ASYMPTOTIC BEHAVIOR OF SOLUTIONS TO VISCOUS CONSERVATION LAWS WITH SLOWLY VARYING EXTERNAL FORCES

MARIA E. SCHONBEK

Abstract. This paper considers the existence and large time behavior of solutions to the convection-diffusion equation $u_t - \Delta u + b(x) \cdot \nabla(u|u|^{q-1}) = f(x,t)$ in $\mathbb{R}^n \times [0,\infty)$, where $f(x,t)$ is slowly decaying and $q \geq 1 + 1/n$ (or in some particular cases $q \geq 1$). The initial condition u_0 is supposed to be in an appropriate L^p space. Uniform and nonuniform decay of the solutions will be established depending on the data and the forcing term.

1. Introduction

In this paper, we study the existence and large time behavior of solutions $u = u(x,t),\ x \in \mathbb{R}^n,\ t > 0,\ n \geq 2$ to the Cauchy problem for the nonlinear convection-diffusion equation

$$\left\{ \begin{array}{l} u_t - \Delta u + b(x) \cdot \nabla(u|u|^{q-1}) = f(x,t), \\ u(x,0) = u_0(x), \end{array} \right. \tag{1.1}$$

where the vector function $b(x) \in \mathbb{R}^n$ is bounded and divergence free. Depending on the question addressed q will satisfy either $q \geq 1 + \frac{1}{n}$ (or in some particular cases $q \geq 1$). We note that the condition $q \geq 1 + \frac{1}{n}$is used in many decay results when the forcing term $f = 0$, see [19]. The initial data will be supposed to satisfy $u_0 \in L^1(\mathbb{R}^n)$ or $u_0 \in L^1 \cap L^\infty(\mathbb{R}^n)$. The aim of this paper is first to establish existence of solutions in the presence of appropriate external functions. Second, to study the decay of these solutions when the external forces are slowly decaying. The results obtained can be extended to the more general case where the convective term has the form $\nabla \cdot g(u)$, where g is a C^1-vector function which satisfies $|g(u)| \leq C|u|^q$, $|g'(u)| \leq C|u|^{q-1}$ for every $u \in \mathbb{R}$, $q \geq 1$, and a constant C.

The typical nonlinear term occurring in hydrodynamics in the one dimensional case has the form $uu_x = (u^2/2)_x$ (as in the case of the viscous Burgers equation). The most obvious generalization of this nonlinearity consists in replacing the square by a power u^q where q is a positive integer. The problem with the definition of u^q for negative u and for non-integer q as usual is avoided by choosing the nonlinear power as $\nabla(u|u|^{q-1})$. The interest for studying these equations lies also in the fact

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that one particularly important example corresponds to the vorticity for the 2-d Navier-Stokes equations.

$$\omega_t - \Delta\omega + u \cdot \nabla\omega = f \tag{1.2}$$

In this case the function $b(x) = u(x)$ is given by the velocity $u = (u_1, u_2)$, the solution to the non-stationary incompressible 2-dimensiona Navier-Stokes. It is well known that for suitable data, the corresponding solution u is bounded and divergence free. The solutions to equation (1.2) has been studied by several authors starting with the work of Kato-Fujita [6] and more recently by Galley-Wayne [3]. The interest in functions that are slowly decaying lies in the fact that when the decay of the forcing term is fast then the methods used in the case that $f = 0$ will apply with small modifications. In particular one is interested in the case when f is time independent. Thus understanding the case when f is slowly decaying is a first step in that direction.

For the existence of solutions fixed point methods and technical a priori estimate will be used. For similar techniques see [2] and [19]. The second part of the paper will focus on the long time behavior of the solutions. Uniform and non-uniform decay will be established depending on the choice of the external function, which either decays very slowly or is in a specified Sobolev space. For fast decaying forcing terms simple extensions of the Fourier-Splitting technique [12, 13, 10, 17, 18] will easily give decay. Two types of decay will be obtained

1. Non–uniform decay in L^p, $2 \leq p \leq \infty$,
2. Decay in L^p, $2 \leq p \leq \infty$ with a slow rate depending on the decay rate of the forcing term.

The first step will be to obtain decay of the energy of the solutions (i.e. decay in the L^2 norm). The general L^p decay will follow by interpolation. The methods for energy decay are based on ideas of [7, 10] and the Fourier–Splitting technique [12, 13, 10, 17, 18]. Specifically non-uniform decay will be established for forces as described below in Assumption **A.1**. This class of forces include forces $f \in L^1(0, \infty; L^2)$.

Uniform decay at slow algebraic rate will be established under Assumptions **A.2** and **A.3** below. The slowness of the decay is due to the influence of the external forces. More precisely we will obtain uniform decay (UD) in L^p for a class of forces that include functions of the type

- [f1] $f \in L^\infty(0, \infty; L^1)$.
- [f2] $\|f(t)\|_2 \leq C(1+t)^{-1-\epsilon}$.
- [f3] $\|rf(t)\|_2 \leq C(1+t)^{-1/2-2\epsilon}$ with $\epsilon > 0$ is a small constant and $r = |x|$.
- [f4] Consideration will also be given to force which are gradients.

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1.1. **Notation.** For $1 \le p \le \infty$, we denote by L^p the usual Banach space consisting of all Lebesgue measurable functions defined on $\mathbb{R}^n$ with norm $\|v\|_p = (\int_{\mathbb{R}^n} |v(\cdot)|^p \, dx)^{1/p} (< \infty)$. We will always denote by $\|\cdot\|_{\mathcal{X}}$ the norm of any other Banach space $\mathcal{X}$ used in this paper.

If k is a nonnegative integer, $W^{k,p}$ will be the Sobolev space consisting of functions in L^p whose generalized derivatives up to order k belong to L^p.

The Fourier transform of v is defined as $\mathcal{F}(v) = \widehat{v}(\xi) \equiv (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-ix\xi} v(x) dx$.

Given a multi-index $\gamma = (\gamma_1, \dots, \gamma_n)$, we denote $\partial^\gamma = \partial^{|\gamma|}/\partial_{x_1}^{\gamma_1} \cdots \partial_{x_n}^{\gamma_n}$. On the other hand, for $\beta > 0$, the operator D^β is defined via the Fourier transform as $\widehat{(D^\beta w)}(\xi) = |\xi|^\beta \widehat{w}(\xi)$. Let L^2 and $\dot{H}_0^1$ denote the completions of $C_0^\infty(\mathbb{R}^n)$ in the L^2-norm $\|\cdot\|_2$ and the Dirichlet (homogeneous $\dot{H}^1$) norm $\|\nabla \cdot\|_2$. We denote

$$L^p(a,b;L^q) = \Big\{ f : (a,b) \times \mathbb{R}^n \to \mathbb{R}^n;\ \|f\|_{L^p(a,b;L^q)} = \left(\int_a^b \|f(\tau)\|_p^q d\tau \right)^{1/q} < \infty \Big\}.$$

The notation of $\|\cdot\|_{p,q}$ will be used as the norm of $L^p(0,\infty;L^q)$. $H^1 = \dot{H}^1 \cap L^2$. The symbol $\langle \cdot, \cdot \rangle$ denotes the inner product in L^2. Various constants are simply denoted by C.

2. Existence results

The focus of this section is the existence of solutions to (1.1). Fixed point theory techniques will yield local existence. A priori estimates will then allow to pass to a global solution. The main result in this section the following:

Theorem 2.1. *Let $u_0 \in L^1$, $q \ge 1$ and $b \equiv (b_1, b_2, \cdots, b_n) \in (L^\infty)^n$ with $div\, b = 0$. Then the following assertions hold:*

•1 *If $f \in L^1(0,\infty;L^1)$ then there exists a unique mild solution $u \in C((0,\infty);L^1)$ of (1.1) such that*

$$\|u(t)\|_1 \le \|u_0\|_1 + \|f\|_{1,1}, \quad \forall t > 0. \tag{2.1}$$

•2.*a If $f \in L^\infty(0,\infty;L^p) \cap L^1(0,\infty;L^1)$, $p \in [1,\infty)$ then there exists a unique mild solution $u \in C((0,\infty);L^p)$ of (1.1), and constants*

$$N_p = N(p,n,\|u_0\|_1 \|f\|_{1,1}),\ M_p = M(p,n,\|u_0\|_1), \quad \text{and} \quad \beta_p = \beta_(p,n)$$

such that

$$\|u\|_p \le \max\{N_p t^{-n/2(1-1/p)}, M_p \|f\|_{p,\infty}^{\beta_p}\}, \quad \forall t > 0, \tag{2.2}$$

where $M_p \to \infty$, $N_p \to \infty$ as $p \to \infty$.

•2.*b If in addition to hypothesis in •2.a, $f \in C(0,\infty;W^{2,p})$ then the solution constructed in •2.a will satisfy $u \in C((0,\infty);W^{2,p} \cap L^1) \cap C^1((0,\infty);L^p)$.*

•3 *If* $f \in L^1(0,\infty; L^1 \cap L^\infty) \cap W^{2,p}$, $u_0 \in L^p$, $p \in [1,\infty)$ *then there exists a unique solution* $u \in C([0,\infty) : W^{2,p} \cap L^1) \cap C^1((0,\infty); L^p)$, *and*

$$\|u(t)\|_p \leq \exp\left\{\frac{p-1}{p}\|f\|_{\infty,1}\right\}\left[\|u_0\|_p + \|f\|_{1,1}^{1/p}\right], \quad \forall t > 0. \tag{2.3}$$

•4 *If in addition to the hypothesis in* •3 $u_0 \in L^\infty$, *then*

$$\|u(t)\|_\infty \leq \exp\{\|f\|_{\infty,1}\}\left[\|u_0\|_\infty + 1\right], \quad \forall t > 0. \tag{2.4}$$

Proof.

Remark 2.2. When $q = 1$ the equation is linear and existence is a well known result.

Part 1: As in [2] we first consider data $u_0 \in L^1 \cap L^\infty$. Supposing that $f \in L^1(0,\infty; L^1 \cap L^\infty) \cap C(0,\infty; W^{2,p})$ we show that $u \in C((0,\infty); W^{2,p} \cap L^1) \cap C^1((0,\infty); L^p)$, where $p \in (1,\infty)$. By appropriate simplifications, the argument presented yields for the less restrictive initial hypothesis given in part •1 that $u \in C([0,\infty); L^1)$ and in part •2 that $u \in C([0,\infty); L^p)$. These simplifications are straightforward and as such omitted.

Let $G = G(x,t)$ be the heat kernel, then

$$\begin{aligned} u(x,t) = \; & G * u_0 + \int_0^t \sum_{j=1}^n \partial_{x_j} G(t-s) * (b_j u(s)|u(s)|^{q-1}) ds \\ & + \int_0^t G(t-s) * f(s) ds. \end{aligned} \tag{2.5}$$

Following standard Banach fixed theorem techniques (see also [2]) introduce the operator:

$$[\Phi(u)] = G * u_0 + \int_0^t \left[\sum_{j=1}^n \partial_{x_j} G(t-s) * (b_j u(s)|u(s)|^{q-1}) + G(t-s) * f(s)\right] ds.$$

Apply fixed point theorem to Φ in the closed subset of $C([0,T]; L^1 \cap L^\infty)$:

$$B = \left\{u \in C((0,T]; L^1 \cap L^\infty); \; \sup_{0<t<T} (\|u(t)\|_1 + \|u(t)\|_\infty) \leq M\right\}$$

with M large enough and T small enough to insure that Φ has a unique fixed point. By the hypothesis on f, standard computations yield that the integral equation (2.5) has a unique local in time solution $u = u(x,t)$ in B, see [2].

As in [2] classical regularity yields

$$u \in C((0,T); W^{2,p}) \cap C^1((0,T); L^p) \tag{2.6}$$

for every $p \in (1,\infty)$. The solution can be extended to a maximal interval $T_{\max}$. To obtain a global solution it suffices to show the a priori estimate

$$\sup_{[0,T_{\max})} (\|u(t)\|_1 + \|u(t)\|_\infty) < C, \tag{2.7}$$

where C is a constant independent of $T_{\max}$. For this we follow in part the steps in [2]. Due to the added forcing term new estimates will be needed.

Note that the since b is divergence free it will enter in the estimates similarly than if it would be a constant. Let $\phi \in C^1$, then the following integral vanishes

$$\begin{aligned}
&\int_{\mathbb{R}^n} b \cdot \nabla(|u|^{q-1}u)\phi(u)dx \\
&= -\int_{\mathbb{R}^n} (\operatorname{div} b)|u|^{q-1}u\phi(u)dx - \int_{\mathbb{R}^n} |u|^{q-1}u\phi'(u)b \cdot \nabla u dx \\
&= -\int_{\mathbb{R}^n} |u|^{q-1}u\phi'(u)b \cdot \nabla u dx = -\int_{\mathbb{R}^n} b \cdot \nabla \int_0^u |s|^{q-1}s\phi'(s)dsdx \\
&= \int_{\mathbb{R}^n} (\operatorname{div} b) \int_0^u |s|^{q-1}s\phi'(s)dsdx = 0.
\end{aligned} \tag{2.8}$$

Since $u(t) \in W^{2,p}$ (for $t > 0$), for any $\phi \in C^1(\mathbb{R})$ we have

$$\int_{\mathbb{R}^n} \phi(u)\Delta u dx = -\int_{\mathbb{R}^n} \phi'(u)|\nabla u|^2 dx. \tag{2.9}$$

By density argument, (2.9) holds for $\phi(s) = \operatorname{sign} s$, hence

$$\int_{\mathbb{R}^n} (\operatorname{sign} u)\Delta u \, dx \leq 0. \tag{2.10}$$

Thus multiplying (1.1) by $\operatorname{sign} u$ and integrating it in $\mathbb{R}^n$ gives

$$\frac{d}{dt}\int_{\mathbb{R}^n} |u(x,t)|dx \leq \int_{\mathbb{R}^n} |f(x,t)| \, dx \tag{2.11}$$

Since $f \in L^1(0,\infty; L^1)$ and $u_0 \in L^1$ it follows that the solution u has the required L^1 a priori bound:

$$\|u(t)\|_1 \leq \|u_0\|_1 + \|f\|_{1,1}. \tag{2.12}$$

This concludes the L^1 estimate.

Remark 2.3. Note that up to now only conditions on $\|u_0\|_1$ and $\|f\|_{1,1}$ have been used. If the hypothesis on the initial data is just given by $\bullet 1$, as noted above simple modifications of the above argument yield $u \in C([0,\infty); L^1)$. The modifications to the argument are straightforward and as such are omitted.

The L^∞ a priori estimate is obtained as follows. Multiply equation (1.1) by $\operatorname{sign}(u - \|u_0\|_\infty - \int_0^t f \, ds)^+$, integrate in space to yield

$$\frac{d}{dt}\int_{R^n} (u - \|u_0\|_\infty - \int_0^t |f| \, ds)^+ dx \leq 0$$

Now multiply equation (1.1) by $\operatorname{sign}(u + \|u_0\|_\infty + \int_0^t f \, ds)^-$, integrating in space gives

$$\frac{d}{dt}\int_{R^n} (u + \|u_0\|_\infty + \int_0^t |f| \, ds)^- dx \leq 0$$

Hence

$$||u||_\infty \leq \|u_0\|_\infty + ||f(\cdot,t)||_{1,1}$$

. This completes the the existence in the case when the data $u_0 \in L^1 \cap L^\infty$

Part 2: The next step is to obtain an L^p a priori bound for $t>0$ which depends only on the L^1 data. Suppose first, that the data u_0 is in $L^1 \cap L^p$ and obtain an estimate on the solution which depends only on the L^1 norm of u_0. Then suppose that the data u_0 is in L^1 alone. Standard approximation theorems insure that it is possible to construct a sequence of data functions $u_0^k \in L^1 \cap L^p$ which converge strongly to u_0 . We show that the solutions corresponding to the data u_0^k converge to the solution with data in L^1 only. Hence the solution constructed via approximation , will be bounded in L^p with the bound depending only on the L^1 norm of the data.

L^p-a priori estimate

Recall first the following known interpolation inequality, which will be needed below:

Lemma 2.4. ([2]) *For every $p \in [2,\infty)$ there exists some constant $C = C(n) > 0$ such that*

$$\|v\|_p^{(n(p-1)+2)p/n(p-1)} \leq C\|v\|_1^{2p/n(p-1)}\|\nabla(|v|^{p/2})\|_2^2 \tag{2.13}$$

for every $v \in W^{2,p}(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$.

To estimate the L^p norm, multiply equation (1.1) by $p|u|^{p-2}u$, use (2.8) and integrate in space

$$\frac{d}{dt}\int_{\mathbb{R}^n} |u|^p dx + 4\frac{(p-1)}{p}\int_{\mathbb{R}^n} |\nabla(|u|^{p/2})|^2 dx = p\int_{\mathbb{R}^n} f|u|^{p-2}u dx. \tag{2.14}$$

Let

$$B_p = 4C\frac{p-1}{p}\Big(\|u_0\|_1 + \|f||_{1,1}\Big)^{-2p/n(p-1)} \geq C\Big(\|u_0\|_1 + \|f||_{1,1}\Big)^{-2p/n(p-1)},$$

where C is constant depending on n. Using Lemma (2.4), equation (2.12) and the last equality yields

$$\frac{d}{dt}\|u\|_p^p + B_p\|u\|_p^{p\gamma} \leq p\int_{\mathbb{R}^n} f|u|^{p-2}u dx = I, \tag{2.15}$$

where $\gamma = \frac{n(p-1)+2}{n(p-1)} = 1 + \frac{2}{n(p-1)}$.

The RHS of the last equation is bounded as follows:

$$I \leq p\int_{\mathbb{R}^n} f|u|^{p-1}dx \leq p\|f\|_p\|u\|_p^{p-1} \leq \frac{A^\alpha}{\alpha}\|u\|_p^{p\gamma} + \frac{\|f\|^\beta p^\beta}{\beta\ A^\beta}, \tag{2.16}$$

where $\alpha(p-1) = p(1+\frac{2}{n(p-1)}) = p\gamma$. Let $\frac{1}{\alpha}+\frac{1}{\beta} = 1$ and choose A so that $\frac{A^\alpha}{\alpha} = \frac{B_p}{2} = C_p$. Combining (2.15) with (2.16) yields

$$\frac{d}{dt}\|u\|_p^p + C_p\|u\|_p^{p\gamma} \leq \frac{\|f\|_p^\beta p^\beta}{\beta\ A^\beta}. \tag{2.17}$$

Denote by $v = \|u\|_p^p$. The last equation can be rewritten as

$$\frac{dv}{dt} + C_p v^\gamma \leq \frac{\|f\|_{p,\infty}^\beta p^\beta}{\beta A^\beta}. \tag{2.18}$$

Let $K_p = \frac{p^\beta \|f\|^\beta_{p,\infty}}{\beta\ A^\beta}$. Now consider two cases:

- Case 1. Intervals where $K_p \geq \frac{C_p}{2} v^\gamma$,
- Case 2. Intervals where $K_p \leq \frac{C_p}{2} v^\gamma$.

We consider the problem separately on intervals where $v \leq \left[\frac{2}{C_p} K_p\right]^{1/\gamma}$ and intervals where $v \geq \left[\frac{2}{C_p} K_p\right]^{1/\gamma}$, then take the maximum as a bound. If for all time Case 1 holds then $v = \|u\|_p^p \leq \left[\frac{2}{C_p} K_p\right]^{1/\gamma}$ and we are done. Thus to estimate the solution of equation (2.18) it is only necessary to address intervals where Case 2 is satisfied. In this situation we work on intervals which start at $t = 0$ if $v(0) \geq K_p$ or on intervals starting at $t = t_1$, where t_1 is such that $v(t_1) = K_p$. For intervals where Case2 holds the L^p norm of the solution satisfies the differential inequality:

$$\begin{aligned} &\frac{dv}{dt} + C_p v^\gamma \leq \frac{C_p}{2} v^\gamma, \\ &v(t_0) = v_0, \end{aligned} \tag{2.19}$$

where $t_0 = 0$, or $t_0 = t_1$ and $v_0 = v(0)$ or $v_0 = K_p$ respectively. We work on intervals where $v \geq K_p > 0$, hence the last equation reduces to

$$v^{-\gamma}\ \frac{dv}{dt} \leq -\frac{C_p}{2}. \tag{2.20}$$

An easy computation yields

$$v^{\gamma-1} \leq \left[v_0^{1-\gamma} + (\gamma-1)\frac{C_p t}{2}\right]^{-1}. \tag{2.21}$$

Note that $\gamma - 1 = \frac{n}{2(p-1)}$. Hence the last inequality can be expressed as

$$v \leq [v_0^{-n/2(p-1)} + tC_p(\frac{n}{4(p-1)}]^{-\frac{2}{n}(p-1)} \leq [tC_p\frac{n}{4(p-1)}]^{-\frac{2}{n}(p-1)}. \tag{2.22}$$

Here one uses that $v_0^{-n/2(p-1)} > 0$, which holds since $v_0 = v(t_1) = K_p$ or $v_0 = v(0)$ and in both cases $v_0 > 0$ (See definition of K_p above).

Let

$$N_p = [C_p(\frac{n}{4(p-1)}]^{-\frac{n}{2}(p-1)}, \quad M_p = \left(\frac{2p^\beta}{C_p \beta A^\beta}\right)^{1/\gamma}.$$

Combining the above estimates yields

$$v \leq \max\{M_p \|f\|^{\beta/\gamma}_{p,\infty}, N_p t^{-\frac{2}{n}(p-1)}\} = H_p. \tag{2.23}$$

This last estimate gives an L^p estimate for all $t > 0$ independent of the L^p norm of the initial data. By their definition it follows that $N_p \to \infty$ and $M_p \to \infty$ as $p \to \infty$. This concludes the proof of Theorem(2.1), part •2a, in the case where the data satisfies $u_0 \in L^1 \cap L^p$.

As remarked above, the proof when the initial data is in L^1 alone uses an approximation argument similar to the one presented in [2]. We need first an L^1

stability estimate. Let $u, v \in C([0,t]; L^1 \cap L^p)$ be two solutions of (1.1) with data $u_0, v_0 \in L^1 \cap L^p$. The following property will be necessary.

$$\frac{d}{dt}\int_{\mathbb{R}^n} |u(t,x) - v(t,x)| dx \leq 0, \quad \forall t > 0. \tag{2.24}$$

Notice that the equation for $u - v$ is the same as the difference of the corresponding homogeneous equations. Hence (2.24) follows as in he homogeneous case, [2], multiplying the equation by $\text{sign}(u - v)$ and integrating in space. Thus the following contraction property holds

$$\|u(t) - v(t)\|_1 \leq \|u_0 - v_0\|_1, \quad \forall t > 0. \tag{2.25}$$

Let $u_0 \in L^1(\mathbb{R}^n)$. Let $u_{o,k} \in L^1(\mathbb{R}^n) \cap L^s(\mathbb{R}^n)$ approximate u_0 in L^1, with s as high as needed. Let $u_k = u_k(x,t)$ be the sequence of solutions with data $u_{o,k}$. Then by the contraction property (2.25) it follows that $\{u_n\}$ is a Cauchy sequence in $C([0,\infty); L^1(\mathbb{R}^n))$. Let $u(t)$ be the limit in L^1. Clearly $u(x,0) = u_0$. The same sequence u_k will be Cauchy in L^p, as can be seen by choosing a to sequence to be bounded in $L^1 \cap L^s$, with $s = 2p - 1$, hence for each fixed p and $t > 0$

(2.26)

$$\begin{aligned}
&(\int |u_n(t) - u_m(t)|^p dx)^{1/p} = (\int |u_n(t) - u_m(t)|^{1/2} |u_n(t) - u_m(t)|^{p-1/2} dx)^{1/p} \\
&\leq (\int |u_n(t) - u_m(t)| dx \int |u_n(t) - u_m(t)|^{2p-1}\, dx)^{1/2p} \leq 2Q_p(t)(\int |u_n - u_m| dx)^{1/2p}
\end{aligned}$$

where we used $\|u\|_{2p-1}^{\frac{2p-1}{2p}} \leq [H_p]^{1/2p}(t) = Q_p(t)$. Thus the sequence of solutions is Cauchy for each p when $t > 0$. The convergence of the u_k to u is strong in $L^1 \cap L^p$, for all $p \in [1, \infty)$. Hence u is a solution of (1.1) which satisfies the required a priori estimates. This concludes part $\bullet 2.a$ of the theorem .

Part $\bullet 2.b$ The additional regularity of f, $f \in C(0,\infty; W^{2,p}$ yields by classical regularity arguments that the solution satisfies

$$u \in C((0, T_{max}); W^{2,p} \cap L^1) \cap C^1((0, T_{max}); L^p),$$

and by $\bullet 2.a$ it follows that $T_{max} = \infty$.

Part $\bullet 3$ From inequality (2.15) it follows that

$$\frac{d}{dt}\|u\|_p^p dx + B_p \|u\|_p^{p\gamma} = p \int_{\mathbb{R}^n} f|u|^{p-2} u dx \leq p \int_{\mathbb{R}^n} |f|(|u|^p + 1) dx. \tag{2.27}$$

Hence

$$\frac{d}{dt}\|u\|_p^p \leq p \int_{\mathbb{R}^n} |f| dx + p \int_{\mathbb{R}^n} |f||u|^p dx \leq p \int_{\mathbb{R}^n} |f| dx + p \|f(t)\|_\infty \|u\|_p^p. \tag{2.28}$$

Gronwall's inequality and taking the p-th root yields

$$\|u(t)\|_p \leq \{exp\|f||_{\infty,1}\}[\|u_0\|_p + p^{1/p}\|f\|_{1,1}^{1/p}].. \tag{2.29}$$

By classical regularity arguments the local estimate of the solution was in the space $C((0, T_{max}); W^{2,p} \cap L^1) \cap C^1((0, T_{max}); L^p)$, hence so is the global solution.

Part •4 To obtain the L^∞ estimate simply pass to the limit as p tends to infinity in the last equality. This completes the proof of the theorem. □

3. Assumptions on the external force

This section describes additional assumptions, on the forcing term f, under which decay of the solutions will be studied. These assumptions require very weak decay for the function f. Typical conditions we are seeking are of the form

- i. $f \in L^1(0,\infty; L^2)$.
- ii $f \in L^2(0,\infty; L^{\frac{2n}{n+2}})$ for $n \geq 3$.
- iii. $\rho f \in L^2(0,\infty; L^2)$ for an appropriate weight function ρ.

Three possible assumptions will be given: **A.1**, **A.2** and **A.3**. Assumption **A.1** yields non–uniform decay, while assumptions **A.2**, **A.3** yield a slow algebraic rate of decay. Functions satisfying i, ii, ii above will be special cases fitting into **A.1**, **A.2** or **A.3** for appropriate parameters.

We remark that the assumptions here are the same as the ones used on forces regarding non-homogeneous solutions to the Navier–Stokes equations ([10]).

Assumptions on f:

1. Assumption **A.1**: For $x_0 \in \mathbb{R}^n$, set
$$\rho = \rho_{x_0}(x) = \begin{cases} |x - x_0|, & \text{if } n \geq 3, \\ |x - x_0|(1 + |\ln|x - x_0||), & \text{if } n = 2. \end{cases}$$
Suppose that for $0 \leq \gamma \leq 1$, $2 \leq p \leq \frac{2n}{n-2+2\gamma}$ ($< \infty$ if $n = 2$) and $\theta = \frac{4p}{2p\gamma + np - 2n}$,
$$\rho^\gamma f \in L^{\theta'}(0,\infty; L^{p'}),$$
where p' and θ' are the Hölder conjugates of p and θ.

   Remark 3.1. (1) Assumption **A.1** includes cases i, ii and iii in the special case where ρ is given above, as follows:
   - For i. let $\gamma = 0,\ p = 2, \theta = 1$.
   - For ii. let $\gamma = 0,\ p = \frac{n-2}{2n},\ \theta = 2$.
   - For iii. let $\gamma = 1,\ p = 2,\ \theta = 2$, and ρ is of the type described in **A.1**.

2. Assumption **A.2**: Let γ, p, θ and ρ be the same as in assumption **A.1**. For small $\epsilon > 0$ suppose that f satisfies
$$(1+t)^\beta \rho^\gamma f \in L^\infty(0,\infty; L^{p'}),$$
where $\beta = \frac{1}{\theta'} + \epsilon\frac{\theta+2}{\theta}$.

   Remark 3.2. By choosing $\gamma = 0, p = p' = 2$ and $\beta = 1 + \epsilon$, **A.2** is specialized as
$$\|f(t)\|_2 \leq C(1+t)^{-1-\epsilon}.$$

The choice of $\gamma = 1$, $p = p' = 2$, yields $\theta = 2$ and $\beta = 1/2 + 2\epsilon$ which gives class [f3.] in the introduction.

$$\|\rho f(t)\|_2 \leq C(1+t)^{-\frac{1}{2}-2\epsilon}.$$

Remark 3.3. Assumption **A.2** implies **A.1**.

3. Assumption **A.3**: Let f satisfy either:
   i. f can be written as $f = Dg$, where D is any first order derivative and $g \in L^\infty(0,\infty; L^1)$ or,
   ii. $f \in L^2(0,\infty; L^1)$.

4. Some preliminaries

The propositions and lemmas in this section are technical and provide several auxiliary estimates that will be needed in the sequel.

Remark 4.1. In the remainder of the paper we will always assume that the external force has enough regularity to insure the existence of a regular solution to equation (1.1). To obtain the L^2 decay it would suffice to work with solutions u described in Remark (7.7). The arguments would be also possible for weak solutions via approximations.

Auxiliary estimates on low and high frequency parts of the solutions

To establish decay of the solutions we will need to analyze separately low and high frequencies of the solution u, in the sense that u can be split as

$$\|u\|_2 \leq \|\phi\widehat{u}\|_2 + \|\psi\widehat{u}\|_2,$$

where $\psi = 1 - \phi$ and ϕ is centered on low frequencies. The next propositions and corollary will be useful for this analysis, since u satisfies the integral equation (2.5) the following corollary is straightforward to establish.

Proposition 4.2. *Let $E \in C^1(\mathbb{R}_+;\mathbb{R})$, $E \geq 0$ and $\psi \in C^1(0,\infty; L^\infty)$. Let u be a regular solution constructed in Theorem (2.1) or in or in Remark (7.7) then u satisfies*

(4.1)
$$\begin{aligned}
&E(t)\|\psi(t)\widehat{u}(t)\|_2^2 \\
&\leq E(s)\|\psi(s)\widehat{u}(s)\|_2^2 + \int_s^t E'(\tau)\|\psi(\tau))\widehat{u}(\tau)\|_2^2 d\tau \\
&\quad +2\int_s^t E(\tau)\big[\langle\, \psi'(\tau)\widehat{u}(\tau), \psi\hat{u}(\tau)\,\rangle - \|\xi\psi(\tau)\widehat{u}(\tau)\|_2^2\big] d\tau \\
&\quad +2\int_s^t E(\tau)\big[\langle\, \mathcal{F}(b\cdot\nabla(u|u|^{q-1}(\tau)), (1-\psi^2)\widehat{u}(\tau)\,\rangle + \langle\, \widehat{f}(\tau), \psi(\tau)^2\widehat{u}(\tau)\,\rangle\big] d\tau,
\end{aligned}$$

for almost all $s > 0$ with $s = 0$ and all $t \geq s$. In particular, the solution satisfies

$$
\begin{aligned}
&E(t)\|u(t)\|_2^2 \\
&\leq E(s)\|u(s)\|_2^2 + \int_s^t E'(\tau)\|u(\tau)\|_2^2 d\tau \\
&\quad -2\int_s^t E(\tau)\|\nabla u(\tau)\|_2^2 d\tau + 2\int_s^t E(\tau)\,\langle\, f(\tau), u(\tau)\,\rangle\, d\tau
\end{aligned}
\tag{4.2}
$$

for almost all $s \geq 0$ and all $t \geq s$.

Proof. The proof is obtained via a slight modification of the proof of a similar statement for solutions to the Navier–Stokes equations. Multiply the Fourier Transform of equation (1.1) by $2E(\tau)\psi^2\widehat{u}(\tau)$ and notice that

$$
\langle\, b \cdot \nabla(u|u|^{q-1}), u\,\rangle = 0. \tag{4.3}
$$

Hence the result follows after integrating over the time interval $[s, t]$. For details we refer the reader to [10].

The second inequality follows from the first one by choosing $\psi = 1$. $\square$

Corollary 4.3. (Low frequency) Let u be a regular solutions constructed in Theorem (2.1) or in Remark (7.7) and $\phi \in L^2 \cap L^\infty$, then

$$
\begin{aligned}
&\|\phi\widehat{u}\|_2^2 = \|\check{\phi} * u(t)\|_2^2 \\
&\leq \|e^{\Delta(t-s)}\check{\phi} * u(s)\|_2^2 + 2\int_s^t \Big(\Big|\,\langle\, b\cdot\nabla(u|u|^{q-1}), e^{2\Delta(t-\tau)}\check{\phi}^2 * u(\tau)\,\rangle\,\Big| \\
&\qquad + \Big|\,\langle\, f, e^{2\Delta(t-\tau)}\check{\phi}^2 * u(\tau)\,\rangle\,\Big|\Big) d\tau.
\end{aligned}
\tag{4.4}
$$

Proof. Define $\Phi(\tau)$ by

$$
\Phi(\tau) = \mathcal{F}^{-1}(e^{-|\xi|^2(t+\eta-\tau)}\phi)(\xi), \quad \eta > 0.
$$

Thenletterviscous

$$
\Phi'(\tau) = \mathcal{F}^{-1}(|\xi|^2 e^{-|\xi|^2(t+\eta-\tau)}\phi)(\xi), \quad \eta > 0,
$$

and $\Phi(\tau) * u(\tau) = e^{\Delta(t+\eta-\tau)}\check{\phi} * u(\tau)$. Thus

$$
\begin{aligned}
&\langle\, \Phi'(\tau) * u(\tau), \Phi(\tau) * u(\tau)\,\rangle - \|\nabla\Phi(\tau) * u(\tau)\|_2^2 \\
&= \langle\, -\Delta e^{\Delta(t+\eta-\tau)}\check{\phi} * u(\tau), e^{\Delta(t+\eta-\tau)}\check{\phi} * u(\tau)\,\rangle - \|\nabla e^{\Delta(t+\eta-\tau)}\check{\phi} * u(\tau)\|_2^2 \\
&= 0.
\end{aligned}
$$

Here as usual the notation $e^{\Delta(t)}g$ is used to indicate the convolution in space of g with the heat Kernel. By (4.1) with $E(t) = 1$ and $\psi = \Phi$ as defined above, using (4.3) yields

$$
\begin{aligned}
&\|e^{\Delta(\eta)}\check{\phi} * u(t)\|_2^2 \\
&\leq \|e^{\Delta(t+\eta-s)}\check{\phi} * u(s)\|_2^2 + 2\int_s^t \Big(\Big|\,\langle\, \mathcal{F}(b\cdot\nabla(u|u|^{q-1})), e^{2\Delta(t+\eta-\tau)}\check{\phi}^2 * u(\tau)\,\rangle\,\Big| \\
&\qquad + \Big|\,\langle\, f, e^{2\Delta(t+\eta-\tau)}\check{\phi}^2 * u(\tau)\,\rangle\,\Big|\Big) d\tau.
\end{aligned}
\tag{4.5}
$$

Letting $\eta \to 0$ in (2.24), we obtain (2.23). $\square$

Auxiliary estimates for the non-uniform decay of the solutions

For these estimates we apply Proposition (4.2) to a particular function ψ and combine it with Fourier Splitting technique. The new estimate will be needed to establish the non uniform decay of the solutions.

Proposition 4.4. *Let u be a regular solution constructed in Theorem (2.1) or in Remark (7.7). Let $E(t)$ be as in (4.2), $\psi = 1 - \exp(-|\xi|^2) = 1 - \phi$ and $b \in L^\infty$, $q \geq 1 + \frac{1}{n}$, $\chi(t) = \{\xi \in \mathbb{R}^n; |\xi| \leq G(t)\}$, where G is a continuos function. Let $\chi(t)^c = \mathbb{R}^n \setminus \chi(t)$. For $n = 2$ suppose additionally that $u_0 \in L^\infty$. Then*

$$
\begin{aligned}
&E(t)\|(1-\phi)\widehat{u}(t)\|_2^2 \\
&\leq E(s)\|(1-\phi)\widehat{u}(s)\|_2^2 + \int_s^t E'(\tau) \int_\chi |(1-\phi)\widehat{u}(\tau)|^2 d\xi \, d\tau \\
&\quad + \int_s^t (E'(\tau) - 2E(\tau)G^2(\tau)) \int_{\chi(t)^c} |(1-\phi)\widehat{u}(\tau)|^2 d\xi d\tau \\
&\quad + C\int_s^t E(\tau)\|\nabla u\|_2^2 d\tau + \int_s^t E(\tau)|\langle f, \{(1-\phi)^2\}^\vee * u(\tau)\rangle \, d\tau.
\end{aligned} \tag{4.6}
$$

Proof. **Case 1:** $n \geq 3$. We use the notation $\Gamma = 1 - (1-\phi)^2$. Note that in this case ψ is independent of t, hence $\psi' = 0$. Thus inequality (4.1) yields that

$$
\begin{aligned}
&E(t)\|(1-\phi)\widehat{u}(t)\|_2^2 \\
&\leq E(s)\|(1-\phi)\widehat{u}(s)\|_2^2 - 2\int_s^t E(\tau)\|\xi(1-\phi)\widehat{u}(\tau)\|_2^2 d\tau \\
&\quad + \int_s^t E'(\tau)\Big[\int_{\chi(t)} |(1-\phi)\widehat{u}(\tau)|^2 d\xi + \int_{\chi^c} |(1-\phi)\widehat{u}(\tau)|^2 d\xi\Big] d\tau \\
&\quad + 2\int_s^t E(\tau)\big(|\langle \mathcal{F}b \cdot \nabla(u|u|^{q-1})), \Gamma\widehat{u}(\tau)\rangle| + |\langle \widehat{f}, (1-\phi)^2\widehat{u}(\tau)\rangle|\big) d\tau \\
&\leq E(s)\|(1-\phi)\widehat{u}(s)\|_2^2 + \int_s^t E'(\tau)\int_{\chi(\tau)} |(1-\phi)\widehat{u}(\tau)|^2 d\xi d\tau \\
&\quad + \int_s^t (E'(\tau) - 2E(t)G^2(t)) \int_{\chi(\tau)^c} |(1-\phi)\widehat{u}(\tau)|^2 d\xi d\tau \\
&\quad + 2\int_s^t E(\tau)\big(|\langle \mathcal{F}(b \cdot \nabla(u|u|^{q-1}), \Gamma\widehat{u}(\tau)\rangle| + |\langle f, \{(1-\phi)^2\}^\vee * u(\tau)\rangle|\big) d\tau.
\end{aligned} \tag{4.7}
$$

Since $\mathcal{F}^{-1}(\Gamma) = \mathcal{F}^{-1}(1-(1-\phi)^2) \equiv \tilde{\psi}$ is a rapidly decreasing function, the fourth term of the right hand side of (4.7) is estimated using Haussdorf-Young's

$$
\begin{aligned}
&\int_s^t E(\tau) |\langle b \cdot \nabla(u|u|^{q-1}), \tilde{\psi} * u(\tau) \rangle| d\tau \\
&\leq \|b\|_\infty \int_s^t E(\tau) \|\nabla u|u|^{q-1}\|_{\frac{2n}{n+2}} \|\tilde{\psi} * u\|_{\frac{2n}{n-2}} d\tau.
\end{aligned} \tag{4.8}
$$

By Hölder's and Gagliardo-Nirenberg inequalities it follows that if $q \geq 1 + \frac{1}{n}$

$$
\|\nabla u|u|^{q-1}\|_{\frac{2n}{n+2}} \|\tilde{\psi} * u\|_{\frac{2n}{n-2}} \leq \|u\|^{q-1}_{n(q-1)} \|\nabla u\|_2 \|\tilde{\psi}\|_1 \|u\|_{\frac{2n}{n-2}} \leq C \|\nabla u\|_2^2 \tag{4.9}
$$

Here we used that since $n(q-1) \geq 1$ we have $u \in L^{n(q-1)}$. Combining the last inequality with (4.8) and (4.7) establishes Proposition (4.4) in the case when $n \geq 3$.

Case 2: Here $n = 2$, $q \geq 1 + 1/2$ and $u_0 \in L^1 \cap L^\infty$.
We replace the function Γ by $1 - \Gamma = (1-\phi)^2 = \psi^2$ in (4.7). As mentioned above $\langle b \cdot \nabla(u|u|^{q-1}), u \rangle = 0$. Hence inequality (4.7) remains the same with this replacement. Define now $\tilde{\psi} = \mathcal{F}^{-1}(1-\phi)$. In this case again we need to estimate the fourth term of (4.7). Since $b \in L^\infty$, by Haussdorf-Young and Gagliardo-Nirenberg it follows that

$$
\begin{aligned}
&\int_s^t E(\tau) |\langle b \cdot \nabla(u|u|^{q-1}), \tilde{\psi} * u(\tau) \rangle| d\tau \\
&\leq \|b\|_\infty \int_s^t E(\tau) \int_{\mathbb{R}^n} |u|^{q-1} |\nabla u(x)| \; \delta\tau \int_{\mathbb{R}^n} |\tilde{\psi}(x-y) u(y)| dy \; dx \\
&\leq \|b\|_\infty \int_s^t E(\tau) \|u\|_\infty \|\tilde{\psi}\|_1 \|\nabla u\|_2 \|u\|^{(q-1)}_{2(q-1)} \; d\tau \leq C \int_s^t E(\tau) \|\nabla \; u\|_2^2 \; d\tau
\end{aligned} \tag{4.10}
$$

Note that $2(q-1) \geq 1$ and hence $u \in L^{2(q-1)}$. The last inequality holds since in 2 dimensions $\|u\|_\infty \leq C\|\nabla u\|_2$. The conclusion of the Proposition follows combining the last inequality with (4.8) and (4.7). □

5. Non-uniform decay

In order to establish non-uniform L^p time decay we first study the decay of energy in L^2-norm. In this case high and low frequencies of the solutions are analyzed separately. That is as described in the last section we split the L^2 norm of u

$$
\|u\|_2 \leq \|\phi \hat{u}\|_2 + \|(1-\phi)\hat{u}\|_2,
$$

where ϕ is centered at low frequencies. Once the L^2 decay is established, interpolation will yield L^p-decay. For the low frequency estimates, ideas developed by Masuda [7] will be used. For high frequency estimates the main tool will be the Fourier Splitting Method. Similar techniques were used for Navier-Stokes in [10]. In what follows all the constants that depend on norms of f, b, u_0 and on q, n will be denoted by C. In some cases C might also depend on functions ϕ which will be introduced below.

We remark that since in the case $n \geq 3$ the initial data is only in L^1 this non-uniform decay might be optimal. We recall that such lack of uniformity can be found even at the Heat equation level, where if the data is only in L^2 there are examples of solution that do not decay uniformly. For a reference see [14] .

Theorem 5.1. *(Non-uniform decay) Let $q \geq 1 + \frac{1}{n}$. If $n \geq 3$ let $u_0 \in L^1$. If $n = 2$, let $u_0 \in L^1 \cap L^\infty$. Suppose f satisfies* **A.1** *and u is a solution constructed in Theorem (2.1). Let $b \in (L^\infty)^n$ with div $b = 0$. Then the solution u to (1.1) satisfies the non-uniform energy decay:*

$$\|u(t)\|_p \to 0 \qquad \text{as } t \to \infty,$$

where $p \in (1, \infty)$, for $n \geq 3$ (and $p \in (1, \infty]$ when $n = 2$).

Proof. To modulate the low frequencies choose $\phi(\xi) = \exp(-|\xi|^2)$. The proof is split into three steps:

- Low frequency estimates : Estimates on $\|\phi(\xi)\widehat{u}\|_2$.
- High frequency estimates : Estimates on $\|[1 - \phi(\xi)]\widehat{u}$
  $|_2$.
- L^p decay.

The proof makes use of the generalized energy inequality and the auxiliary estimates obtained in the Section on Preliminaries and in the Appendix.

Low Frequency decay

Corollary 4.3 and Plancherel's identity yield

$$\begin{aligned} &\|\phi\widehat{u}(t)\|_2^2 \\ &\leq \|e^{-|\xi|^2(t-s)}\phi\widehat{u}(s)\|^2 + 2\int_s^t |\langle\, e^{2\Delta(t-\tau)}\check{\phi}^2 * \big(b \cdot \nabla(u|u|^{q-1}), u\,\rangle\,|d\tau \\ &\quad +2\int_s^t |\langle\, f, e^{2\Delta(t-\tau)}\check{\phi}^2 * u\,\rangle\,|d\tau \equiv I + II + III. \end{aligned} \tag{5.1}$$

To bound term II in the RHS of the last equation, two cases have to be considered separately: $n = 2$, and $n \geq 3$.

Case: $n = 2$ Since div $b = 0$, the derivative can be passed onto u after an integration by parts. Using Hölder and Haussdorf-Young inequalities yields

$$\begin{aligned} II &\leq 2\int_s^t |\langle \sum_{j=1}^n \check{\phi}^2 * b_j(x)(u|u|^{q-1}), e^{2\Delta(t-\tau)}\partial_j u\,\rangle\,|d\tau \\ &\leq C\int_s^t \sum_{j=1}^n \|\check{\phi}^2 * b_j(x)u|u|^{q-1}\|_2\|\partial_j u\|_2 d\tau \leq C\int_s^t \|\check{\phi}^2\|_1\|u^q\|_2\|\nabla u\|_2 d\tau. \end{aligned} \tag{5.2}$$

Here $q \geq 1+1/2 = 3/2$,and since all the L^p, $1 \leq p \leq \infty$ norms of u are bounded,

$$\|u^q\|_2 \leq \|u\|_\infty \|u\|_{2q-1}^{q-1/2} \leq\leq C\|\nabla u\|_2. \tag{5.3}$$

Note also that by the definition of $\check{\phi}$ it follows that $\check{\phi} \in L^2$. Combining the last two bounds yields when $n = 2$

$$II \leq C \int_s^t \|\nabla u\|_2^2. \tag{5.4}$$

Case: $n \geq 3$ Since div $b = 0$ the derivative can be passed onto u after an integration by parts. Using Hölder and Haussdorf-Young inequalities it follows that

$$II \leq C \int_s^t \|b\|_\infty \|\check{\phi}^2 * u|u|^{q-1}\|_2 \|\nabla u\|_2 \, ds \leq C \int_s^t \|\check{\phi}^2\|_{\frac{4}{3}} \|u^q\|_{\frac{4}{3}} \|\nabla u\|_2 \, d\tau \tag{5.5}$$

To bound the last integral use

$$\|u^{(q-1)+1}\|_{\frac{4}{3}} \leq (\|u\|_{(q-1)n}^{\frac{4}{3}(q-1)} \|u\|_{\frac{2n}{n-2}} \leq C\|\nabla u\|_2 \tag{5.6}$$

Here we used that $(q-1)n \geq 1$ by hypothesis. and that all the L^p norms of u are bounded, combining (5.5) and (5.6), yields for $n \geq 3$,

$$II \leq C \int_s^t \|\nabla u\|_2^2 \, d\tau. \tag{5.7}$$

To bound III we proceed as follows

(5.8)

$$III = 2 \int_s^t |\langle f, e^{2\Delta(t-\tau)} \check{\phi}^2 * u \rangle| d\tau \leq C \int_s^t |\langle f, u \rangle \int_{\mathbb{R}^n} e^{2\Delta(t-\tau)} \check{\phi}^2 \, dy \leq C \int_s^t \langle |f|, |u| \rangle \, d\tau$$

Hence by (5.1), (5.4), (5.7), (5.8) and Lemma 7.1 ,

$$\|\phi \widehat{u}(t)\|_2^2 \leq \|e^{-|\xi|^2(t-s)} \phi \widehat{u}(s)\|_2^2 + C \int_s^t \|\nabla u\|_2^2 d\tau + C\Big(\int_s^t \|\nabla u\|_2^2 d\tau\Big)^{1/\theta}. \tag{5.9}$$

where θ was defined in Assumption **A.1**. Since $\overline{\lim}_{t\to\infty} \|e^{-|\xi|^2(t-s)} \phi \widehat{u}(s)\|_2^2 = 0$, we have by letting $t \to \infty$ in (5.9), that

$$\overline{\lim_{t\to\infty}} \|\phi \widehat{u}(t)\|_2^2 \leq C \int_s^\infty \|\nabla u\|_2^2 d\tau + C\Big(\int_s^\infty \|\nabla u\|_2^2 d\tau\Big)^{1/\theta}. \tag{5.10}$$

By Proposition (7.4) in the Appendix the right hand side of (5.10) converges 0 as $s \to \infty$.

Remark 5.2. Dealing with the solutions constructed in Remark (7.7), then in the inequality (5.10) the last integral would be replaced by

$$\int_s^\infty \|f\|_2 d\tau,$$

which also converges to zero as $sto\infty$.

High Frequency decay

To estimate the high frequency part of the energy, Fourier splitting techniques will be used, [12, 13, 15, 17, 18]. From Proposition (4.4) in the Section on Preliminaries, we need inequality (4.6), we record this inequality to make it easier to follow the arguments below

$$
\begin{aligned}
&E(t)\|(1-\phi)\hat{u}(t)\|_2^2\\
&\leq E(s)\|(1-\phi)\widehat{u}(s)\|_2^2 + + \int_s^t E'(\tau)\int_{\chi(t)} |(1-\phi)\widehat{u}(\tau)|^2 d\xi\, d\tau\\
&\quad + \int_s^t (E'(\tau) - 2E(t)G^2(t)) \int_{\chi(t)^c} |(1-\phi)\widehat{u}(\tau)|^2 d\xi d\tau\\
&\quad + C\int_s^t E(\tau)\|\nabla u\|_2^2 d\tau + \int_s^t E(\tau) |\langle f, \{(1-\phi)^2\}^\vee * u(\tau)\rangle| d\tau.
\end{aligned}
\tag{5.11}
$$

Let $\chi(t) = \{\xi \in \mathbb{R}^n; |\xi| \leq G(t)\}$ Choose $E(t) = (1+t)^\alpha$ with $\alpha > 0$, $G^2(t) = \frac{\alpha}{2(1+t)}$ in (5.11). Since $E'(t) = 2E(t)G^2(t)$, it follows from inequality (5.11) dividing by $E(t)$

(5.12)

$$
\begin{aligned}
&\|(1-\phi)\hat{u}(t)\|_2^2\\
&\leq (\frac{1+s}{1+t})^\alpha \|(1-\phi)\hat{u}(s)\|_2^2\\
&\quad + \frac{\alpha}{(1+t)^\alpha}\int_s^t (1+\tau)^{\alpha-1}\int_{\chi(\tau)} |(1-\phi)\widehat{u}|^2 d\xi d\tau\\
&\quad + C(\phi)\frac{1}{(1+t)^\alpha}\int_s^t (1+\tau)^\alpha \|\nabla u\|_2^2 d\tau + C\int_s^t (\frac{1+\tau}{1+t})^\alpha |\langle f, \{(1-\phi)^2\}^\vee * u\rangle| d\tau.
\end{aligned}
$$

Lemma 7.1 in the Appendix is used to bound the last integral on the RHS of the last inequality, then it is easy to see that

$$
\begin{aligned}
&\|(1-\phi)\hat{u}(t)\|_2^2\\
&\leq (\frac{1+s}{1+t})^\alpha \|(1-\phi)\hat{u}(s)\|_2^2 + \frac{\alpha}{(1+t)^\alpha}\int_s^t (1+\tau)^{\alpha-1}\int_{\chi(\tau)} |(1-\phi)\widehat{u}|^2 d\xi d\tau\\
&\quad + C\int_s^t \|\nabla u\|_2^2 d\tau + C(\int_s^t \|\nabla u\|_2^2 d\tau)^{1/\theta}\\
&= I + II + III + IV,
\end{aligned}
\tag{5.13}
$$

where θ is defined in assumption **A.1**. Here $C = C(\phi, u_0)$.

Remark 5.3. If we would have used the solution constructed in Remark (7.7), the the last integral in inequality (5.13) would be replaced by $\int_s^\infty \|f\|_2 d\tau$ and this term tends to zero if we let first $t \to \infty$ and then $s \to \infty$.

It is immediate that the first term tends to zero as time goes to infinity in (5.13). Just as with the Low Frequency the last two terms are of the form $\int_s^t \|\nabla u\|_2^2\, d\tau$ and hence tend to zero as we let first $t \to \infty$ and then $s \to \infty$.

To bound term II, observe that $|1-\phi| \leq C|\xi|^2$. Hence

$$\int_{\chi(\tau)} |(1-\phi)\widehat{u}|^2 d\xi \leq CG(\tau)^4 \int_{\chi(\tau)} |\widehat{u}|^2 d\xi \leq C(1+\tau)^{-2}\|u\|_2. \tag{5.14}$$

Thus

$$II \leq \frac{C}{(1+t)^\alpha} \int_s^t (1+\tau)^{\alpha-3}\, d\tau. \tag{5.15}$$

It is clear that $\lim_{s\to\infty}[lim_{t\to\infty} II] = 0$. Hence all terms on the RHS of (5.13) tend to zero, thus $\|\widehat{u}(1-\phi)\|_2 \to 0$ as $t \to \infty$ Since both the low frequency part and the high frequency part decay to zero the L^2 norm tends to zero. That is

$$\lim_{t\ to\infty} \|u(t)\|_2 \leq \lim_{t\to infty} \|\phi\widehat{u}(t)\|_2 + \lim_{t\to\infty} \|(1-\phi)\widehat{u}(t)\|_2 = 0.$$

This establishes L^2- norm decay in Theorem (5.1).

We proceed now with the proof for L^p. We consider three subcases

- Decay in L^p, $p \in (1,2)$, $n \geq 2$. Follows by interpolation between L^1 and L^2.
- Decay in L^p, $p \in (2,\infty)$, $n \geq 2$. Follows by interpolation between L^2 and L^{p+1}.
- Decay in L^∞, $n = 2$. Follows by interpolation between L^2 and H^1.

This concludes the proof of Theorem (5.1) . □

Corollary 5.4. Under the hypothesis of Theorem (5.1). The solutions decay in H^1

$$\lim_{t\to\infty} \|\nabla u(t)\|_2 = 0.$$

Proof. Follows by interpolating between L^2 and $W^{2,2}$.

□

Corollary 5.5. Suppose that f satisfies the same conditions as in Theorem (5.1). Let u be a regular solutions constructed in Theorem (2.1) or in Remark (7.7). Then,

$$\frac{1}{t}\int_0^t \|u(\tau)\|_p d\tau \to 0 \text{ as } t \to \infty. \tag{5.16}$$

Proof. For any $\epsilon > 0$, we can choose s sufficiently large so that

$$\|u(t)\|_p \leq \epsilon \text{ for } t \geq s.$$

Then

$$\begin{aligned} &\frac{1}{t}\int_0^t \|u(\tau)\|_p d\tau = \frac{1}{t}\int_0^s \|u(\tau)\|_p d\tau + \frac{1}{t}\int_s^t \|u(\tau)\|_p d\tau \\ &\leq \frac{1}{t}\int_0^s \|u(\tau)\|_p d\tau + \epsilon\frac{t-s}{t} \to \epsilon \quad \text{ as } t \to \infty. \end{aligned} \tag{5.17}$$

□

6. Uniform decay

In this section several uniform rates of decay for regular solutions of (1.1) are established. The main tool will be Fourier Splitting. Since the decay of the forcing function is slow the solutions will also only have slow decay. The main results of the section are comprised in Theorems 6.1 and 6.2. As before C denotes constants that might depend on $f, u_0, b, q,$ and, n. We note here that the main estimate needed for Theorem 6.1 is contained in inequality (4.2), hence there are no restrictions on the size of q other than that $q \geq 1$. For Theorem 6.2, besides of estimates from the preliminary section we need some Sobolev estimates that will put stronger restrictions on the lower bound of the size of q for the part on algebraic rate.

Theorem 6.1. *Let u be a regular solution constructed in Theorem (2.1) (or in Remark (7.7) for L^2 norm decay) . Let $q \geq 1$. Then*

1)

$$\|u(t)\|_2 \leq C(1+t)^{-\alpha},$$

where

$$\alpha = \begin{cases} \min\left(\frac{n}{2}, \epsilon\frac{\theta+2}{\theta}\right), & \text{if } n \geq 2,\ f \text{ satisfies } \mathbf{A.2,A.3i}, \\ \min\left(1, \frac{n-2}{2}, \epsilon\frac{\theta+2}{\theta}\right) & \text{if } n \geq 3,\ f \text{ satisfies } \mathbf{A.2,A.3ii}. \end{cases}$$

2) If $r \in (1,2)$ then $\|u(t)\|_r \leq C(1+t)^{-2\alpha(1-1/r)}$.

3) If $r \geq 2$ then $\|u(t)\|_r \leq C(1+t)^{-\alpha\frac{2}{r}+\delta}$, $\delta > 0$.

Proof. Case 1. Assume f satisfies assumptions A.2 and A.3 (i). Let $f = Dg$, $g \in L^\infty((0,\infty); L^1)$. We first establish the decay in L^2. For easier understanding of the argument we record here inequality (4.2) from the section on Preliminaries.

$$\begin{aligned} &E(t)\|u(t)\|_2^2 \\ &\leq E(s)\|u(s)\|_2^2 + \int_s^t E'(\tau)\|u(\tau)\|_2^2 d\tau - 2\int_s^t E(\tau)\|\nabla u(\tau)\|_2^2 d\tau \\ &\quad +2\int_s^t E(\tau)|\langle f(\tau), u(\tau)\rangle| d\tau. \end{aligned} \tag{6.1}$$

As before let $\chi(t) = \{\xi \in \mathbb{R}^n; |\xi| < G(t)\}$. By Parseval the second and third terms on the right hand side of (4.2) can be treated in Fourier space. The integral over $\mathbb{R}^n$ in frequency domain can be split into $\mathbb{R}^n = \chi \cup \chi^c$. Let $E(t)$, $G(t)$ be functions of t alone and satisfy

$$E'(t) = 2G^2(t)E(t). \tag{6.2}$$

Note that the part of the L^2 norm of the gradient integrated on $\chi(t)^c$ is negative and hence can be dropped. With the above choice of $E(t)$ and $G(t)$ we have

$$\int_{\chi^c} \left[E'(\tau)|u(\tau)|^2 - 2|\nabla u(\tau)|^2 E(t)\right] d\tau \leq \int_{\chi^c} [E'(\tau) - 2G^2(\tau)E(\tau)]|u(\tau)|^2\, d\tau = 0,$$

thus inequality (6.1) yields

$$\begin{aligned} E(t)\|u(t)\|_2^2 &\le E(0)\|u_0\|_2^2 + 2\int_0^t E'(\tau)\int_{\chi(\tau)} |\hat{u}|^2 d\xi d\tau \\ &\quad +2\int_0^t E(\tau)|\langle f(\tau), u(\tau)\rangle| d\tau \\ &= I + II + III. \end{aligned} \tag{6.3}$$

To bound the second term on the RHS of (6.3) we need to estimate the Fourier Transform of the solution. Taking the Fourier Transform of equation (1.1) and solving in frequency space yields

$$\begin{aligned} |\mathcal{F}u(\tau)| &\le |e^{-|\xi|^2\tau}\widehat{u_0}| + C\int_0^\tau e^{-|\xi|^2(\tau-\sigma)}[|\xi||\mathcal{F}(b\cdot u|u|^{q-1})| + |\widehat{g}|]d\sigma \\ &\le |e^{-|\xi|^2\tau}\widehat{u_0}| + C|\xi|\int_0^\tau e^{-|\xi|^2(\tau-\sigma)}d\sigma. \end{aligned} \tag{6.4}$$

Here we used that the L^q norm of the solution can be bounded and $b \in L^\infty$ and $\hat{g} = \xi\hat{f}$. By Jensen's inequality, and since $|\xi| \le G(t)$ in the ball $\chi(t)$, the second term of the right hand side of (6.3), can be bounded by

$$|| \le C\int_0^t E'(\tau)\big[\|e^{\Delta\tau}u_0\|_2^2 + \tau G(\tau)^{n+2}\big]d\tau. \tag{6.5}$$

Combining this last estimate with (6.3) yields

$$\begin{aligned} &E(t)\|\hat{u}(t)\|_2^2 \\ &\le E(0)\|u_0\|_2^2 + 2\int_0^t E'(\tau)\|e^{\Delta\tau}u_0\|_2^2 d\tau \\ &\quad +C(\|u_0\|_1, \|g\|_{\infty,1})\int_0^t \tau E'(\tau)G(\tau)^{n+2}d\tau + 2\int_0^t E(\tau)|\langle f(\tau), u(\tau)\rangle| d\tau. \end{aligned} \tag{6.6}$$

Let $E(t) = (1+t)^\alpha$. Then by equation (6.2) $G^2(t) = \frac{\alpha}{2}(t+1)^{-1}$. Hence the integrand of the third term on the RHS of the last inequality, since $\xi \in \chi$, can be bounded by

$$\tau E'(\tau)G(\tau)^n|\xi|^2 \le \tau E'(\tau)G(\tau)^{n+2} \le C(1+\tau)^{1+\alpha-1-(\frac{n}{2}+1)}.$$

Recall that standard L^2-L^1 estimates, [11], [5], for the Heat operator, if $u_0 \in L^1$, give

$$\|e^{t\Delta}u_0\|_2 \le C_1 t^{-\frac{n}{4}}, \tag{6.7}$$

where $C_1 = C_1(\|u_0\|_1)$. The assumptions on the initial data combined with the estimate (6.7),(6.3) and (6.6) yield

$$\begin{aligned}(1+t)^{\alpha}\|\widehat{u}(t)\|_2^2 &\leq C(\|u_0\|_2^2 + \int_0^t (1+\tau)^{\alpha-1-n/2} d\tau \\ &\quad + \int_0^t (1+\tau)^{\alpha-n/2-1} d\tau + \int_0^t (1+\tau)^{\alpha} |\langle f(\tau), u(\tau)\rangle| d\tau) \\ &\equiv I_1 + I_2 + I_3 + I_4.\end{aligned} \tag{6.8}$$

Note that I_2 and I_3 are the same, and an easy calculation shows that I_2 and I_3 are bounded under the choice $\alpha < \frac{n}{2}$. To estimate I_4 Hypothesis **A.2** and Lemma (7.3) are needed. Let $h(t+1) = (1+t)^{\alpha}$, with $\alpha < \beta - \frac{1}{\theta'} = \epsilon\frac{\theta+2}{\theta}$. Then $(\alpha-\beta)\theta' < -1$, hence the hypothesis of Lemma (7.1) are satisfied

$$I_4 \leq C. \tag{6.9}$$

Let now

$$\alpha < \min\left\{\frac{n}{2}, \epsilon(\frac{\theta+2}{\theta})\right\}.$$

Then (6.8) yields

$$(1+t)^{\alpha}\|u(t)\|_2^2 \leq C. \tag{6.10}$$

This concludes the first part of the theorem in the norm L^2. The decay in the norm L^r follows by a straightforward interpolation between $L^1 - L^2$ if $r \in (1,2)$ and an interpolation between $L^2 - L^{mr}$ when $r > 2$, with any $m > 1$, this yields with the decay in L^2.

$$\|u(t)\|_r \leq C(t+1)^{-2\frac{m-1}{mr-2}}.$$

Now let $m \to \infty$, yielding the decay $\|u(t)\|_r \leq C(t+1)^{-\frac{2}{r}+\delta}$ for any $\delta > 0$, provided m is chosen sufficiently large.

Case 2. Here we have **A.2**, **A.3**.(ii), and $n \geq 3$. Hence $f \in L^2(0,\infty; L^1)$. We first analyze the decay in L^2. The starting point is inequality (6.1). The second term on the right-hand side of (6.1) can be bounded splitting it into high and low frequencies via appropriate functions ϕ,

$$\begin{aligned}\int_0^t E'(\tau)\int_{\mathbb{R}^n} |\widehat{u}(\xi)|^2 d\xi d\tau &\leq \int_0^t E'(\tau)\int_{\mathbb{R}^n} |\phi(\xi)\hat{u}(\xi)|^2 d\xi d\tau \\ &\quad + \int_0^t E'(\tau)\int_{\mathbb{R}^n} |(1-\phi^2(\xi))^{1/2}\hat{u}(\xi)|^2 d\xi d\tau.\end{aligned} \tag{6.11}$$

We let $\phi(\xi) = e^{-h(t)|\xi|^2}$, where $h(t)$ will be determined below. Note that provided $h(t) \geq 0$ it follows that $w(r) = 1 - e^{-h(t)r^2} - 2h(t)r^2 \leq 0$. Thus choosing $E(t)$ so that

$$h(\tau)E'(\tau) \leq E(\tau), \tag{6.12}$$

yields

$$\int_0^t E'(\tau)\int_{\mathbb{R}^n} |(1-\phi^2(\xi))^{1/2}\hat{u}(\xi)|^2 d\xi d\tau - 2\int_s^t E(\tau)\|\nabla u(\tau)\|_2^2 d\tau \leq 0. \tag{6.13}$$

Using estimates (6.11) and (6.13) in (6.1) gives

$$\begin{aligned} E(t)\|u(t)\|_2^2 \leq\ & E(0)\|u_0\|_2^2 + \int_0^t E'(\tau)\int_{\mathbb{R}^n}|\phi(\xi)\hat{u}(\tau)|^2 d\xi d\tau \\ & +2\int_0^t E(\tau)|\langle f(\tau),u(\tau)\rangle|d\tau. \end{aligned} \tag{6.14}$$

To estimate the second term of right hand side in (6.14), write the solution to equation (1.1) in frequency space and use Jensen inequality (2.5), hence

$$\begin{aligned} & \int_{\mathbb{R}^n}|\phi(\xi)\mathcal{F}u(\tau)|^2 d\xi \\ & \leq \int_{\mathbb{R}^n}\phi(\xi)^2 e^{-2\tau|\xi|^2}|\widehat{u_0}|^2 d\xi \\ & + \int_{\mathbb{R}^n}\Big|\int_0^\tau \phi(\xi)^2 e^{-|\xi|^2(\tau-\sigma)}\mathcal{F}(b\cdot\nabla(u|u|^{q-1})+f)d\sigma\Big|^2 d\xi \\ & \leq C\Big(\int_{\mathbb{R}^n}\phi(\xi)^2|e^{-|\xi|^2\tau}\widehat{u_0}|^2 d\xi + \tau\int_0^\tau\int_{\mathbb{R}^n}\phi(\xi)^2 e^{-2(\tau-\sigma)|\xi|^2}|\xi|^2\|u\|_q^{2q}|d\xi d\sigma \\ & \quad +\tau\int_0^\tau\int_{\mathbb{R}^n}\phi(\xi)^2 e^{-2(\tau-\sigma)|\xi|^2}|\widehat{f}|^2 d\sigma d\xi\Big) \equiv I_1+I_2+I_3. \end{aligned} \tag{6.15}$$

One has

$$I_1 \leq C\int_{\mathbb{R}^n} e^{-2(\tau+h(\tau))|\xi|^2}|\widehat{u_0}|^2 d\xi \leq C(\tau+h(\tau))^{-\frac{n}{2}}\|u_0\|_1^2, \tag{6.16}$$

$$\begin{aligned} I_2 & \leq C\tau\int_0^\tau\int_{\mathbb{R}^n} e^{-2(\tau-\sigma+h(\tau))|\xi|^2}|\xi|^2\|u\|_q^{2q}d\xi\, d\sigma \\ & \leq C\tau\sup_\tau\|u\|_q^{2q}\int_0^\tau (h(\tau)+\tau-\sigma)^{-\frac{n}{2}-1}d\sigma, \end{aligned} \tag{6.17}$$

and

$$\begin{aligned} I_3 & \leq C\tau\int_0^\tau\int_{\mathbb{R}^n} e^{-2(\tau-\sigma+h(\tau))|\xi|^2}|\hat{f}|^2 d\xi d\sigma \\ & \leq C\tau\int_0^\tau (\tau-\sigma+h(\tau))^{-\frac{n}{2}}\|f(\sigma)\|_1^2 d\sigma. \end{aligned} \tag{6.18}$$

Choose $h(\tau)=(1+\tau)$ and $E(\tau)=(1+\tau)^\alpha$, due to condition (6.12) on the function h, it is necessary to suppose $\alpha\leq 1$. Then from the bounds for $I_i, i=1,2,3$ it follows that

$$\begin{aligned} I_1+I_2+I_3 \leq\ & C(1+2\tau)^{-\frac{n}{2}}\|u_0\|_1^2 + C\tau(1+\tau)^{-\frac{n}{2}}\sup_\tau\|u\|_q^{2q} \\ & +C\tau(1+\tau)^{-\frac{n}{2}}\int_0^\tau\|f(\sigma)\|_1^2 d\sigma \leq C(1+\tau)^{-\frac{n}{2}+1}. \end{aligned} \tag{6.19}$$

We recall that from Lemma (7.3), if $\alpha < \beta - \frac{1}{\theta'} = \epsilon\frac{\theta+2}{\theta}$, then

$$\int_0^t (1+\tau)^\alpha|\langle f(\tau),u(\tau)\rangle|d\tau \leq C.$$

By our choice of $E(t)$ it follows by (6.14) and ((6.19) that

$$(6.20)\qquad \begin{aligned}(1+t)^{\alpha}\|u(t)\|_2^2 &\leq \|u_0\|_2^2 + \int_0^t (1+\tau)^{\alpha} |\,\langle\, f(\tau), u(\tau)\,\rangle\,| d\tau \\ &+C\int_0^t (1+\tau)^{\alpha-1}\big[(1+\tau)^{-\frac{n}{2}+1}\big] d\tau \\ &\leq \|u_0\|_2^2 + C\left(\int_0^t (1+\tau)^{\alpha-\frac{n}{2}} d\tau + 1\right).\end{aligned}$$

Here C depends only on the data f, n, u_0 and q. This last bound follows provided $\alpha < \min(1, \frac{n}{2}-1, \epsilon\frac{\theta+2}{\theta})$. Recall that we only consider the case $n \geq 3$. The desired rate of decay in L^2 follows. As in case 1, the L^p decay rates result by appropriate interpolation. □

The following theorem considers the case when $n = 2$ and f satisfies **A.2, A.3ii**.

Theorem 6.2. *Let u be a regular solution constructed in Theorem (2.1) (or in Remark (7.7) for the L^2 norm decay). Let $n = 2$, $q \geq 1$.*
Suppose f satisfies **A.2, A.3ii**. *Then*
1) $\|u(t)\|_2 \leq C[\ln(e+t)]^{-1}$, for any $\alpha < 1$.
2) If $r \in (1,2)$ then $\|u(t)\|_r \leq [\ln(e+t)]^{-2(1-1/r)}$.
3) If $r > 2$ then $\|u(t)\|_r \leq C\ [\ln(e+t)]^{-\frac{2}{r}+\delta}$, $\delta > 0$.
4) If in addition $q \geq 1 + 1/n = 1 + 1/2$, $f \in L^1((0,\infty); L^1)$ and $u_0 \in L^\infty$, then

$$\begin{cases} \|u(t)\|_2 \,\leq\, C(t+e)^{-2\epsilon}, \\ \|u(t)\|_r \leq C(t+e)^{-2\epsilon(1-1/r)}, \ \text{if } r \in (1,2), \\ \|u(t)\|_r \leq C(t+e)^{-\frac{4\epsilon}{r}}, \ \text{if } r \in (2,\infty), \end{cases}$$

where ϵ was defined in **A.2**.

Proof.
The starting point is inequality (6.1). As before let $\phi = \exp\left(-|\xi|^2 h(t)\right)$,where now $h(\tau) = (e+\tau)(\ln(e+\tau))$ and $E(\tau) = \ln(e+\tau)$. Since $h(t)E'(t) - E(t) = 0$, as in the proof of Theorem (6.1) inequality (6.1) reduces to inequality (6.14). Thus the second term in (6.1) can be bounded again by integrals I, II, III where the old h is replaced by the new function h given above. This analysis yields

$$(6.21)\qquad \begin{aligned}\int_{\mathbb{R}^n} |\phi(\xi)\mathcal{F}u(\tau)|^2 d\xi \ &\leq C(\tau + (\tau+e)\ln(e+\tau))^{-1}\|u_0\|_1^2 \\ &+C\tau(\tau + (\tau+e)\ln(e+\tau))^{-1}\sup\|u\|_q^{2q} \\ &+C\tau(\tau + (\tau+e)\ln(e+\tau))^{-1}\|f\|_{1,2}^2.\end{aligned}$$

Since $f \in L^2(0,\infty;L^1)$ and $\|u\|_q^{2q}$ are bounded by hypothesis, combining (6.14) and (6.21) yields

$$\begin{aligned}(\ln(e+\tau))\|u(t)\|_2^2 &\le \|u_0\|_2^2 + C\int_0^t (\ln(e+\tau))^{-1}(e+\tau)^{-2}\,d\tau \\ &+ \int_0^t (\ln(e+\tau))|\langle f(\tau),u(\tau)\rangle|d\tau = I + II.\end{aligned} \tag{6.22}$$

Integral I can be estimated by straightforward integration

$$\int_0^t (e+\tau)^{-1}(\ln(e+\tau))^{-1}d\tau = \int_1^{\ln(e+t)} \frac{d\omega}{\omega e^{\omega}} \le \int_1^{\ln(e+t)} \frac{d\omega}{e^{\omega}} \le C.$$

To estimate first integral II we use Lemma (7.3) with

$$h(t) = \ln(e+t), \ \text{ where } \ \int_0^\infty \Big(\frac{\ln(e+\tau)}{(e+\tau)^\beta}\Big)^{\theta'} d\tau \le C,$$

since $\beta\theta' = 1 + \epsilon\frac{\tau+2}{\tau} > 1$. Hence by Lemma (7.3) it follows that

$$II = \int_0^t (\ln(e+\tau))|\langle f(\tau),u(\tau)\rangle|d\tau \le C. \tag{6.23}$$

Hence the RHS of inequality (6.22) is bounded by a constant C. Thus,

$$\|u(t)\|_2 \le C(\ln(e+t))^{-1}. \tag{6.24}$$

The L^p estimates follow by standard interpolation between L^1 and L^2 for part 2, and between L^2 and L^{mr} for part 3 and then choosing m as large as necessary.

To prove the last part of the theorem, we start by inequality (6.3) and use a modification of Fourier Splitting combined with an appropriate Gronwall inequality as was done in ([18]). To bound the terms in this inequality it is necessary to estimate the square of Fourier Transform of the solution integrated over $\chi(t) = \{\xi : |\xi| \le G(t)\}$, with G defined below. We express equation (1.1) in Fourier form and write the solution in its integral form. Since the function $b(x)$ is in L^∞, we simply bound it by a constant. Thus taking the square of $\hat{u}$ and integrating over $\chi(\tau)$ yield

$$\begin{aligned}&\int_{\chi(\tau)} |\mathcal{F}u(\tau)|^2\,d\xi \\ &\le 2\int_{\chi(\tau)} |e^{-|\xi|^2\tau}\widehat{u_0}|^2 d\xi + 2\int_{\chi(\tau)} \Big\{\int_0^\tau e^{-|\xi|^2(\tau-\sigma)}C(|\xi|\|u\|_q^q + \|\widehat{f}\|_\infty)d\tau\Big\}^2 d\xi \\ &\le C|G(\tau)^2|\Big\{\|u_0\|_1^2 + (\int_0^\tau \|f(\sigma)\|_1 d\sigma)^2\Big\} + C\tau\int_{\chi(\tau)} |\xi|^2(\int_0^\tau \|u(\sigma)\|_q^{2q} d\sigma)\,d\xi.\end{aligned} \tag{6.25}$$

We are going to consider two separate cases:

Case: $q \ge 2$

Using the logarithmic decay established in first part of the Theorem we have

$$\|u(t)\|_q^{2q} \le C\|u\|_2^4\|u\|_\infty^{2q-4} \le C\|u\|_2^2 \ln(t+e)^{-1}. \tag{6.26}$$

Here the constant C depends on $\|u_0\|_\infty$. Now set $E(t) = (e+t)^2$ and $G^2(t) = \frac{2}{e+t}$ and use the last estimate in inequality (6.25) combined with (6.3), yielding

$$
\begin{aligned}
&(e+t)^2\|u(t)\|_2^2\\
&\leq \|u_0\|_1^2 + C\int_0^t (e+\tau)(e+\tau)^{-1}\Big\{\|u_0\|_1^2 + (\int_0^\tau \|f(\sigma)\|_1 d\sigma)^2\Big\}\, d\tau\\
&\quad +2C\int_0^t \tau(e+\tau)^{-2}\int_0^\tau \|u(\sigma)\|_2^2(\ln(e+\sigma))^{-1}d\sigma d\tau\\
&\quad +2\int_0^t (e+\tau)^2|\langle f(\tau), u(\tau)\rangle|d\tau.
\end{aligned}
\tag{6.27}
$$

To bound the last integral on the RHS of (6.27), we use Lemma (7.3). Note that $\epsilon < \epsilon\frac{\theta+2}{\theta} = \beta - \frac{1}{\theta'}$. Define $h(t+e) = (t+e)^\epsilon$. Since $(\epsilon-\beta)\theta' < -1$, the hypothesis of Lemma (7.3) hold, thus we have the bound

$$
\begin{aligned}
&\int_0^t (e+\tau)^2|\langle f(\tau), u(\tau)\rangle|d\tau\\
&\leq (e+t)^{2-\epsilon}\int_0^t (e+\tau)^\epsilon|\langle f(\tau), u(\tau)\rangle|d\tau \leq C(e+t)^{2-\epsilon}.
\end{aligned}
\tag{6.28}
$$

Hence using this last estimate in (6.27) yields, after dividing by $(e+t)$, one concludes that

$$
\begin{aligned}
&(e+t)\|u(t)\|_2^2\\
&\leq C\left(\frac{1}{e+t} + 1 + \int_0^t \|u(\sigma)\|_2^2(\ln(e+\sigma))^{-1}d\sigma + (e+t)^{1-\epsilon}\right).
\end{aligned}
\tag{6.29}
$$

We use the following version of Gronwall inequality, [4], (page 36).

$$
\phi(t) \leq \mu(t) + \int_a^t \nu(s)\phi(s)ds,
\tag{6.30}
$$

then, provided $\mu > 0$, it follows that

$$
\phi \leq \mu(t) + \int_a^t \nu(s)\mu(s)\exp\int_a^s \nu(r)drds.
\tag{6.31}
$$

Now put

$$
\begin{aligned}
&\phi(t) = (e+t)\|u(t)\|_2^2, \quad \mu(t) = C(1 + \frac{1}{e+t} + (e+t)^{1-\epsilon}),\\
&\nu(t) = C\big[\ln(e+t)(e+t)\big]^{-1}.
\end{aligned}
\tag{6.32}
$$

Then

$$
\exp\int_0^t \nu(r)dr = \ln(t+e),
\tag{6.33}
$$

and

$$\begin{aligned}
\int_0^t \nu(s)\mu(s)\exp^{\int_0^s \nu(r)dr}\,ds &= C\int_0^t \frac{1+\frac{1}{e+s}+(e+s)^{1-\epsilon}}{\ln(e+s)(e+s)}\ln(e+s)ds \\
&= C\int_s^t \frac{1}{e+s}+\frac{1}{(e+s)^2}+(e+s)^{-\epsilon}ds \qquad (6.34)\\
&= C\big(\ln(e+t)+1-\frac{1}{e+t}+(e+t)^{1-\epsilon}\big).
\end{aligned}$$

Now apply Gronwall to (6.29), with ϕ,μ and ν defined in (6.32) and use the computations of (6.34). It follows that

$$(e+t)\|u(t)\|_2^2 \le C\left(1+(e+t)^{1-\epsilon}+\ln(e+t)+(e+t)^{1-\epsilon}\right). \qquad (6.35)$$

Hence it follows that

$$\|u(t)\|_2^2 \le C(e+t)^{-\epsilon}.$$

The decay of the L^p norms follows by a straightforward interpolation. The proof of the theorem is now complete. □

Case: $1+1/2<q<2$
In this case we replace inequality (6.26) by the interpolation inequality

$$\|u\|_q^{2q} \le \|u\|_1^{4-2q}\|u\|_2^{4(q-1)}$$

For the range of q for this case we have $4(q-1)=2+\epsilon$ for some $\epsilon>0$. The logarithmic decay in this case will be of order $(\ln(e+t))^{-\epsilon}$. The arguments of the last case can be used with obvious straightforward modifications.

This concludes the proof of the theorem.

□

7. Appendix

The proof of the first Lemma can be found in [10].

Lemma 7.1. [10]. *Let $0\le\gamma\le 1$ and $2\le p\le\frac{2n}{n-2+2\gamma}$. If $n=2$ and $\gamma=0$ then $p<\infty$. Let f satisfy* **A.1**. *Then for $u\in L^\infty(0,T;L^2)\cap L^2(0,T;\dot{H}^1)$ and $0\le s<t<\infty$, we have*

$$\int_s^t <|f|,|u|> d\tau \le C\mathcal{E}(t)^\mu\Big(\int_s^t \|\nabla u(\tau)\|_2^2 d\tau\Big)^{\frac{1}{\theta}}\|\rho(|x-x_0|)^\gamma f\|_{L^{\theta',p'}},$$

where $x_0\in\mathbb{R}^n$, $\mathcal{E}(t)=\sup_{\tau<t}\|u(\tau)\|_2^2$ and $\mu=n[\frac{1}{2}-\frac{1}{p}]$. The weight function ρ and θ are the same as the one defined in assumption **A.1**.

Remark 7.2. Note that for our solutions $\mathcal{E}(t)<\infty$ provided f satisfies the appropriate hypothesis in Theorem (2.1).

The following auxiliary estimate is an extension of Lemma (7.1) in [10].

Lemma 7.3. *Let $0 \le \gamma \le 1$ and $2 \le p \le \frac{2n}{n-2+2\gamma}$. If $n = 2$ and $\gamma = 0$ then $p < \infty$. Let f satisfy* **A.2**. *Let $h(\tau)$ be a function that satisfies*

$$\int_0^\infty \Big|\frac{h(\tau+b)}{(\tau+b)^\beta}\Big|^{\theta'} dt < \infty. \tag{7.1}$$

Then for $u \in L^\infty((0,T);L^2) \cap L^2((0,T);\dot H^1)$ and $0 \le s < t < \infty$, we have for $b > 0$

$$\int_s^t |h(\tau+b)| \langle\, |f|, |u|\,\rangle\, d\tau \le C\Big(\int_s^t \|\nabla u(\tau)\|_2^2 d\tau\Big)^{\frac{1}{\theta}} \tag{7.2}$$

Proof. The proof follows a modified version of Lemma (7.1) [10]. Let $r = |x - x_0|$. Recall that

$$\rho = \rho_{x_0}(r) = \begin{cases} r(1 + |\ln r|) & (n = 2), \\ r \quad (n = 3, 4). \end{cases}$$

Case 1: $0 \le \gamma < 1$ Hölder inequality yields,

$$\int_s^t |h(\tau+b)||\,\langle\, f, u\,\rangle\,| d\tau \le \int_s^t |h(\tau+b)| \|\rho^\gamma f\|_{p'} \|\frac{u}{\rho^\gamma}\|_p d\tau. \tag{7.3}$$

Let $p = \alpha + \beta$, $\alpha = p(1-\gamma)$, $\beta = p\gamma$. Note that $\gamma p \le 2$, hence choose m such that $\frac{\gamma p}{2} + \frac{1}{m} = 1$.

By Hölder inequality it follows that

$$\|\frac{u}{\rho^\gamma}\|_p \le \|\frac{u}{\rho}\|_2^\gamma \|u\|_{\alpha m}^{(1-\gamma)}. \tag{7.4}$$

where $\alpha m = \frac{2p(1-\gamma)}{2-\gamma p}$. To bound the RHS of (7.4) we use Gagliardo-Nirenberg inequality which yields

$$\|u\|_{\alpha m}^{(1-\gamma)} \le \|u\|_2^\mu \|\nabla u\|_2^{(1-\gamma)(\lambda)}. \tag{7.5}$$

where $\lambda = \frac{n}{1-\gamma}(\frac{1}{2} - \frac{1}{p})$ and $\mu = (1-\gamma)(1-\lambda)$. Because of the bounds of γ and p in the hypothesis it follows that $\lambda \le 1$.

Next we use the Sobolev-Hardy inequality [1] we have $\|\frac{u}{\rho}\|_2^\gamma \le C\|\nabla u\|_2^\gamma$. Combining inequalities (7.3), (7.4), (7.5), with Hardy's inequality yields

$$\begin{aligned} &\int_s^t h(\tau+b)||\,\langle\, |f|, |u|\,\rangle\, d\tau \\ &\le \int_s^t \frac{|h(\tau+b)|}{(\tau+b)^\beta}((\tau+b)^\beta)\|\rho^\gamma f\|_{p'} \|u\|_2^\mu \|\nabla u\|_2^{(1-\gamma)\lambda+\gamma} d\tau = I. \end{aligned} \tag{7.6}$$

By hypothesis we have

$$\sup_{\{0<t<\infty\}} (\tau+b)^\beta \|\rho^\gamma f\|_{p'} \|u\|_2^\mu \le C.$$

Hence it follows by Hölder's inequality that

$$I \le C(\int_s^t \|\nabla u(\tau)\|_2^2 d\tau)^{\frac{1}{\theta}} (\int_s^t \Big|\frac{|h(\tau+b)|}{(\tau+b)^\beta}\Big|^{\theta'} d\tau)^{\frac{1}{\theta'}}. \tag{7.7}$$

Here we used that $\theta(\lambda(1-\gamma)+\gamma) = 2$. This completes the lemma in the case $\gamma \in [0, 1)$.

Case 2 : $\gamma = 1$ Note that when $\gamma = 1$ then $p = 2$ and $\theta = 2$. Combining inequalities (7.4), (7.5), (7.6) and a Hölder inequality yields

$$\int_s^t \frac{|h(\tau+b)|}{(\tau+b)^\beta}(t+b)^\beta \|\rho f\|_2 \|\frac{u}{\rho}\|_2 d\tau \le C\int_s^t \|\nabla u(\tau)\|_2 \delta\tau \Big|\frac{|h(\tau+b)|}{(\tau+b)^\beta}\Big| \tag{7.8}$$
$$\le \left(\int_s^t |\nabla u\|_2^2 d\tau \;\; \int_s^t \Big|\tfrac{|h(\tau+b)|}{(\tau+b)^\beta}\Big|^2 \delta\tau\right)^{\frac{1}{2}}.$$

Which completes the proof of the Lemma □

Next we derive some straightforward Sobolev estimates on the solution. The new estimate from this proposition is that $\nabla u \in L^2(0,\infty; L^2)$.

Proposition 7.4. *Let $u(x,t)$ be a solution constructed in Theorem (2.1) in part $\bullet 2b$ or $\bullet 3$, let $0 \le s \le t$. Then*

1) The u satisfies the energy inequality:

$$\|u(t)\|_2^2 + 2\int_s^t \|\nabla u(\tau)\|_2^2 d\tau \le \|u(s)\|_2^2 + \int_s^t |\langle f(\tau), u(\tau)\rangle| d\tau. \tag{7.9}$$

2) Suppose that f satisfies assumption **A.1**. *Then for $s = 0$ and for almost all $s > 0$ and all $t \ge s$ the following a-priori estimate on u holds:*

$$\mathcal{E}(t) = \sup_{0\le\tau\le t} \|u(\tau)\|_2 \le C_1, \tag{7.10}$$

$$\int_s^t \|\nabla u(\tau)\|_2^2 d\tau \le C_1. \tag{7.11}$$

3) Interpolating (7.10) and (7.11), for $2 \le q \le 2n/(n-2)$ and $\frac{n}{q} + \frac{2}{\sigma} = \frac{n}{2}$, yields

$$\|u\|_{L^{\sigma,q}} \le C_1,$$

where C_1 is a constant only depends on $\|u_0\|_2$ and $\|\rho^\gamma f\|_{L^{\theta',p'}}$.

Proof. To obtain (7.9) multiply the convection diffusion (1.1) by u and integrate over $\mathbb{R}^n \times [s,t]$. Note that the convection term integrates to zero since b is divergence free. Equalities (7.10) and (7.11) are an immediate consequence now of Lemma (7.1) and the hypothesis on the function f. □

Remark 7.5. Since from the hypothesis in Theorem 2.1 in part $\bullet 3$, we substitute the hypothesis on f by $f \in L^1([0,\infty); L^2 \cap L^1)$. Hence the following a priori estimate on the $L^2((0,\infty) : L^2)$ norm of the gradient is straightforward as is shown in the following Lemma.

Lemma 7.6. *Let $u_0 \in L^1 \cap L^2$, $f \in L^1([0,\infty); L^2 \cap L^1)$. Let u be a solution to equation (1.1) with data u_0, then for $s \in [0,t]$*

$$\|u(t)\|_2^2 + 2\int_s^t \|\nabla u(\tau)\|_2^2 d\tau \le \|u(s)\|_2^2 + C\|f\|_{2,1}, \tag{7.12}$$

where the constant C depends on u_0.

Proof. By fixed point arguments we can get a local solution, the method is the same as for part •2 of Theorem (2.1). Hence we only need an a priori estimate to extend the solution globally. For this multiply the convection diffusion (1.1) by u and integrating over $\mathbb{R}^n$ it follows that

$$\frac{d}{dt}\|u(t)\|_2^2 + 2\|\nabla u(t)\|_2^2 \leq 2\|f\|_2\|u\|_2. \tag{7.13}$$

Hence

$$\frac{d}{dt}\|u(t)\|_2 \leq 2\|f\|_2 \tag{7.14}$$

and hence

$$\|u\|_2 \leq \|u_0\|_2 + \|f\|_{2,1}.$$

With this estimate in hand repeat the previous steps to get by multiplying equation (1.1) by u and integrate in space. Now using the estimate on the L^2 norm that we just obtained it follows that

$$\begin{aligned} \|u(t)\|_2^2 + 2\int_s^t \|\nabla u(\tau)\|_2^2 d\tau &\leq \|u(s)\|_2^2 + C\int_s^t |\langle f(\tau), u(\tau)\rangle| d\tau \\ &\leq \|u(s)\|_2^2 + C\|f\|_{2,1}. \end{aligned} \tag{7.15}$$

□

Remark 7.7. If in addition to the hypothesis in the last lemma we suppose that $f \in W^{2,2}$ the the solution obtained will belong to $C((0,\infty); L^1 \cap W^{2,2}) \cap C^1((0,\infty); L^2)$.

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Department of Mathematics, UC Santa Cruz, Santa Cruz, CA 95064, USA
E-mail address: schonbek@math.ucsc.edu