]\}$.
- *if in addition for $\beta > 0$ it holds that $I_\beta \theta_0 \in L^1(\mathbb{R}^2)$, then*

$$\|x_j u(t)\|_r \leq C(t+1)^{-\tau},$$

where $\tau = \min\{\lambda_r, [\frac{1}{\alpha} + \mu][\frac{1}{r'} - \frac{1}{2} + \frac{2}{r}\nu]\}$ and ν was defined in Theorem 5.

Proof. Proceeding as in the proof of Theorem 4we write $x_j \mathcal{R}_k = \mathcal{R}_k x_j + L_{kj}$ for $j, k \in \{1, 2\}$. As before in the case $r > 2$, we use the fact that I_1 is bounded from L^q to L^r, where $1/r = 1/q - 1/2$; thus by (25)

$$\|L_{kj}\theta(t)\|_r \leq \|\theta(t)\|_q.$$

Note that $1 < q < 2$, hence we can interpolate between L^1 and L^2 to obtain

$$\|\theta\|_q \leq C\|\theta\|_1^{2/q-1}\|\theta\|_2^{2/q'} = C\|\theta\|_1^{2/r}\|\theta\|_2^{1-2/r}$$

The conclusion of the theorem follows combining this last inequality with the decay rates of the L^2 norms obtained in [4] when $\mu = 0$, the decay rates of L^2 obtained in Lemma (4) when $\mu > 0$, the decay rates for the L^1 norm obtained in Theorem 5 when $\beta > 0$. As in Theorem 4 we have to consider separately the case when $r = 2$ and when $2 < r < 1/(1-\alpha)$. We omit the details since they are straightforward.

□

4. **Estimates.** In this section we assume that the space dimension is 2 and that $\alpha \in (1/2, 1]$. We suppose that $\theta_0 \in H^1 \cap L^1$. As before we write the solution θ in integral form

$$\begin{aligned} \theta(t) &= \Theta_0(t) - \int_0^t K_\alpha(t-s) * [u(s)\cdot\nabla\theta(s)]\,ds \\ &= \Theta_0(t) - \int_0^t \int_{\mathbb{R}^2} K_\alpha(t-s,\cdot - y)u(y,s)\cdot\nabla\theta(y,s)\,dy\,ds, \end{aligned} \tag{36}$$

where

$$\Theta_0(t) = e^{-t(-\Delta)^\alpha}\theta_0 = K_\alpha(t) * \theta_0 \tag{37}$$

and $u = (u_1, u_2) = (-\mathcal{R}_2\theta, \mathcal{R}_1\theta)$; $\mathcal{R}_1, \mathcal{R}_2$ being the Riesz transforms in $\mathbb{R}^2$.

We start with some preliminary observations that will be needed later. For this we recall Taylor's formula (as used by Fujigaki and Miyakawa [6]) in the form

$$K_\alpha(x-y,t-s) = \sum_{|\beta|+j\le m} \frac{(-1)^{|\beta|+j}}{\beta!j!} D_t^j D^\beta K_\alpha(x,t) y^\beta s^j + H(x,y,t,s),$$

where

$$H(x,y,t,s) = \sum_{|\beta|+j=m} \frac{(-1)^{|\beta|+j} y^\beta s^j}{\beta!j!} \int_0^1 [D_t^j D^\beta K_\alpha(x-\sigma y, t-\sigma s) - D_t^j D^\beta K_\alpha(x,t)]\,d\sigma.$$

From here it follows choosing $m=1$

$$K_\alpha(x-y,t-s) \tag{38}$$

$$= K_\alpha(x,t) - \sum_{j=1}^2 \frac{\partial K_\alpha}{\partial x_j}(x,t) y_j - \frac{\partial K_\alpha}{\partial t}(x,t)s + H(x,t,y,s) \tag{39}$$

with

$$\begin{aligned} H(x,t,y,s) &= \int_0^1 \left[\sum_{j=1}^2 \left(\frac{\partial K_\alpha}{\partial x_j}(x,t) - \frac{\partial K_\alpha}{\partial x_j}(x-\sigma y, t-\sigma s) \right) y_j \right. \\ &\quad + \left. \left(\frac{\partial K_\alpha}{\partial t}(x,t) - \frac{\partial K_\alpha}{\partial t}(x-\sigma y, t-\sigma s) \right) s \right] d\sigma. \end{aligned}$$

Observe that

$$\int_{\mathbb{R}^2} u(s)\cdot\nabla\theta(s)\,dy\,ds = \int_{\mathbb{R}^2} \mathrm{div}\ [\theta(s)u(s)]\,dy = 0.$$

One also has

$$\int_{\mathbb{R}^2} y_j[u(s)\cdot\nabla\theta(s)]\,dy = 0,\ j = 1,2.$$

In fact,

$$\int_{\mathbb{R}^2} y_j[u(s)\cdot\nabla\theta(s)]\,dy = \int_{\mathbb{R}^2} y_j \mathrm{div}\ (\theta u)\,dy = -\int_{\mathbb{R}^2} u_j\theta\,dy.$$

The last integral can be written in the form

$$\pm\int_{\mathbb{R}^2} (\mathcal{R}_k\theta)\theta\,dy\,(k \neq j)$$

and is zero by the skew-adjointness of the Riesz transforms. Using the vanishing of these integrals when multiplying (38) by $u(y,s)\cdot\nabla\theta(y,s)$ and integrating over $\mathbb{R}^2\times[0,t]$ with respect to (y,s), one gets

$$\begin{aligned}\theta(x,t)-\Theta_0(x,t) &= -\int_0^t\int_{\mathbb{R}^2} K_\alpha(t-s)[u(y,s)\cdot\nabla\theta(y,s)]\,dy\,ds\\ &= -\int_0^t\int_{\mathbb{R}^2} H(x,t,y,s)u(y,s)\cdot\nabla\theta(y,s)]\,dy\,ds.\end{aligned}\tag{40}$$

Theorem 7. *Assume $\theta_0\in L^1\cap L^2_w\cap H^1$, and $\alpha\in(1/2,1)$.In the case $\alpha=1$ we suppose additionally that $\|\widehat{\theta}(\xi)\|\le c|\xi|^\mu$, with $\mu>0$. Then*

$$\lim_{t\to\infty} t^{1/\alpha}\|\theta(t)-\Theta_0(t)\|_2=0.$$

Proof. By (40), it suffices to prove

$$\lim_{t\to\infty} t^{1/\alpha}\int_0^t\left\|\int_{\mathbb{R}^2} H(\cdot,t,y,s)u(y,s)\cdot\nabla\theta(y,s)\,dy\right\|_2 ds=0.\tag{41}$$

From the definition of H,in (39) the fact that $u\nabla\theta=\operatorname{div}(\theta u)$, and some integration by parts, one can write

$$\int_{\mathbb{R}^2} H(x,t,y,s)u(y,s)\cdot\nabla\theta(y,s)\,dy=\sum_{1\le j,k\le 2}A_{jk}+B_1+B_2+B_3$$

where

$$\begin{aligned}A_{jk} &= A_{jk}(x,t,s)=-\int_0^1\int_{\mathbb{R}^2}\frac{\partial^2K_\alpha}{\partial x_j\partial x_k}(x-\sigma y,t-\sigma s)(\sigma y_j)u_k(y,s)\theta(y,s)\,dy\,d\sigma\\ &= -\int_0^1\frac{1}{\sigma}\frac{\partial^2K_\alpha}{\partial x_j\partial x_k}(\cdot,t-\sigma s)*[(\frac{y_j}{\sigma})u_k(\frac{\cdot}{\sigma},s)\theta(\frac{\cdot}{\sigma},s)]\,d\sigma, j,k=1,2.\\ B_j &= B_j(x,t,s)=\int_0^1\int_{\mathbb{R}^2}\left(\frac{\partial K_\alpha}{\partial x_j}(x,t)-\frac{\partial K_\alpha}{\partial x_j}(x-\sigma y,t-\sigma s)\right)u_j(y,s)\theta(y,s)\,dy\,d\sigma,\ \ j=1,2;\\ B_3 &= B_3(x,t,s)=\int_0^1\int_{\mathbb{R}^2}\left(\frac{\partial K_\alpha}{\partial t}(\cdot,t)-\frac{\partial K_\alpha}{\partial t}(x-\sigma y,t-\sigma s)\right)su(y,s)\cdot\nabla\theta(y,s)\,dy\;d\sigma.\end{aligned}$$

Then (41) is equivalent to

$$\lim_{t\to\infty} t^{\frac{1}{\alpha}}\int_0^t\|A_{jk}(t,s)\|_2\,ds = 0,\ \ j,k=1,2,\tag{42}$$

$$\lim_{t\to\infty} t^{\frac{1}{\alpha}}\int_0^t\|B_j(t,s)\|_2\,ds = 0,\ \ j=1,2,3.\tag{43}$$

Let $2\le r<\infty$; by Lemma 2, with $p=r'$, $q=2$, and by lemma 1,

$$\begin{aligned}\|A_{jk}(t,s)\|_2 &\le \int_0^1\frac{1}{\sigma}\|\frac{\partial^2K_\alpha}{\partial x_j\partial x_k}(t-\sigma s)\|_{r'}\|u_k(\frac{\cdot}{\sigma},s)\|_r\|(\frac{y_j}{\sigma})\theta(\frac{\cdot}{\sigma},s)\|_2\,d\sigma\\ &= \int_0^1\sigma^{2/r}\|\frac{\partial^2K_\alpha}{\partial x_j\partial x_k}(t-\sigma s)\|_{r'}\|u_k(s)\|_r\|y_j\theta(s)\|_2\,d\sigma\\ &\le C\left(\int_0^1\sigma^{2/r}(t-\sigma s)^{-\frac{1}{\alpha}-\frac{1}{\alpha r}}\,d\sigma\right)\|u(s)\|_r\|y_j\theta(s)\|_2;\end{aligned}$$

estimating $\sigma^{2/r} \le 1$, performing the integration with respect to σ and using (26) to estimate the L^r-norm of u and Theorem 2 for the L^2-norm of the moment $y_j\theta$, we get

$$\|A_{jk}(t,s)\|_2 \le \frac{C}{s}\left((t-s)^{1-\frac{r+1}{\alpha r}} - t^{1-\frac{r+1}{\alpha r}}\right)(1+s)^{-\frac{1}{\alpha}\left(\frac{1}{r'}+\mu\right)}. \tag{44}$$

We assume that $\mu = 0$ if $1/2 < \alpha < 1$, $0 < \mu < 1$ if $\alpha = 1$. Let r be large enough so that its conjugate exponent satisfies

$$1 < r' < \min\left(\frac{1}{\alpha-\mu}, \frac{1}{2(1-\alpha)}\right).$$

Notice that $1/(\alpha-\mu) = 1/\alpha > 1$ if $\alpha < 1$, $1/(\alpha-\mu) = 1/(1-\mu) > 1$ if $\alpha = 1$. Notice also that $1/(2(1-\alpha)) > 1$. The condition $r' < 1/(2(1-\alpha))$ is equivalent to $1-((r+1)/(\alpha r) > -1$. The condition $r' < 1/(\alpha-\mu)$ is equivalent to having the exponent of $1+s$ in (44) be less than -1; i.e., to

$$\frac{1}{\alpha}\left(\frac{1}{r'}+\mu\right) > 1.$$

We will estimate

$$\frac{(t-s)^{1-\frac{r+1}{\alpha r}} - t^{1-\frac{r+1}{\alpha r}}}{s} \le \begin{cases} Ct^{-\frac{r+1}{\alpha r}}, \text{if} & 0 \le s \le t/2, \\ C\frac{(t-s)^{1-\frac{r+1}{\alpha r}}}{t} \quad \text{if} & t/2 < s < t. \end{cases}$$

Thus

$$\int_0^{t/2} \frac{\left((t-s)^{1-\frac{r+1}{\alpha r}} - t^{1-\frac{r+1}{\alpha r}}\right)}{s}(1+s)^{-\frac{1}{\alpha}\left(\frac{1}{r'}+\mu\right)}\,ds$$
$$\le Ct^{-\frac{r+1}{\alpha r}}\int_0^\infty (1+s)^{-\frac{1}{\alpha}\left(\frac{1}{r'}+\mu\right)}\,ds \le Ct^{-\frac{1}{\alpha}-\frac{1}{\alpha r}},$$

while

$$\int_{t/2}^{t} \frac{\left((t-s)^{1-\frac{r+1}{\alpha r}} - t^{1-\frac{r+1}{\alpha r}}\right)}{s}(1+s)^{-\frac{1}{\alpha}\left(\frac{1}{r'}+\mu\right)}\,ds$$
$$\le Ct^{-1-\frac{1}{\alpha}\left(\frac{1}{r'}+\mu\right)}\int_{t/2}^t (t-s)^{1-\frac{r+1}{\alpha r}}\,ds \le Ct^{1-\frac{2}{\alpha}-\frac{\mu}{\alpha}}.$$

It follows that

$$t^{\frac{1}{\alpha}}\int_0^t \|A_{jk}(t,s)\|_2\,ds \le C\left(t^{-\frac{1}{\alpha r}} + t^{1-\frac{1}{\alpha}-\frac{\mu}{\alpha}}\right).$$

Notice that $1-\frac{1}{\alpha}-\frac{\mu}{\alpha} < 0$ because $1-1/\alpha < 0$ if $\alpha < 1$, $-\mu < 0$ if $\alpha = 1$. Thus

$$\lim_{t\to\infty} t^{\frac{1}{\alpha}}\int_0^t \|A_{jk}(t,s)\|_2\,ds = 0,$$

proving (42).

Consider next the terms B_j, $j = 1,2,3$. For convenience we set $\partial/\partial t = \partial/\partial x_3$ and introduce

$$k_j(t,y,s,\sigma) = \left\|\frac{\partial K_\alpha}{\partial x_j}(\cdot,t) - \frac{\partial K_\alpha}{\partial x_j}(\cdot-\sigma y, t-\sigma s)\right\|_2$$

for $j = 1, 2, 3$; then

$$\|B_j(t,s)\|_2 \le \begin{cases} \int_0^1 \int_{\mathbb{R}^2} k_j(t,y,s,\sigma)|u(y,s)||\theta(y,s)|\,dy\,d\sigma, & j = 1,2, \\ \\ \int_0^1 \int_{\mathbb{R}^2} k_j(t,y,s,\sigma)s|u(y,s)||\nabla\theta(y,s)|\,dy\,d\sigma, & j = 3. \end{cases} \tag{45}$$

Using that

$$\left\{ \frac{\partial \widehat{K_\alpha}}{\partial x_j}(\cdot,t) - \frac{\partial \widehat{K_\alpha}}{\partial x_j}(\cdot - \sigma y, t - \sigma s) \right\}(\xi) = \begin{cases} i\xi_j e^{-t|\xi|^{2\alpha}} \left(1 - e^{-i\sigma y\cdot\xi} e^{\sigma s|\xi|^{2\alpha}}\right), & j = 1,2, \\ \\ -|\xi|^{2\alpha} e^{-t|\xi|^{2\alpha}} \left(1 - e^{-i\sigma y\cdot\xi} e^{\sigma s|\xi|^{2\alpha}}\right), & j = 3, \end{cases}$$

it follows by Parseval and changing variables by $\eta = t^{\frac{1}{2\alpha}}\xi$ that

$$\begin{aligned} k_j(t,y,s,\sigma) &\le \left(\int_{\mathbb{R}^2} |\xi|^{\lambda_j} e^{-2t|\xi|^{2\alpha}} \left|1 - e^{-i\sigma y\cdot\xi} e^{\sigma s|\xi|^{2\alpha}}\right|^2 d\xi \right)^{1/2} \\ &= t^{-\left(\frac{2}{\alpha} + \frac{\lambda_j}{4\alpha}\right)} \left(\int_{\mathbb{R}^2} |\eta|^{\lambda_j} e^{-2|\eta|^{2\alpha}} \left|1 - e^{-it^{-1/2\alpha}\sigma y\cdot\eta} e^{\sigma(s/t)|\eta|^{2\alpha}}\right|^2 d\eta \right)^{1/2} \end{aligned} \tag{46}$$

where $\lambda_1 = \lambda_2 = 2$, $\lambda_3 = 4\alpha$. The integrand of the last integral in (46) goes pointwise to 0 as $t \to \infty$ and is bounded by

$$C|\eta|^{\lambda_j} e^{-2(1-\frac{\sigma s}{t})|\eta|^{2\alpha}} \le C|\eta|^{\lambda_j} e^{-|\eta|^{2\alpha}}$$

if $t \ge 2s$, which is integrable over $\mathbb{R}^2$ hence, by Lebesgue's dominated convergence theorem,

$$\lim_{t\to\infty} t^{\frac{1}{2\alpha} + \frac{\lambda_j}{4\alpha}} k_j(t,y,s,\sigma) = 0 \tag{47}$$

for all fixed values of y, s, σ, $j = 1, 2, 3$.

Assume now $0 \le s \le t/2$. Then the integrand of the final integral in (46) can be estimated, as mentioned, by the integrable expression $C|\eta|^{\lambda_j} e^{-|\eta|^{2\alpha}}$, hence

$$t^{\frac{1}{\alpha}} k_j(t,y,s,\sigma) \le C t^{\frac{1}{2\alpha} - \frac{\lambda_j}{4\alpha}} = \begin{cases} C, & j = 1,2, \\ Ct^{-1+\frac{1}{2\alpha}}, & j = 3. \end{cases} \tag{48}$$

Let $j = 1, 2$; then

$$t^{\frac{1}{\alpha}} \int_0^{t/2} \|B_j(t,s)\|_2\,ds \le \int_0^\infty \int_0^1 \int_{\mathbb{R}^2} t^{1/\alpha} k_j(t,y,s,\sigma)\chi_{[0,t/2]}(s)|u(y,s)||\theta(y,s)|\,dy\,d\sigma\,ds.$$

The integrand of this integral converges to 0 for $t \to \infty$, by (47) ($1/(2\alpha) + \lambda_j/(4\alpha) = 1/\alpha$ if $j = 1, 2$). By (48) it is bounded by $C|u(y,s)||\theta(y,s)|$, which is integrable over the domain of integration; in fact, by Theorem 2 and Corollary 1 it follows that

$$\int_0^\infty \int_0^1 \int_{\mathbb{R}^2} |u(y,s)||\theta(y,s)|\,dy\,d\sigma\,ds \le \int_0^\infty \|u(s)\|_2\|\theta(s)\|_2\,ds \le C\int_0^\infty (1+s)^{-\frac{1+\mu}{\alpha}}\,ds < \infty$$

this last integral is bounded for $\alpha \le 1$ since for $\alpha = 1$ we assume that $\mu > 0$.

Invoking again the Lebesgue dominated convergence theorem it follows from (47) that

$$\lim_{t\to\infty} t^{1/\alpha} \int_0^{t/2} \|B_j(t,s)\|_2\,ds = 0, \tag{49}$$

for $j = 1, 2$.

If $j = 3$, then by (48),

$$
\begin{aligned}
t^{\frac{1}{\alpha}} \int_0^{t/2} \|B_3(t,s)\|_2\, ds &\leq \int_0^{t/2} \int_0^1 \int_{\mathbb{R}^2} t^{1/\alpha} k_j(t,y,s,\sigma) s |u(y,s)||\nabla\theta(y,s)|\, dy\, d\sigma\, ds \\
&\leq Ct^{-1+\frac{1}{2\alpha}} \int_0^{t/2} s\|u(s)\|_2 \|\nabla\theta(s)\|_2\, ds;
\end{aligned}
$$

by Theorem 2 and Corollary 1,

$$
t^{\frac{1}{\alpha}} \int_0^{t/2} \|B_3(t,s)\|_2\, ds \leq Ct^{-1+\frac{1}{2\alpha}} \int_0^{t/2} s(1+s)^{-\frac{3+\mu}{2\alpha}}\, ds.
$$

The integral in this last expression is largest if $\mu = 0$, in which case it is uniformly bounded in t if $1/2 < \alpha < 3/4$, and it is of order $t^{2-(3/2\alpha)}$ if $3/4 \leq \alpha < 1$. If $\alpha = 1$ we assume $\mu > 0$ and the integral in question is of order $t^{2-(3/2)\mu}$. In each case, multiplying by $t^{-1+(1/2\alpha)}$ gives an expression going to 0 for $t \to \infty$, proving that (49) is also valid for $j = 3$.

Assume now $t/2 < s < t$. We claim that

$$
\int_0^1 k_j(t,y,s,\sigma)\, d\sigma \leq \begin{cases} Ct^{-1}(t-s)^{-\frac{1}{\alpha}+1}, & j = 1,2; \alpha < 1 \\ Ct^{-1}\log(t/(t-s)), & j = 1,2; \alpha = 1, \\ Ct^{-1}(t-s)^{-\frac{1}{2\alpha}}, & j = 3. \end{cases} \tag{50}
$$

In fact, by the first inequality in (46), if we bound

$$
|\xi|^{\lambda_j} e^{-2t|\xi|^{2\alpha}} \left|1 - e^{-i\sigma y\cdot\xi} e^{\sigma s|\xi|^{2\alpha}}\right|^2 \leq C|\xi|^{\lambda_j} e^{-2(t-\sigma s)|\xi|^{2\alpha}},
$$

then

$$
\begin{aligned}
\int_0^1 k_j(t,y,s,\sigma)\, d\sigma &\leq C \int_0^1 \left(\int_{\mathbb{R}^2} |\xi|^{\lambda_j} e^{-2(t-\sigma s)|\xi|^{2\alpha}}\, d\xi\right)^{1/2} d\sigma \\
&= C\int_0^1 (t-\sigma s)^{-\frac{1}{\alpha}-\frac{\lambda_j}{2\alpha}}\, d\sigma.
\end{aligned}
$$

The claim follows performing this last integral, recalling that $\lambda_j = 2$ if $j = 1,2$, $\lambda_3 = 4\alpha$, and $t/2 < s < t$.

Let $j = 1,2$. By Theorem 2and Corollary 1, and by (50), we get if $\alpha < 1$,

$$
\begin{aligned}
t^{\frac{1}{\alpha}} \int_{t/2}^t \|B_j(t,s)\|_2\, ds &\leq \int_{t/2}^t t^{\frac{1}{\alpha}-1}(t-s)^{1-\frac{1}{\alpha}}(1+s)^{-\frac{1+\mu}{\alpha}}\, ds \\
&\leq Ct^{1-\frac{1+\mu}{\alpha}} \int_{1/2}^1 s^{1/\alpha}(1-s)^{1-\frac{1}{\alpha}}\, ds \leq Ct^{1-\frac{1+\mu}{\alpha}}
\end{aligned}
$$

which goes to 0 as $t \to \infty$. The same conclusion follows if $\alpha = 1$; in this case the factor $(t-2)^{-(1/\alpha)+1}$ is replaced by $\log(t/(t-s)$ and the final estimate is

$$
t^{\frac{1}{\alpha}} \int_{t/2}^t \|B_j(t,s)\|_2\, ds \leq Ct^{-\mu} \int_{1/2}^1 \log(1/(1-s))\, ds \leq Ct^{-\mu} \to 0
$$

as $t \to \infty$. Assuming finally $j = 3$, then by (50)

$$
\begin{aligned}
t^{\frac{1}{\alpha}} \int_{t/2} \|B_3(t,s)\|_2\, ds &\leq t^{1/\alpha} \int_{t/2}^t \int_{\mathbb{R}^2} \int_0^1 k_3(t,y,s,\sigma) s|u(y,s)||\nabla\theta(y,s)|\, d\sigma\, dy\, ds \\
&\leq Ct^{-1+\frac{1}{\alpha}} \int_{t/2}^t (t-s)^{-\frac{1}{2\alpha}} s\|u(s)\|_2 \|\nabla\theta(s)\|_2\, ds.
\end{aligned}
$$

Applying Theorem 2and Corollary 1,

$$t^{\frac{1}{\alpha}}\int_{t/2}\|B_3(t,s)\|_2\,ds \le Ct^{-1+1/\alpha}\int_{t/2}^{t}(t-s)^{-\frac{1}{2\alpha}}s(1+s)^{-\frac{3+\mu}{2\alpha}}\,ds \le Ct^{1-\frac{2+\mu}{\alpha}} \to 0$$

as $t\to\infty$. We proved that

$$\lim_{t\to\infty} t^{\frac{1}{\alpha}}\int_{t/2}^{t}\|B_j(t,s)\|_2\,ds = 0$$

for $j=1,2,3$. Together with (49) (which was proved for $j=1,2,3$) it follows that (43) holds for $j=1,2,3$. This concludes the proof of the theorem. □

Remark 4. While Theorem 7 has somewhat stronger hypotheses than [Theorem 4.3] in [4], the conclusion is also considerably stronger.

The next theorem establishes the lower bounds of rates of decay for solutions with zero or non zero initial mass.

Theorem 8. *Assume the hypothesis of Theorem 7. Let θ be a solution to equation (1) with initial datum θ_0 and Θ_0 be the solution of the linear geostrophic equation with the same initial datum. Then for any $\tau\in[0,1/\alpha]$,*

$$C_0(t+1)^{-\tau} \le \|\Theta_0(t)\|_2 \le C_1(t+1)^{-\tau}$$

if and only if

$$C_0(t+1)^{-\tau} \le \|\theta(t)\|_2 \le C_1(t+1)^{-\tau}.$$

In particular, if $0\le\mu\le 1$ and $\hat\theta_0(\xi)$ is of order $|\xi|^\mu$ near the origin; i.e., satisfies that there exists constants $c_1\ge c_0>0$ such that

$$c_0|\xi|^\mu \le |\hat\theta_0(\xi)| \le c_1|\xi|^\mu$$

in a neighborhood of the origin, then there exist constants C_0, C_1 such that

$$C_0(t+1)^{-\frac{\mu+1}{2\alpha}} \le \|\theta(t)\|_2 \le C_1(t+1)^{-\frac{\mu+1}{2\alpha}}.$$

for $t\ge 0$.

Proof. The first part of the theorem is an immediate consequence of Theorem 7 and appropriate triangle inequalities. The second part follows from the first part in view of Lemma 7. □

Appendix. In this appendix we prove some of the properties of the one-parameter semigroup of operators generated by $(-\Delta)^\alpha$. We will be working in $\mathbb{R}^n$ since there is nothing to gain by assuming $n=2$. That is, we define $K_\alpha:\mathbb{R}^n\times\mathbb{R}\to\mathbb{C}$ by

$$\hat K_\alpha(\xi,t) = (2\pi)^{-n/2}e^{-t|\xi|^{2\alpha}}.$$

We notice that

$$K_\alpha(x,t) = (2\pi)^{-n/2}t^{-\frac{n}{2\alpha}}g_\alpha\left(t^{-\frac{1}{2\alpha}}x\right)$$

where $g_\alpha:\mathbb{R}^n\to\mathbb{C}$ is defined by

$$\hat g_\alpha(\xi) = e^{-|\xi|^{2\alpha}}.$$

We will also consider the function $g_{\alpha,j}$ defined for $j=0,1,\ldots$ by

$$\hat g_{\alpha,j}(\xi) = (-1)^j|\xi|^{2j\alpha}e^{-|\xi|^{2\alpha}},$$

(so $g_\alpha = g_{\alpha,0}$). Then

$$D_t^j K_\alpha(x,t) = (2\pi)^{-n/2} t^{-\frac{n}{2\alpha}-j} g_{\alpha,j}\left(t^{-\frac{1}{2\alpha}}x\right). \tag{51}$$

Lemma 5. *Let $j \in \mathbb{N}\cup\{0\}$. Then $g_{\alpha,j} \in C^\infty(\mathbb{R}^n)$ and $x^\gamma D^\beta g_{a,j} \in L^1(\mathbb{R}^n)\cap L^\infty(\mathbb{R}^n)$ for all multi-indices β, γ such that $|\gamma| < |\beta| + 2\alpha\max(1,j)$.*

Proof. We have

$$g_{\alpha,j}(x) = (2\pi)^{-n/2}(-1)^j \int_{\mathbb{R}^n} e^{ix\cdot\xi}|\xi|^{2j\alpha}e^{-|\xi|^{2\alpha}}\,d\xi,$$

from which it is obvious that $g_a \in C^\infty$, $D^\beta g_\alpha \in L^\infty$ for all β. Next, we claim that for all multi-indices $\lambda = (\lambda_1,\ldots,\lambda_n) \neq 0$,

$$D^\lambda\left(|\xi|^{2j\alpha}e^{-|\xi|^{2\alpha}}\right) = \sum_{\ell=0}^m p_\ell(\xi)|\xi|^{2(\ell+j)\alpha-2|\lambda|}e^{-|\xi|^{2\alpha}} \quad (\xi \neq 0)$$

where m is a positive integer depending on λ and each p_ℓ is a homogeneous polynomial, depending on λ, of degree $|\lambda|$. Moreover, $p_0(\xi) \equiv 0$ if $j = 0$, $|\lambda| \geq 1$. We prove this by induction on $|\lambda|$. The case $|\lambda| = 0$ is, of course, obvious. If $j = 0$ and $|\lambda| = 1$; say, $\lambda = e_j = (\delta_{j1},\ldots,\delta_{jn})$, Then

$$D^\lambda\left(e^{-|\xi|^{2\alpha}}\right) = -2\alpha|\xi|^{2\alpha-2}\xi_j e^{-|\xi|^{2\alpha}}$$

which is of the claimed form, with $m = 1$, $p_1(\xi) = \xi_j$ and $p_0 = 0$. Assuming the result proved for $|\lambda| = k$, some $k \geq 0$, to see it implies the result for $k+1$ it suffices to see that differentiating with respect to ξ_j a term of the form

$$p_\ell(\xi)|\xi|^{2(\ell+j)\alpha-2k}e^{-|\xi|^{2\alpha}},$$

where $\ell \in \mathbb{N}$ and p_ℓ is a homogeneous polynomial of degree k, gives rise to a sum of similar terms, with k replaced by $k+1$. We have

$$\begin{aligned}
\frac{\partial}{\partial\xi_j}\left(p_\ell(\xi)|\xi|^{2(\ell+j)\alpha-2k}e^{-|\xi|^{2\alpha}}\right) \quad &= \quad \left(\frac{\partial p_\ell}{\partial \xi_j}(\xi)|\xi|^{2(\ell+j)\alpha-2k}\right.\\
&\left.+(2(\ell+j)\alpha-2k)p_\ell(\xi)|\xi|^{2(\ell+j)\alpha-2k-2}\xi_j - 2\alpha p_\ell(\xi)|\xi|^{2(\ell+j)\alpha-2k+2\alpha-2}\xi_j\right)e^{-|\xi|^{2\alpha}}\\
&= q_1(\xi)|\xi|^{2(\ell+j)\alpha-2(k+1)}e^{-|\xi|^{2\alpha}} + q_2(\xi)|\xi|^{2(\ell\alpha+j)-2(k+1)}e^{-|\xi|^{2\alpha}}\\
&+q_3(\xi)|\xi|^{2(\ell+j+1)\alpha-2(k+1)}e^{-|\xi|^{2\alpha}},
\end{aligned}$$

where q_1, q_2, q_3 are given by

$$\begin{aligned}
q_1(\xi) &= \frac{\partial p_\ell}{\partial\xi_j}(\xi)|\xi|^2,\\
q_2(\xi) &= (2(\ell+j)\alpha-2k)p_\ell(\xi)\xi_j,\\
q_3(\xi) &= -2\alpha p_\ell(\xi)\xi_j,
\end{aligned}$$

hence are homogeneous polynomials of degree $k+1$. The claim is established. It is now an easy consequence of Leibniz' formula, if we estimate $|p_\ell(\xi)|$ by const$|\xi|^{|\lambda|}$, that for all multi-indices β, λ we can write

$$D^\lambda\left(\xi^\beta e^{-|\xi|^{2\alpha}}\right) = \begin{cases} |\xi|^{2\alpha+|\beta|-|\lambda|}h(\xi) & \text{if} \quad j = 0,\\ |\xi|^{2j\alpha+|\beta|-|\lambda|}h(\xi) & \text{if} \quad j \geq 1.\end{cases} \tag{52}$$

where

$$|h(\xi)| \leq |\xi|^\nu e^{-|\xi|^{2\alpha}}$$

for some $\nu \geq 0$ (depending, of course, on β, λ, and j). We thus proved, replacing λ by $\lambda + \gamma$ and setting $g_{\alpha,j,\beta,\gamma} = x^\gamma D^\beta g_{\alpha,j}$, that

$$\left|\left\{x^\lambda g_{\alpha,j,\beta,\gamma}\right\}\widehat{\ }(\xi)\right| = \left|D^{\lambda+\gamma}\left(\xi^\beta \hat{g}_{\alpha,j}\right)(\xi)\right| = |\xi|^\mu h(\xi) \tag{53}$$

where h is as described and

$$\mu = \begin{cases} 2\alpha + |\beta| - |\gamma| - |\lambda| & \text{if } \quad j = 0, \\ 2\alpha j + |\beta| - |\gamma| - |\lambda| & \text{if } \quad j = 0. \end{cases}$$

By the assumption on γ we conclude that $|\mu| > -|\lambda|$ so that taking now $|\lambda| = n$ we see that $D^{\lambda+\gamma}\left(\xi^\beta \hat{g}_{\alpha,j}\right)$ is locally integrable in $\mathbb{R}^n$, in particular, the classical derivatives we computed for $\xi \neq 0$, coincide with the derivatives in the sense of distributions. Moreover, we can select r such that $1 < r \leq 2$ and $r\mu > -n$; then $|D^{\lambda+\gamma}\left(\xi^\beta \hat{g}_{\alpha,j}\right)|^r$ is also locally integrable, hence integrable since at infinity it decays faster than any negative power of $|\xi|$. In other words, the Fourier transform of $x^\lambda g_{\alpha,j,\beta,\gamma}$ is in L^r hence, since $1 < r \leq 2$, $x^\lambda g_{\alpha,j,\beta,\gamma} \in L^{r'}$. Since this holds for all λ with $|\lambda| = n$, we proved that $|x|^n g_{\alpha,j,\beta,\gamma} \in L^{r'}$. On the other hand, it is clear that $\hat{g}_{\alpha,j,\beta,\gamma} \in L^1 \cap L^2$, thus $g_{\alpha,j,\beta,\gamma} \in L^\infty \cap L^2$, in particular, $g_{\alpha,j,\beta,\gamma} \in L^{r'}$ proving that $(1 + |x|)^n g_{\alpha,j,\beta,\gamma} \in L^{r'}$. Thus

$$\begin{aligned} \int_{\mathbb{R}^n} |g_{\alpha,j,\beta,\gamma}(x)|\,dx &= \int_{\mathbb{R}^n} (1+|x|)^{-n}(1+|x|)^n |g_{\alpha,j,\beta,\gamma}(x)|\,dx \\ &\leq \left(\int_{\mathbb{R}^n} (1+|x|)^{-rn}\,dx\right)^{\frac{1}{r}} \left(\int_{\mathbb{R}^n} |(1+|x|)^n g_{\alpha,j,\beta,\gamma}(x)|^{r'}\,dx\right)^{\frac{1}{r'}} < \infty. \end{aligned}$$

This completes the proof that $x^\gamma D^\beta g_{\alpha,j} = (i)^{|\beta|+|\gamma|} g_{\alpha,j,\beta,\gamma} \in L^1$. □

As an immediate corollary to Lemma 5 we obtain

Lemma 6. *Let β, γ be multi-indices, $|\gamma| < |\beta| + 2\alpha \max(j, 1)$, $j = 0, 1, 2, \ldots$, $1 \leq p \leq \infty$. Then*

$$\|x^\gamma D_t^j D^\beta K_\alpha(t)\|_p = C t^{\frac{|\gamma|-|\beta|}{2\alpha} - j - \frac{n(p-1)}{2\alpha p}}$$

for some constant C depending only on α, β, γ, j, p, and the space dimension n.

Proof. In view of (51),

$$x^\gamma D_t^j D^\beta K_\alpha(t) = (2\pi)^{-n/2} t^{-\frac{n+|\beta|}{2\alpha} - j} x^\gamma (D^\beta g_{\alpha,j})\left(t^{-\frac{1}{2\alpha}} x\right).$$

By Lemma 5, $x^\gamma D^\beta g_{\alpha,j} \in L^1 \cap L^\infty \subset L^p$, hence

$$\|x^\gamma D_t^j D^\beta K_\alpha(t)\|_p = (2\pi)^{-n/2} t^{\frac{|\gamma|-|\beta|}{2\alpha} - j - \frac{n(p-1)}{2\alpha p}} \|x^\gamma D^\beta g_{\alpha,j}\|_p$$

proving the lemma with $C = (2\pi)^{-n/2}\|x^\gamma D^\beta g_{\alpha,j}\|_p$.

□

In particular, we see that $K_\alpha(t) \in L^1(\mathbb{R}^n)$ for all $t > 0$ and

$$\|K_\alpha(t)\|_1 \leq C$$

for some constant C depending only on α and the space dimension n. It is also easy to see that $K_\alpha(t) * K_\alpha(s) = K_\alpha(s + t)$ for all $s, t > 0$ and that

$$\lim_{t \to 0} K_\alpha(t) * f = f$$

uniformly for every continuous function of compact support $f : \mathbb{R}^n \to \mathbb{R}$. It follows that if we define

$$e^{-t(-\Delta)^\alpha} f = K_\alpha(t) * f$$

for $f \in L^p(\mathbb{R}^n)$, then $\{e^{-t(-\Delta)^\alpha}\}$ is a one-parameter, strongly continuous, semi-group of uniformly bounded operators in $L^p(\mathbb{R}^n)$ for $1 \le p < \infty$. As mentioned before, we can then define $(-\Delta)^\alpha$ as the generator of this semi-group.

Let $1 < r < \infty$. If $\theta_0 \in L^1(\mathbb{R}^n) \cap L^r(R^n)$ then

$$\|K_\alpha(t) * \theta_0\|_2 \le C(1+t)^{-n/2\alpha r'} \|\theta_0\|_1 \tag{54}$$

for all $t \ge 0$. This is, of course an immediate consequence of Lemma 6, according to which $\|K_\alpha(t)\|_r = Ct^{-n/2\alpha r'}$ (and $\|K_\alpha(t)\|_1$ is constant in t). A better rate of decay can be obtained if one knows something about the behavior of $\hat{\theta}_0(\xi)$ near the origin. In fact, from

$$\|K_\alpha(t) * \theta_0\|_2^2 = \|\hat{K}_\alpha(t)\hat{\theta}_0\|_2^2 = \int_{\mathbb{R}^n} e^{-2t|\xi|^{2\alpha}} |\theta_0(\xi)|^2 \, d\xi$$

it is easy to derive the following result. We omit the simple proof.

Lemma 7. *Assume $\theta_0 \in L^2 \cap L^1$ and there exist constants $c_1 \ge c_0 > 0$, $\mu \ge 0$ such that*

$$c_0|\xi|^\mu \le |\hat{\theta}_0(\xi)| \le c_1|\xi|^\mu$$

for all ξ in a neighborhood of the origin. Then there exist constants C_1, C_2 such that

$$C_1(1+t)^{-\frac{2\mu+n}{4\alpha}} \le \|K_\alpha(t) * \theta_0\|_2 \le C_2(1+t)^{-\frac{2\mu+n}{4\alpha}}.$$

We use (54) to estimate the L^r norm of the moments of $K_\alpha(t) * \theta_0$.

Lemma 8. *Let $\theta_0 \in L^1_w(\mathbb{R}^n) \cap L^r_w(\mathbb{R}^n)$, where $2 \le r < \infty$, and assume that $|\hat{\theta}_0(\xi)| \le C|\xi|^\mu$ for some constants $C, \mu \ge 0$ and all ξ in a neighborhood of 0 in $\mathbb{R}^n$. Then*

$$\|x_j\left(K_\alpha(t) * \theta_0\right)\|_2 \le C(1+t)^{-\min\left(\frac{n}{2\alpha r'}, \frac{n+(\mu-1)r'}{2\alpha r'}\right)}$$

for all $t \ge 0$, and some constant C depending on θ_0 (j=1,2).

Proof. We have

$$x_j\left(K_\alpha(t) * \theta_0\right) = \left(x_j K_\alpha(t)\right) * \theta_0 + K_\alpha(t) * \left(x_j \theta_0\right).$$

By (54), the second term on the right hand side has L^r norm bounded by $C(1+t)^{-n/2\alpha r'}$. It suffices to estimate the L^r norm of the first term. Since $r \ge 2$,

$$\begin{aligned} \|\left(x_j K_\alpha(t)\right) * \theta_0\|_r &\le C\|\frac{\partial K_\alpha}{\partial \xi_j}\hat{\theta}_0\|_{r'} \\ &\le C\left(\int_{\mathbb{R}^2} t^{r'}|\xi|^{(2\alpha-1)r'} e^{-r't|\xi|^{2\alpha}} |\hat{\theta}_0(\xi)|^{r'} \, d\xi\right)^{1/r'}. \end{aligned}$$

Since $\theta \in L^1$ we have $\hat{\theta} \in L^\infty$, and we may assume that $|\theta(\xi)| leC|\xi|^\mu$ for all $\xi \in \mathbb{R}^n$. Thus, going over to polar coordinates $\rho = |\xi|$ and then changing variables by $\rho' = t^{1/2\alpha}\rho$,

$$\begin{aligned} \|\left(x_j K_\alpha(t)\right) * \theta_0\|_r &\le Ct\left(\int_0^\infty \rho^{n-1+(2\alpha-1)r'} e^{-r't\rho^{2\alpha}} \, d\rho\right)^{1/r'} \\ &= Ct^{-\frac{n+(\mu-1)r'}{2\alpha r'}}. \end{aligned}$$

The Lemma follows. □

The hypotheses $\theta_0 \in L^1_w$ of Lemma 8 implies $\theta_0 \in L^1$, hence $\hat{\theta}_0 \in L^\infty$ and if in addition $\theta_0 \in L^r_w$ then all hypotheses of Lemma 8 hold with $\mu = 0$. In this case $n/2\alpha r' \geq (n + (\mu - 1)r')/2\alpha r' = (n - r')/2\alpha r'$. We thus have

Corollary 4. *Let $\theta_0 \in L^1_w(\mathbb{R}^n) \cap L^r_w(\mathbb{R}^n)$, where $2 \leq r < \infty$. Then*

$$\|x_j\,(K_\alpha(t) * \theta_0)\,\|_r \leq C(1+t)^{-\frac{n-r'}{2\alpha r'}}$$

for all $t \geq 0$, and some constant C depending on θ_0 (j=1,2).

Appendix 2: A modified Gronwall inequality. Let $0 < \delta < 1$ and define $\Phi : \mathbb{C} \to \mathbb{C}$ by

$$\Phi(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma\left(n(1-\delta)+1\right)}.$$

It is easy to see that this series converges for all $z \in \mathbb{C}$; for example, by Stirling's formula, $\Gamma(x) \geq c e^{-x} x^{x-1/2}$ for some constant $c > 0$, thus

$$\lim_{x\to\infty} \Gamma(x)^{1/x} \geq \lim_{x\to\infty} e^{-1} c^{1/x} x^{1-\frac{1}{2x}} = \infty.$$

Thus also

$$\lim_{n\to\infty} \Gamma\left(n(1-\delta)+1\right)^{1/n} = \lim_{n\to\infty} \left(\Gamma\left(n(1-\delta)+1\right)^{1/[n(1-\delta)+1]}\right)^{[n(1-\delta)+1]/n} = \infty.$$

This proves that the series has infinite radius of convergence. The following lemma can now be stated.

Lemma Let $0 \leq T \leq \infty$ and let $f : [0,T) \to [0,\infty)$ be continuous and satisfy

$$f(t) \leq A + B\int_0^t (t-s)^{-\delta} f(s)\,ds \tag{55}$$

for all $t \in [0,T)$. Then

$$f(t) \leq A\Phi(B\Gamma(1-\delta)t^{1-\delta})$$

for $t \in [0,T)$.

Proof. Let $0 \leq T_1 < T$, which will remain fixed for a while, and let $M = \sup_{0\leq t\leq T_1} f(t)$. Claim

$$f(t) \leq A\sum_{k=0}^{n-1} \frac{\left(B\Gamma(1-\delta)t^{1-\delta}\right)^k}{\Gamma(k(1-\delta)+1)} + M\frac{\left(B\Gamma(1-\delta)t^{1-\delta}\right)^n}{\Gamma(n(1-\delta)+1)} \tag{56}$$

for $n = 0, 1, 2, \ldots$, $0 \leq t \leq T_1$.

In fact, if $n = 0$ the claim reduces to $f(t) \leq M$ in $[0,T_1]$, which is just the definition of T_1. Assume proved for some $n \geq 0$. Then (55) implies

$$\begin{aligned} f(t) \quad &\leq \quad A + BA\sum_{k=0}^{n-1} \frac{(B\Gamma(1-\delta))^k}{\Gamma(k(1-\delta)+1)} \int_0^t (t-s)^{-\delta} s^{k(1-\delta)}\,ds \\ &\quad + BM\frac{(B\Gamma(1-\delta))^n}{\Gamma(n(1-\delta)+1)} \int_0^t (t-s)^{-\delta} s^{n(1-\delta)}\,ds. \end{aligned}$$

Now

$$\int_0^t (t-s)^{-\delta} s^{k(1-\delta)}\,ds = t^{(k+1)(1-\delta)} B(1-\delta, k(1-\delta)+1)$$

where

$$B(1-\delta, k(1-\delta)+1) = \frac{\Gamma(1-\delta)\Gamma(k(1-\delta)+1)}{\Gamma((k+1)(1-\delta)+1)}.$$

Using this in (57) gives (56) with n replaced by $n+1$. The claim is established. Because the series converges, the last term in (56) goes to 0 as $n \to \infty$. Letting $n \to \infty$ in (56) proves the inequality of the lemma for $0 \leq t \leq T_1$. Since $T_1 \in [0, T)$ was arbitrary, the lemma is proved. $\square$

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