# ON QUESTIONS OF DECAY AND EXISTENCE FOR THE VISCOUS CAMASSA-HOLM EQUATIONS

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ABSTRACT. We consider the viscous $n$-dimensional Camassa-Holm equations, with $n = 2, 3, 4$ in the whole space. We establish existence and regularity of the solutions and study the large time behavior of the solutions in several Sobolev spaces. We first show that if the data is only in $L^2$ then the solution decays without a rate and that this is the best that can be expected for data in $L^2$. For solutions with data in $H^m \cap L^1$ we obtain decay at an algebraic rate which is optimal in the sense that it coincides with the rate of the underlying linear part.

**Quelques questions de decroissance et existence pour les equations visqueuses de Camassa-Holm.**
Résumé : On considère les équations visqueuses de Camassa–Holm dans $\mathbb{R}^n$, $n = 2, 3, 4$. Nous établissons l'existence et regularité des solutuions. Nous étudions le comportament asymptotique des solutions dans plusieurs espaces de Sobolev quand le temps tend vers l'infini. On montre que si la donnée est seulement dans $L^2$ la solution decroît vers zero, mais la decroissance ne peux être uniforme. Pour les solutions avec de donnée dans $L^1 \cap H^m$ on obtient une decroissance algébrique avec une vitesse qui est optimale dans le sens que c'est la même que pour les solutions correspondant a l'équation linéaire.

## 1. Introduction

The Viscous Camassa-Holm equations (VCHE) are commonly written as

$$v_t + u \cdot \nabla v + v \cdot \nabla u^T + \nabla \pi = \nu \triangle v \tag{1.1}$$
$$u - \alpha^2 \triangle u = v$$
$$\nabla \cdot v = 0$$

These equations rose from work on shallow water equations [3], which led to [11], [16], where the equations are derived by considering variational principles and Lagrangian averaging. In light of this derivation the equations are sometimes called the Lagrangian Averaged Navier-Stokes equations. In [9], the equations were derived

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as a "filtered" Navier-Stokes equation, which obeys a modified Kelvin circulation theorem along filtered velocities. In this setting they are sometimes referred to as the Navier-Stokes-$\alpha$ equations, where $\alpha$ is the parameter in the filter. Solutions to the VCHE are closely related to solutions of the famous Navier-Stokes equation (NSE), but the filter allows bounds that are currently unobtainable for the NSE, making them in some ways better suited for computational turbulence study, see [12].

In [9], [10] these equations were studied in relation to turbulence theory. This treatment includes existence and uniqueness theorems on the torus in three dimensions. The two dimensional case was considered on the torus and the sphere in [14]. Global existence and uniqueness in three dimensions was proved on bounded domains with zero (non-slip) boundary conditions in [16]. These equations have also been studied in terms of large eddy simulation and turbulent pipe flow in [4],[5],[6], and [8]. In this paper we extend the current existence theorems and study the large time behavior of solutions.

In section four we extend the known existence and uniqueness theorems to the whole space in dimensions $2 \leq n \leq 4$ using the Galerkin method. In the next two sections we continue the decay program of M. E. Schonbek, [20], [21], [22], [24],[26]. In the whole space we prove that the energy of a solution corresponding to data only in $L^2(\mathbb{R}^n)$ decays to zero following the arguments in [18], and then demonstrate that no uniform rate of decay can exists which depends only on the initial energy. If the initial data is assumed to be in $L^1 \cap L^2$ we show, using the Fourier Splitting Method, that the energy of a solution decays at the rate expected from the linear part, this is the same rate of decay as solutions to the NSE. For solution with initial data in $H^m \cap L^1$ we calculate the decay of derivatives using again the Fourier Splitting Method with an inductive argument. Section seven contains analysis of the Helmholtz equation in the whole space using the heat kernel leading to bounds on the filter of the VCHE. In the final section we use the analysis of the filter in section seven to provide conditions under which we can expect a solution of the VCHE to approach a strong solution of the NSE in the whole space.

## 2. Notation

In this paper, $L^p$ denotes the standard Lebesgue space with norm $\|\phi\|_p = (\int |\phi|^p)^{1/p}$. We use $< u, v >= \int uv$ to denote the standard inner product on the Hilbert space $L^2$. Solenoidal vector fields (subsets of $\Sigma = \{\phi \in C_0^\infty(\Omega) | \nabla\phi = 0\}$) will be needed to describe incompressible solutions. $L^p_\sigma$ will denote the completion of $\Sigma$ in the norm $\|\cdot\|_p$. $W^{m,p}$ will be used to denote the standard Sobolev spaces with the convention that $H^m = W^{m,2}$. The completion of $\Sigma$ under the $H^m$ norm will be denoted by $H^m_\sigma$ and $(H^m_\sigma)'$ will be the dual space. To denote the Fourier Transform of a function $\phi$ we will use either $\hat{\phi}$ or $\mathcal{F}(\phi)$, with $\check{\phi}$ or $\mathcal{F}^{-1}(\phi)$ the inverse transform.

## 3. Preliminaries

The Kelvin-filtered Navier-Stokes equations (KFNSE) are given by the formula

$$\begin{aligned} \frac{\partial v}{\partial t} + u \cdot \nabla v + v \cdot \nabla u^T + \nabla \pi &= \nu \triangle v \\ \nabla \cdot v = \nabla \cdot u &= 0 \\ v &= \mathcal{O} u \end{aligned}$$

In the above, $u = g * v$ denotes a spatially filtered fluid velocity and $\mathcal{O}$ is the inverse of this convolution. The term $u \cdot \nabla v$ is similar to "mollifying" the Navier-Stokes equations, originally done by Leray, [15], to approximate solutions. The term $v \cdot \nabla u^T = \sum v_j \nabla u_j$ allows the solution to obey a modification of the Kelvin circulation theorem where circulation is conserved around a loop moving with the filtered velocity $u$. In two and three dimensions, using the identity

$$u \cdot \nabla v + \sum v_j \nabla u_j = -u \times (\nabla \times v) + \nabla (v \cdot u) \tag{3.1}$$

and including the term $\nabla(v \cdot u)$ in the pressure, the KFNSE can be written as

$$\begin{aligned} \frac{\partial v}{\partial t} + \nabla \pi &= u \times (\nabla \times v) + \nu \triangle v \\ \nabla \cdot u = \nabla \cdot v &= 0 \\ v &= \mathcal{O} u \end{aligned}$$

The following lemma will show that the bilinear term in the Kelvin-filtered Navier-Stokes equations behaves similar to the bilinear term in the Navier-Stokes equations.

**Lemma 3.1.** *Let $u$ and $v$ be smooth divergence free functions with compact support, then*

$$\begin{aligned} < u \cdot \nabla v, u > + < v \cdot \nabla u^T, u > &= 0 \\ < u \times (\nabla \times v), u > &= 0 \end{aligned}$$

*Proof.* The second equality is a consequence of the first, the identity (3.1), and the fact that $u$ is divergence free. To see the first inequality we just need to rearrange the terms and then integrate by parts

$$\sum_{i,j} \int_{\mathbb{R}^n} v_j \partial_i u_j u_i \, dx = - \sum_{i,j} \int_{\mathbb{R}^n} u_i \partial_i v_j u_j \, dx$$

□

Using this lemma, we can formally multiply the KFNSE by $u$ to find

$$< \frac{\partial}{\partial t} v, u > + \nu < \nabla v, \nabla u > = 0 \tag{3.2}$$

By choosing $\mathcal{O}$ to be the Helmholtz operator $\mathcal{O} = 1 - \alpha^2 \triangle$ we recover the Viscous Camassa-Holm equations

$$\begin{aligned} v_t + u \cdot \nabla v + v \cdot \nabla u^T + \nabla \pi &= \nu \triangle v \\ u - \alpha^2 \triangle u &= v \\ \nabla \cdot v &= 0 \end{aligned}$$

In the case of the VCHE, (3.2) becomes

$$\frac{1}{2}\frac{d}{dt}(< u, u > +\alpha^2 < \nabla u, \nabla u >) + \nu(< \nabla u, \nabla u > +\alpha^2 < \triangle u, \triangle u >) = 0 \tag{3.3}$$

This relation gives a priori estimates on $u$:

$$\begin{aligned}\|u(\cdot,t)\|_2^2 + \alpha^2\|\nabla u(\cdot,t)\|_2^2 + 2\nu\int_0^t \|\nabla u(\cdot,t)\|_2^2\,dt + 2\nu\alpha^2\int_0^t \|\nabla^2 u(\cdot,t)\|_2^2\,dt \\ \le \|u_0\|_2^2 + \alpha^2\|\nabla u_0\|_2^2\end{aligned} \tag{3.4}$$

## 4. Existence of Solutions for the VCHE

Existence and uniqueness of solutions for the VCHE was proved first on periodic domains in three dimensions, in [10], using the Galerkin method. The most general existence and uniqueness theorems in three dimensions are provided in [16] which relies on a fixed point argument. The theorems in [16] assume the initial data $u_0 \in H_0^1 \cap H^s$ with $s \in [3, 5)$ and $u = Au = 0$ on the boundary, where $A$ is the Stokes operator. We extend these results by providing theorems which cover the whole space in dimensions $2 \le n \le 4$. As an intermediate step, we provide a new existence proof on bounded domains in dimensions $2 \le n \le 4$, using the Galerkin Method, with initial data $v_0 \in L^2$, and $u = v = 0$ on the boundary. Our bounded result in three dimensions is slightly stronger then [16], by assuming $v_0 \in L^2$ we have only implied $u \in H_\sigma^2$.

**Definition 4.1.** A weak solution to the VCHE (1.1) with zero (no-slip) boundary conditions on any open bounded set $\Omega \subset \mathbb{R}^n$, $n = 2, 3, 4$, with smooth boundary is a pair of functions, $u$, $v$, such that

$$\begin{aligned} v &\in L^\infty([0,T]; L_\sigma^2(\Omega)) \cap L^2([0,T]; H_\sigma^1(\Omega)) \\ \partial_t v &\in L^2([0,T]; (H_\sigma^1)'(\Omega)) \end{aligned}$$

as well as $u|_{\partial\Omega} = v|_{\partial\Omega} = 0$, $v(x, 0) = v_0$, and for any $\phi \in L^2([0,T]; H_\sigma^1(\Omega))$ with $\phi(T) = 0$ the following equalities are satisfied:

$$\begin{aligned}\int_0^T < v, \partial_t\phi > \, ds + \int_0^T < u\cdot\nabla v, \phi > \, ds \\ + \int_0^T < \phi\cdot\nabla u, v > \, ds + \int_0^T < \nabla v, \nabla\phi > \, ds =< v_0, \phi(0) >\end{aligned}$$

and for $t$ a.e.

$$< u, \phi > +\alpha^2 < \nabla u, \nabla\phi >=< v, \phi >$$

**Definition 4.2.** A weak solution to the VCHE (1.1) in the whole space $\mathbb{R}^n$, $n = 2, 3, 4$, with initial data $v_0$, is a pair of functions, $u$, $v$, where

$$\begin{aligned} v &\in L^\infty([0,T]; L_\sigma^2(\mathbb{R}^n)) \cap L^2([0,T]; H_\sigma^1(\mathbb{R}^n)) \\ \partial_t v &\in L^2([0,T]; (H_\sigma^1)'(\mathbb{R}^n)) \end{aligned}$$

such that $v(x,0) = v_0$, and for any $\phi \in L^2([0,T]; H^1_\sigma(\mathbb{R}^n))$ with $\phi(T) = 0$ the following equalities are satisfied:

$$\int_0^T < v, \partial_t \phi > \, ds + \int_0^T < u \cdot \nabla v, \phi > \, ds$$
$$+ \int_0^T < \phi \cdot \nabla u, v > \, ds + \int_0^T < \nabla v, \nabla \phi > \, ds = < v_0, \phi(0) >$$

and for $t$ a.e.

$$< u, \phi > + \alpha^2 < \nabla u, \nabla \phi > = < v, \phi >$$

**Theorem 4.3.** *Given initial data $v_0 \in H^M_\sigma(\Omega)$,where $\Omega \subset \mathbb{R}^n$ is an open bounded set with smooth boundary and $v_0|_{\partial\Omega} = 0$ or $\Omega = \mathbb{R}^n$, $n = 2, 3, 4$, $M \geq 0$, ($H^0_\sigma = L^2_\sigma$), there exists a unique weak solution to the VCHE (1.1) in the sense of Definition 4.1 or 4.2 respectively. This solutions satisfies the estimate (3.4) as well as*

$$\|\partial_t^k \nabla^m v(t)\|_2^2 + \int_0^t \|\partial_t^k \nabla^{m+1} v(s)\|_2^2 \, ds \leq C(\|v_0\|_{H^M_0}) \tag{4.1}$$

*for all $m + 2k \leq M$.*

*Proof.* Existence is given by Theorems 8.4 and 8.6 in the appendix. The regularity statement is Theorem 8.8 and uniqueness is Theorem 8.9. We construct approximate solutions using the Galerkin method on bounded domains, and obtain uniform bounds through energy methods. Using a compactness lemma we are able to find a strongly convergent subsequence which allows the limit of the approximate solutions to pass through the non-linearity. To extend to unbounded domains we solve the problem in balls of radius $R_i$ (a sequence tending to infinity), and then invoke a diagonal argument. The regularity is then established through an inductive argument relying on energy methods. □

Next, we will state a Corollary that describes the action of the filter and will be used many times in the following two sections.

*Corollary* 4.4.

$$\|\partial_t^k \nabla^m u\|_2^2 + 2\alpha^2 \|\partial_t^k \nabla^{m+1} u\|_2^2 + \alpha^4 \|\partial_t^k \nabla^{m+2} u\|_2^2 = \|\partial_t^k \nabla^m v\|_2^2$$
$$\|\partial_t^k \nabla^m u\|_n^2 + \|\partial_t^k \nabla^{m+1} u\|_n^2 \leq C \|\partial_t^k \nabla^m v\|_2^2$$
$$\|\partial_t^k \nabla^m u(t)\|_n^2 + \int_0^t \|\partial_t^k \nabla^{m+1} u(s)\|_n^2 \, ds \leq C(\|v_0\|_{H^M_0})$$

for all $m + 2k \leq M$, where $C$ is a constant which depends only on $\alpha$,$n$, $m$, and $k$.

*Proof.* This is an application of the Gagliardo-Nirenberg-Sobolev inequality to the bounds in the previous theorem. Differentiating the filter relation shows

$$\partial_t^k \nabla^m u - \alpha^2 \partial_t^k \nabla^m \triangle u = \partial_t^k \nabla^m v$$

Squaring this relation then integrating by parts gives

$$\|\partial_t^k \nabla^m u\|_2^2 + 2\alpha^2 \|\partial_t^k \nabla^{m+1} u\|_2^2 + \alpha^4 \|\partial_t^k \nabla^{m+2} u\|_2^2 = \|\partial_t^k \nabla^m v\|_2^2$$

This is the first bound in the corollary. Applying the Gagliardo-Nirenberg-Sobolev inequality to $\|u\|_n$ and using the previous equality shows

$$\|\partial_t^k \nabla^m u\|_n^2 \leq C\|\partial_t^k \nabla^m v\|_2^2$$
$$\|\partial_t^k \nabla^{m+1} u\|_n^2 \leq C\|\partial_t^k \nabla^m v\|_2^2$$

This is the second set of bounds. Combining this with the regularity bounds in the theorem give the final set of bounds. □

## 5. Large Time Behavior of the VCHE: Non-Uniform Decay

In bounded domains it is easy to see that the energy of a solution decays exponentially using the Poincaré inequality

$$\|u\|_2^2 \leq C(\Omega)\|\nabla u\|_2^2$$

Indeed, start with the energy estimate (3.3) and apply the Poincaré inequality to find

$$\frac{1}{2}\frac{d}{dt}(< u, u > +\alpha^2 < \nabla u, \nabla u >) + C(\Omega)\nu(< u, u > +\alpha^2 < \nabla u, \nabla u >) \leq 0$$

This differential inequality implies

$$< u, u > +\alpha^2 < \nabla u, \nabla u > \leq Ce^{-t}$$

The situation in unbounded domains is more delicate. In this section it is shown that solutions with data only in $L^2_\sigma$ decay to zero with no algebraic rate and this is the optimal decay rate.

We will follow [18] to show that the solutions in the whole space, constructed in Theorem 4.3, approach zero as time becomes large. The idea is to split the solution into low and high frequency parts using a cut-off function and generalized energy inequalities to show that both the high and low frequency terms approach zero. The idea of splitting into low and high frequency was first used in [17].

**Lemma 5.1.** *Solutions of the VCHE constructed in Theorem 4.3 with $\Omega = \mathbb{R}^n$ satisfy the following generalized energy inequalities. Let $E \in C^1([0,\infty))$, $\psi \in C^1([0,\infty); C^1 \cap L^2(\mathbb{R}^n))$, and $\tilde{\psi} \in C^1(0,\infty; L^\infty(\mathbb{R}^n))$.*

$$\begin{aligned}
E(t)\|\psi(t) * v(t)\|_2^2 = {} & E(s)\|\psi(s) * v(s)\|_2^2 + \int_s^t E'(\tau)\|\psi(\tau) * v(\tau)\|_2^2\, d\tau \\
& + 2\int_s^t E(\tau) < \psi'(\tau) * v(\tau), \psi(\tau) * v(\tau) > d\tau \\
& - 2\int_s^t E(\tau)\|\nabla\psi(\tau) * v(\tau)\|_2^2\, d\tau \\
& - 2\int_s^t E(\tau) < u \cdot \nabla v, \psi(\tau) * \psi(\tau) * v(\tau) > d\tau \\
& - 2\int_s^t E(\tau) < v \cdot \nabla u^T, \psi(\tau) * \psi(\tau) * v(\tau) > d\tau
\end{aligned} \tag{5.1}$$

*and*

$$
\begin{aligned}
E(t)\|\tilde{\psi}(t)\hat{v}(t)\|_2^2 &= E(s)\|\tilde{\psi}(s)\hat{v}(s)\|_2^2 + \int_s^t E'(\tau)\|\tilde{\psi}(\tau)\hat{v}(\tau)\|_2^2\,d\tau \\
&+ 2\int_s^t E(\tau) < \tilde{\psi}'(\tau)\hat{v}(\tau), \tilde{\psi}(\tau)\hat{v}(\tau) > \,d\tau \\
&- 2\int_s^t E(\tau)\|\xi\tilde{\psi}(\tau)\hat{v}(\tau)\|_2^2\,d\tau \\
&- 2\int_s^t E(\tau) < \mathcal{F}(u\cdot\nabla v), \tilde{\psi}^2(\tau)\hat{v}(\tau) > \,d\tau \\
&- 2\int_s^t E(\tau) < \mathcal{F}(v\cdot\nabla u^T), \tilde{\psi}^2(\tau)\hat{v}(\tau) > \,d\tau
\end{aligned}
\tag{5.2}
$$

*Proof.* The proof of the first inequality is accomplished by multiplying the VCHE by $E(t)\psi * \psi * v$ then integrating by parts and in time. The second follows by the Plancherel theorem and (5.1). □

**Theorem 5.2.** *Let $v$ be the solution of the VCHE (1.1) constructed in Theorem 4.3 with $\Omega = \mathbb{R}^n$ and $v_0 \in L^2_\sigma(\mathbb{R}^n)$, then*

$$
\lim_{t\to 0}\|v(t)\|_2 = 0 \tag{5.3}
$$

*Proof.* We work in frequency space . We split the energy into low and high frequency parts

$$
\|\hat{v}\|_2 \le \|\phi\hat{v}\|_2 + \|(1-\phi)\hat{v}\|_2 \tag{5.4}
$$

where $\phi$ will be chosen below. To estimate the low frequency part of the energy, begin with the generalized energy estimate (5.1), choose $E(t) = 1$ and

$$
\psi(\tau) = \mathcal{F}^{-1}\left[e^{-|\xi|^2(t+1-\tau)}\right]
$$

Note that $\psi$ and $\mathcal{F}(\psi)$ are rapidly decreasing functions for $\tau < t+1$, and since $\psi' = |\xi|\psi$ the third and fourth terms in (5.1) add to zero. Let $\phi = e^{-|\xi|^2}$ and apply the Plancherel Theorem to see

$$
\begin{aligned}
&\|\phi\hat{v}(t)\|_2^2 \le \|e^{|\xi|^2(t-s)}\phi\hat{v}(s)\|_2^2 \\
&\qquad + 2\int_s^t | < \check{\phi}^2 * (u\cdot\nabla v - v\cdot\nabla u^T), e^{2\triangle(t-\tau)}v(\tau) > |\,d\tau
\end{aligned}
\tag{5.5}
$$

Now apply the Hölder inequality, Young's inequality, and the Gagliardo-Nirenberg-Sobolev inequality

$$
\begin{aligned}
| < \check{\phi}^2 * u\cdot\nabla v, e^{2\triangle(t-\tau)}v(\tau) > | &\le \|\check{\phi}^2 * u\cdot\nabla v\|_2\|e^{2\triangle(t-\tau)}v(\tau)\|_2 \\
&\le C\|\check{\phi}^2\|_{\frac{2n}{(n+2)}}\|u\|_{\frac{2n}{(n-2)}}\|\nabla v\|_2\|v\|_2 \\
&\le C(\phi)\|v\|_2\|\nabla u\|_2\|\nabla v\|_2
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
| < \check{\phi}^2 * v\cdot\nabla u^T, e^{2\triangle(t-\tau)}v(\tau) > | &\le \|\check{\phi}^2 * v\cdot\nabla u^T\|_2\|e^{2\triangle(t-\tau)}v(\tau)\|_2 \\
&\le C\|\check{\phi}^2\|_{\frac{2n}{(n+2)}}\|v\|_{\frac{2n}{(n-2)}}\|\nabla u\|_2\|v\|_2 \\
&\le C(\phi)\|v\|_2\|\nabla u\|_2\|\nabla v\|_2
\end{aligned}
$$

Using the triangle inequality, Hölder's inequality, and (4.1) in (5.5) yields

$$\begin{aligned}\|\phi\hat{v}(t)\|_2^2 &\leq \|e^{|\xi|^2(t-s)}\phi\hat{v}(s)\|_2^2 \\ &+ 2C(\phi)\|v_0\|_2 \left(\int_s^t \|\nabla u\|_2^2\, d\tau\right)^{1/2} \left(\int_s^t \|\nabla v\|_2^2\, d\tau\right)^{1/2}\end{aligned}$$

Taking the limit $t \to \infty$ shows, since the first term on the RHS tends to zero,

$$\limsup_{t\to\infty} \|\phi\hat{v}(t)\|_2^2 \leq 2C(\phi)\|v_0\|_2 \left(\int_s^\infty \|\nabla u\|_2^2\, d\tau\right)^{1/2} \left(\int_s^\infty \|\nabla v\|_2^2\, d\tau\right)^{1/2}$$

The bounds (3.4) and (4.1) show $\|\nabla u\|_2^2$ and $\|\nabla v\|_2^2$ are integrable on the positive real line, so taking $s \to \infty$ leaves

$$\limsup_{t\to\infty} \|\phi\hat{v}(t)\|_2^2 \to 0 \tag{5.6}$$

To estimate the high frequency, start with the generalized energy inequality (5.2) and chose $\tilde{\psi} = 1 - e^{-|\xi|^2}$. Let $A(t) = \{\xi : |\xi| \leq G(t)\}$, then, use $< u \cdot \nabla v, v >= 0$ to replace $\tilde{\psi}^2$ by $1 - \tilde{\psi}^2$ in the 5th term of (5.2).

(5.7)

$$\begin{aligned}E(t)\|(1-\phi)\hat{v}(t)\|_2^2 &\leq E(s)\|(1-\phi)\hat{v}(s)\|_2^2 + \int_s^t E'(\tau)\int_{A(\tau)} |(1-\phi)\hat{v}(\tau)|^2\, d\xi\, d\tau \\ &+ \int_s^t (E'(\tau) - 2E(\tau)G^2(\tau)) \int_{A^C(\tau)} |(1-\phi)\hat{v}(\tau)|^2\, d\xi\, d\tau \\ &+ 2\int_s^t E(\tau)| < \mathcal{F}(u\cdot\nabla v + v\cdot\nabla u^T), (1-\tilde{\psi}^2(\tau))\hat{v}(\tau) > |\, d\tau \\ &+ 2\int_s^t E(\tau)| < \mathcal{F}(v\cdot\nabla u^T), \hat{v}(\tau) > |\, d\tau\end{aligned}$$

Note that both $(1-\tilde{\psi}^2)$ and $\phi = \mathcal{F}^{-1}(1-\tilde{\psi}^2)$ are rapidly decreasing functions. Using again Hölder's inequality and the Plancherel theorem, then Young's inequality and the Gagliardo-Nirenberg-Sobolev inequality leaves

$$\begin{aligned}&| < \mathcal{F}(u\cdot\nabla v + v\cdot\nabla u^T), (1-\tilde{\psi}^2(\tau))\hat{v}(\tau) > | \\ &\qquad = | < (1-\tilde{\psi}^2(\tau))\mathcal{F}(u\cdot\nabla v + v\cdot\nabla u^T), \hat{v}(\tau) > | \\ &\qquad \leq \|\mathcal{F}^{-1}(1-\tilde{\psi}^2(\tau)) * (u\cdot\nabla v + v\cdot\nabla u^T)\|_2\|v\|_2 \\ &\qquad \leq C\|1-\tilde{\psi}\|_{\frac{2n}{n+2}}(\|u\|_{\frac{2n}{n-2}}\|\nabla v\|_2 + \|v\|_{\frac{2n}{n-2}}\|\nabla u\|_2)\|v\|_2 \\ &\qquad \leq C(\phi)\|v\|_2\|\nabla u\|_2\|\nabla v\|_2\end{aligned}$$

Similarly use Hölder's inequality with the Plancherel theorem, then the Gagliardo-Nirenberg-Sobolev inequality, and Corollary 4.4 to find

$$\begin{aligned}| < \mathcal{F}(v\cdot\nabla u^T), \hat{v}(\tau) > | &\leq \|v\cdot\nabla u^T\|_2\|v\|_2 \\ &\leq C\|v\|_{\frac{2n}{n-2}}\|\nabla u\|_n\|v\|_2 \\ &\leq C\|v\|_2\|\nabla v\|_2^2\end{aligned}$$

Choosing $E(t) = (1+t)^\beta$ and $G^2(t) = \beta/2(1+t)$ with $\beta > 0$ sufficiently large in (5.7), so that $E' - 2EG^2 = 0$, leaves

$$\begin{aligned}\|(1-\phi)\hat{v}(t)\|_2^2 \leq{}& \frac{(1+s)^\beta}{(1+t)^\beta}\|(1-\phi)\hat{v}(s)\|_2^2 \\ &+ \int_s^t \frac{\beta(1+\tau)^{\beta-1}}{(1+t)^\beta}\int_{A(\tau)} |(1-\phi)\hat{v}(\tau)|^2\, d\xi\, d\tau \\ &+ C\|v_0\|_2 \int_s^t \frac{(1+\tau)^\beta}{(1+t)^\beta}\|\nabla v\|_2(\|\nabla v\|_2 + \|\nabla u\|_2)\, d\tau\end{aligned}$$

Since $|1-\phi| \leq |\xi|^2$ if $\xi \in A(t)$ and $t$ is sufficiently large, therefore $|1-\phi|^2 \leq \beta^2/4(1+t)^2$, and the second term on the RHS can be bounded as

$$\begin{aligned}\int_s^t \frac{\beta^3(1+\tau)^{-3}}{4}\int_{A(\tau)} |\hat{v}(\tau)|^2\, d\xi\, d\tau &\leq \int_s^t \frac{\beta^3(1+\tau)^{-3}}{4}\|v(\tau)\|_2^2\, d\tau \\ &\leq \frac{\beta^3}{4}\|v_0\|_2^2 \int_s^t (1+\tau)^{-3}\, d\tau \\ &\leq \frac{\beta^3}{8}\|v_0\|_2^2(1+s)^{-2}\end{aligned}$$

Letting $t \to \infty$ shows

$$\begin{aligned}\limsup_{t\to\infty} \|(1-\phi)\hat{v}(t)\|_2^2 \leq{}& \frac{\alpha^3}{8}\|v_0\|_2^2(1+s)^{-2} \\ &+ C\|v_0\|_2\Big(\int_s^\infty \|\nabla v\|_2^2\, d\tau + \int_s^\infty \|\nabla u\|_2^2\, d\tau\Big)\end{aligned} \tag{5.8}$$

Note that all integrals on the RHS exist due to (3.4) and (4.1), then taking the limit $s \to \infty$ proves

$$\limsup_{t\to\infty} \|(1-\phi)\hat{v}(t)\|_2^2 = 0$$

Combining this with (5.6) and the Plancherel Theorem establishes the theorem. □

*Corollary* 5.3. Let $v$ be the solution of the VCHE (1.1) constructed in Theorem 4.3 with $\Omega = \mathbb{R}^n$ corresponding to $v_0 \in H_0^1(\mathbb{R}^n)$. Then

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t \|v(\tau)\|_2\, d\tau = 0$$

*Proof.* Given an $\epsilon > 0$, since $\|u\|_2 \to 0$ as $t \to \infty$ we can choose a large $s$ such that $\|u\|_2 \leq \epsilon$ for $\tau > s$. Then

$$\begin{aligned}\frac{1}{t}\int_0^t \|u(\tau)\|_2\, d\tau &= \frac{1}{t}\int_0^s \|u(\tau)\|_2\, d\tau + \frac{1}{t}\int_s^t \int \|u(\tau)\|_2\, d\tau \\ &\leq \frac{1}{t}\int_0^s \|u(\tau)\|_2\, d\tau + \epsilon\frac{t-s}{t}\end{aligned} \tag{5.9}$$

Note that $\epsilon$ was chosen arbitrarily and let $t \to \infty$ to finish the proof. □

We have shown that the energy of a solution to the VCHE will tend to zero as time becomes large, now we will provide a counter example to show that there is no uniform rate of decay based only on the initial energy of the system. This is analogous to a result proved in [23]. The idea is to take a family of initial data with a parameter $\epsilon$ that have constant $L^2$ norm, but norms of higher derivatives of the initial data can be taken arbitrarily small by picking $\epsilon$ sufficiently small. It is then possible to keep the norms of higher derivatives arbitrarily small in the solution. Combining this with the energy relation (3.3) allows us to place a lower bound on the energy of the solution which depends on $\epsilon$. By choosing $\epsilon$ small we can guarantee that a solution will remain away from zero for any finite amount of time.

**Theorem 5.4.** *There exists no function $G(t, \beta)$ with the following two properties. If $v$ be the solution of the VCHE (1.1) constructed in Theorem 4.3 with $\Omega = \mathbb{R}^n$ and $v_0 \in L^2_\sigma(\mathbb{R}^n)$, then*

$$\begin{aligned} &\|v\|_2 \leq G(t, \|v_0\|_2) \\ &\lim_{t\to\infty} G(t,\beta) = 0 \ \forall \beta \end{aligned} \tag{5.10}$$

*Proof.* Fix $u_0(x)$ to be any smooth function of compact support. Let $u_0^\epsilon(x) = \epsilon^{n/2}u_0(\epsilon x)$ and $v_0^\epsilon = u_0^\epsilon - \alpha^2\triangle u_0^\epsilon$. Note that $\|u_0^\epsilon\|_2 = \|u_0\|_2$ and $\|\nabla^m u_0^\epsilon\|_2 = \epsilon^m\|\nabla u_0\|_2$ for all $\epsilon > 0$. Also,

$$\begin{aligned} \|v_0^\epsilon\|_2^2 &= \|u_0^\epsilon\|_2^2 + \alpha^2\|\nabla u_0^\epsilon\|_2^2 + \alpha^4\|\triangle u_0^\epsilon\|_2^2 \\ &= \|u_0^\epsilon\|_2^2 + \alpha^2\epsilon^2\|\nabla u_0^\epsilon\|_2^2 + \alpha^4\epsilon^4\|\triangle u_0^\epsilon\|_2^2 \end{aligned} \tag{5.11}$$

and

$$\begin{aligned} \|\nabla v_0^\epsilon\|_2^2 &= \|\nabla u_0^\epsilon\|_2^2 + \alpha^2\|\triangle u_0^\epsilon\|_2^2 + \alpha^4\|\nabla\triangle u_0^\epsilon\|_2^2 \\ &= \epsilon^2\|\nabla u_0^\epsilon\|_2^2 + \alpha^2\epsilon^4\|\triangle u_0^\epsilon\|_2^2 + \alpha^4\epsilon^6\|\nabla\triangle u_0^\epsilon\|_2^2 \end{aligned} \tag{5.12}$$

From the two previous inequalities and Corollary 4.4 we obtain a constant $C = C(\|u_0\|_{H_0^3})$, such that for all $\epsilon > 0$

$$\begin{aligned} \|v^\epsilon\|_2^2 &\leq C \\ \|\nabla v^\epsilon\|_2^2 &\leq C\epsilon^2 \end{aligned} \tag{5.13}$$

Multiply the VCHE (1.1) by $\triangle v$, then integrating by parts yields

$$\frac{1}{2}\frac{d}{dt}\|\nabla v\|_2^2 + \nu\|\triangle^2 v\|_2^2 =< u\cdot\nabla v, \triangle v > + < \triangle v\cdot\nabla u, v >$$

The relation $< u, \nabla v, v >= 0$, the Hölder inequality, Sobolev inequality, and then the Cauchy Inequality shows

$$\begin{aligned} | < u\cdot\nabla v, \triangle v > | &= |(-1) < (\nabla u)\cdot\nabla v, \nabla v > | \\ &\leq C\|\nabla u\|_n\|\nabla v\|_2\|\nabla v\|_{\frac{2n}{n-2}} \\ &\leq \frac{\nu}{4}\|\triangle v\|_2^2 + C\|\nabla u\|_n^2\|\nabla v\|_2^2 \end{aligned}$$

Similarly,

$$\begin{aligned} | < \triangle v\cdot\nabla u, v > | &\leq C\|\triangle v\|_2\|\nabla u\|_n\|v\|_{\frac{2n}{n-2}} \\ &\leq \frac{\nu}{4}\|\triangle v\|_2^2 + C\|\nabla v\|_2^2\|\nabla u\|_n^2 \end{aligned}$$

This leaves

$$\frac{1}{2}\frac{d}{dt}\|\nabla v\|_2^2 + \frac{\nu}{2}\|\triangle v\|_2^2 \leq C\|\nabla v\|_2^2\|\nabla u\|_n^2 \tag{5.14}$$

By (3.4) and Corollary 4.4,

$$\int_0^\infty \|\nabla u^\epsilon\|_n^2 \, dt \le \|u_0^\epsilon\|_2^2 + \|\nabla u_0^\epsilon\|_2^2 \le \|v_0^\epsilon\|_2^2$$

This bound, combined with (5.14) yields

$$\|\nabla v^\epsilon\|_2^2 \le \|\nabla v_0^\epsilon\|_2^2 e^{C\|v_0^\epsilon\|_2^2}$$

Again, apply Corollary 4.4

$$\|\nabla u^\epsilon\|_2^2 + \alpha^2 \|\triangle u^\epsilon\|_2^2 \le \|\nabla v^\epsilon\|_2^2 \le \|\nabla v_0^\epsilon\|_2^2 e^{C\|v_0^\epsilon\|_2^2}$$

This together with the energy estimate (3.3) implies

$$\frac{1}{2}\frac{d}{dt}(\|u^\epsilon\|_2^2 + \alpha^2\|\nabla u^\epsilon\|_2^2) \ge -C\epsilon^2$$

Or,

$$\begin{aligned} \|u^\epsilon\|_2^2 + \alpha^2\|\nabla u^\epsilon\|_2^2 &\ge \|u_0^\epsilon\|_2^2 + \alpha^2\|\nabla u_0^\epsilon\|_2^2 - C\epsilon^2 t \\ &= \|u_0\|_2^2 + \epsilon^2\alpha^2\|\nabla u_0\|_2^2 - C\epsilon^2 t \\ &\ge \|u_0\|_2^2 - C\epsilon^2 t \end{aligned} \tag{5.15}$$

From this we can deduce that there is no function $G(\|u_0\|_2, t)$, continuous in $t$, which approaches zero and depends only on $\|u_0\|_2$ and $t$, such that $\|u\|_2 \le G(\|u_0\|_2, t)$. If there was such a function, then at some $t_0$ it would satisfy the bound $G(\|u_0\|_2, t_0) \le \|u_0\|_2/2$. By choosing $\epsilon$ sufficiently small in (5.15), i.e. $\epsilon^2 < \|u_0\|_2^2/4Ct_0$, we have found initial data with a solution which cannot satisfy this estimate. □

## 6. Large Time Behavior of the VCHE: Algebraic Decay

Although there is no uniform rate of decay for solutions with data exclusively in $L^2$, we now show that there is a uniform rate of decay depending on the $L^2$ and $L^1$ norm of the initial data. Moreover, using the Fourier Splitting Method it will be shown that solutions in the whole space decay algebraically as $t \to \infty$ for initial data in $L^1 \cap H^M$, $M \ge 0$. The decay obtained is the same as for the linear part (the heat equation). Note that the initial conditions can be weakened to require only that $v_0 \in X$ where $X = \{v_0 | v_0(t) \le C(1+t)^{-\beta}\}$ where $v_0(t)$ is the solution of the heat equation with initial data $v_0$. The decay rate will depend on the relation between $\beta$ and the number of dimensions. For similar results corresponding to the Navier-Stokes equations see [26], [28], and [29].

The Fourier Splitting Method was originally applied to the NSE in [22]. In [23] the decay rate was made sharp in dimension $n > 2$ through a bootstrap method and logarithmic decay was shown for $n = 2$. In [30] the decay rate for $n = 2$ was made sharp through a bootstrap argument involving the Gronwall inequality. In this section we combine ideas from all of these papers in a slightly different way which allows us to prove the optimal decay rate in dimensions $n \ge 2$ without appealing to a bootstrap argument. This same argument is also applicable the the NSE.

The first goal of this section is to obtain a decay rate for the filtered velocity $u$, which is accomplished by applying the Fourier Splitting Method to the natural energy relation (3.3). This decay rate is then used with an inductive argument to obtain decay rates for the unfiltered velocity $v$ and all of its derivatives. We start by finding estimates on $\|\hat{v}\|_\infty$.

**Lemma 6.1.** *Let $v$ be the solution of the VCHE (1.1) constructed in Theorem 4.3 with $\Omega = \mathbb{R}^n$, corresponding to $v_0 \in L^2_\sigma \cap L^1(\mathbb{R}^n)$. Then,*

$$|\mathcal{F}(v)| \leq C\left[1 + \left(\int_0^t \|u(s)\|_2^2\right)^{1/2}\left(\int_0^t \|\nabla v(s)\|_2^2\right)^{1/2}\right]$$

*Where the constant depends only on the initial data, the dimension of space, and the constants in the VCHE.*

*Proof.* Use the identity

$$\sum_i \nabla(u_i v_i) = \sum_i u_i \nabla v_i + \sum_i v_i \nabla u_i \tag{6.1}$$

and write the Fourier transform of the solution $\mathcal{F}(v)$ as

$$\mathcal{F}(v) = e^{-\nu t|\xi|^2}\mathcal{F}(v_0) + \int_0^t e^{-\nu(t-s)|\xi|^2}\Psi(\xi, s)\, ds$$

where

$$\Psi(\xi, t) = \xi \cdot \mathcal{F}(\pi + \sum_i u_i v_i) - \mathcal{F}(u\nabla v - u\nabla v^T)$$

We would like first to bound $\Psi$, in that direction we have the following estimates

$$|\mathcal{F}(u\nabla v - u\nabla v^T)| \leq C\|u\|_2\|\nabla v\|_2$$

Taking the divergence of the VCHE (1.1) shows

$$\triangle(\pi + \sum_i u_i v_i) = \operatorname{div}(u\nabla v - u\nabla v^T)$$

Using the estimate immediately above and the Fourier transform leaves

$$|\xi\mathcal{F}(\pi + \sum_i u_i v_i)| \leq C\|u\|_2\|\nabla v\|_2$$

Now we can bound the integrand

$$|\Psi(\xi, t)| \leq C\|u\|_2\|\nabla v\|_2$$

Using the Cauchy-Schwartz Inequality,

$$|\mathcal{F}(v)| \leq |\mathcal{F}(v_0)| + C\left(\int_0^t \|u(s)\|_2^2\, ds\right)^{1/2}\left(\int_0^t \|\nabla v(s)\|_2^2\, ds\right)^{1/2}$$

The estimate $|\mathcal{F}(v_0)| \leq \|v_0\|_1$ finishes the proof. □

**Theorem 6.2.** *Let $v$ be the solution of the VCHE (1.1) constructed in Theorem 4.3 with $\Omega = \mathbb{R}^n$, corresponding to $v_0 \in L^2_\sigma \cap L^1(\mathbb{R}^n)$. The solution satisfies the "energy" decay rate*

$$\int_{\mathbb{R}^n} v \cdot u\, dx = \|u\|_2^2 + \alpha^2\|\nabla u\|_2^2 \leq C(t+1)^{-n/2}$$

*Proof.* The previous lemma, with the bound (4.1), yields

$$|\hat{v}|^2 \leq C\left[1 + \int_0^t \|u(s)\|_2^2\, ds\right] \tag{6.2}$$

Now we begin work with the energy estimate (3.3). Using the Plancherel Theorem we rewrite it as

$$\frac{d}{dt}\int_{\mathbb{R}^n}(1+\alpha^2|\xi|^2)\hat{u}^2\,d\xi + 2\nu\int_{\mathbb{R}^n}|\xi|^2(1+\alpha^2|\xi|^2)\hat{u}^2\,d\xi = 0$$

Let $B(\rho)$ be the ball of radius $\rho$ where $\rho^2 = f'(t)/(2\nu f(t))$, and $f$ is a positive, increasing function to be specified later. Let $E^2 = \hat{u}\cdot\hat{v} = (1+\alpha^2|\xi|^2)\hat{u}^2$, then

$$\frac{d}{dt}\int_{\mathbb{R}^n}E^2\,d\xi + 2\nu\rho^2\int_{B^C(\rho)}E^2\,d\xi \le 0$$

or

$$\frac{d}{dt}\int_{\mathbb{R}^n}E^2\,d\xi + 2\nu\rho^2\int_{\mathbb{R}^n}E^2\,d\xi \le 2\nu\rho^2\int_{B(\rho)}E^2\,d\xi \tag{6.3}$$

Recall the relation between $u$ and $v$, that is $v = u - \alpha^2\triangle u$ which has Fourier Transform $\hat{u} = \hat{v}/(1+\alpha^2|\xi|^2)$. Combining this with the definition of $E$ and (6.2) we see

$$E(\xi)^2 \le C\left[1+\int_0^t \|u(s)\|_2^2\,ds\right]$$

Use this bound to evaluate the integral on the right hand side of (6.3)

$$\frac{d}{dt}\int_{\mathbb{R}^n}E^2\,d\xi + 2\nu\rho^2\int_{\mathbb{R}^n}E^2\,d\xi \le C\rho^{2+n}\left[1+\int_0^t \|u(s)\|_2^2\,ds\right]$$

We now have a differential inequality which can be solved using the integrating factor $f$ to find

$$\frac{d}{dt}\left(f\int_{\mathbb{R}^n}E^2\,d\xi\right) \le Cf'\left(\frac{f'}{f}\right)^{n/2}\left[1+\int_0^t \|u(s)\|_2^2\,ds\right]$$

Choose $f = (1+t)^{n/2+1}$ so that $f'/f = (n/2+1)/(1+t)$and integrate in time from 0 to $r$.

$$(1+r)^{n/2+1}\int_{\mathbb{R}^n}E^2(\xi,r)\,d\xi \le \int_{\mathbb{R}^n}E^2(\xi,0)\,d\xi + C\int_0^r(1+\int_0^t \|u(s)\|_2^2\,ds)\,dt$$

Since $\|u\|_2^2 \le \int_{\mathbb{R}^n}E^2\,d\xi$, after using the Tonelli theorem we can bound the integral on the RHS as

$$\begin{aligned}\int_0^r(1+\int_0^t \|u(s)\|_2^2\,ds)\,dt &\le \int_0^r(1+\int_0^r \|u(s)\|_2^2\,ds)\,dt\\ &\le r(1+\int_0^r\int_{\mathbb{R}^n}E^2(\xi,s)\,d\xi\,ds)\end{aligned}$$

which leaves

$$\begin{aligned}(1+r)^{n/2+1}\int_{\mathbb{R}^n}E^2(\xi,r)\,d\xi &\le \int_{\mathbb{R}^n}E^2(\xi,0)\,d\xi + Cr(1+\int_0^r \|u\|_2^2\,ds)\\ &\le C(1+r) + Cr\int_0^r\int_{\mathbb{R}^n}E^2(\xi,s)\,d\xi\,ds\end{aligned}$$

The Gronwall inequality now shows

$$(1+r)^{n/2+1}\int_{\mathbb{R}^n}E^2(\xi,r)\,d\xi \le C(1+r)\exp(Cr\int_0^r(1+s)^{-n/2-1}\,ds)$$

Note that for $n \geq 2$,

$$r \int_0^r (1+s)^{-n/2-1}\, ds \leq C$$

Applying the Plancherel Theorem one more time finishes the proof. $\square$

Next we work out of order and establish the decay rate for the homogeneous $H^1$ norm of $v$ using a similar argument as the previous theorem.

**Theorem 6.3.** *Let $v$ be the solution of the VCHE (1.1) constructed in Theorem 4.3 with $\Omega = \mathbb{R}^n$ corresponding to $v_0 \in H^1_\sigma \cap L^1(\mathbb{R}^n)$. The solution satisfies the decay rate*

$$\|\nabla v\|_2^2 \leq C(t+1)^{-1-n/2}$$

*Proof.* Multiply the VCHE by $\triangle v$, use the relation (6.1), integrate by parts then use the Hölder inequality to obtain

$$\frac{1}{2}\frac{d}{dt}\|\nabla v\|_2^2 + \nu\|\triangle v\|_2^2 \leq C\|u\|_n \|\nabla v\|_{\frac{2n}{n-2}} \|\triangle v\|_2$$

After using the Sobolev inequality, Corollary 4.4, and the previous theorem, this becomes

$$\frac{1}{2}\frac{d}{dt}\|\nabla v\|_2^2 + \nu\|\triangle v\|_2^2 \leq C(1+t)^{-n/2}\|\triangle v\|_2^2$$

We will now restrict ourselves to $t$ large enough so that $C(1+t)^{-1} < \nu/2$, which leaves

$$\frac{d}{dt}\|\nabla v\|_2^2 + \nu\|\triangle v\|_2^2 \leq 0$$

We now apply the Fourier Splitting method as in the previous theorem. Let $B(\rho)$ be the ball of radius $\rho$ where $\rho^2 = f'/(\nu f)$ and $f$ is a positive increasing function to be specified later, then, using the Plancherel theorem,

$$\frac{d}{dt}\|\xi\hat{v}\|_2^2 + \nu\rho^2\|\xi\hat{v}\|_2^2 \leq \nu\rho^4 \int_{B(\rho)} |\hat{v}|^2\, \xi$$

Lemma 6.1 with Theorem 6.2 imply

$$|\hat{v}|^2 \leq C\left[1 + \left(\int_0^t (1+s)^{-n/2}\, ds\right)\left(\int_0^t \|\nabla v\|_2^2\, ds\right)\right]$$

This bound with the previous line leaves

$$\frac{d}{dt}\|\xi\hat{v}\|_2^2 + \nu\rho^2\|\xi\hat{v}\|_2^2 \leq C\nu\rho^{4+n}\left[1 + \left(\int_0^t (1+s)^{-n/2}\, ds\right)\left(\int_0^t \|\nabla v\|_2^2\, ds\right)\right]$$

Set $f = (1+t)^{n/2+2}$ and use it as an integrating factor

$$\frac{d}{dt}\left((1+t)^{n/2+2}\|\xi\hat{v}\|_2^2\right) \leq C\left[1 + \left(\int_0^t (1+s)^{-n/2}\, ds\right)\left(\int_0^t \|\nabla v\|_2^2\, ds\right)\right]$$

Again, as in the previous theorem, integrate in time from 0 to $r$, then use the Tonelli theorem and the Plancherel theorem

$$(1+r)^{n/2+2}\|\nabla v\|_2^2 \leq C(1+r)\left(\int_o^r \left(\int_0^t (1+s)^{-n/2}\, ds\right) dt\right)\int_0^r \|\nabla v(s)\|_2^2\, ds$$

The Gronwall inequality now shows

$$(1+r)^{n/2+2}\|\nabla v\|_2^2 \leq C(1+r)e^A$$

Where

$$A = \left[ \left( \int_o^r \left( \int_0^t (1+s)^{-n/2} \, ds \right) dt \right) \left( \int_o^r (1+t)^{-n/2-2} \, dt \right) \right]$$

Note

$$A = \left( \int_o^r \left( \int_0^t (1+s)^{-n/2} \, ds \right) dt \right) \left( \int_o^r (1+t)^{-n/2-2} \, dt \right) \le C$$

hence,

$$\|\nabla v(r)\|_2^2 \le C(1+r)^{-n/2-1}$$

□

*Corollary* 6.4. Let $v$ be the solution of the VCHE (1.1) constructed in Theorem 4.3 with $\Omega = \mathbb{R}^n$, corresponding to $v_0 \in L^2_\sigma \cap L^1(\mathbb{R}^n)$. Then,

$$|\mathcal{F}(v)| \le C$$
$$|\mathcal{F}(u)| \le C$$

*Proof.* Lemma 6.1, Theorem 6.2, and 6.3. □

**Theorem 6.5.** *Let $v$ be the solution of the VCHE (1.1) constructed in Theorem 4.3 with $\Omega = \mathbb{R}^n$, corresponding to $v_0 \in L^2_\sigma \cap L^1(\mathbb{R}^n)$. Then*

$$\|v\|_2^2 \le C(t+1)^{-n/2}$$

*Proof.* In Theorem 6.2 we have shown that

$$\|u\|_2^2 + \alpha^2 \|\nabla u\|_2^2 \le C(t+1)^{-n/2} \tag{6.4}$$

Differentiating the Helmholtz equation, then squaring and integrating it shows

$$\|\nabla u\|_2^2 + 2\alpha^2 \|\nabla^2 u\|_2^2 + \alpha^4 \|\nabla^3 u\|_2^2 = \|\nabla v\|_2^2$$

Combining this with Theorem 6.3 leaves

$$\|\nabla^2 u\|_2^2 \le C(t+1)^{-n/2-1}$$

With (6.4) we see

$$\|v\|_2^2 = \|u\|_2^2 + 2\alpha^2 \|\nabla u\|_2^2 + \alpha^4 \|\triangle u\|_2^2 \le C(t+1)^{-n/2}$$

□

We now turn our attention to a more general situation involving the Fourier Splitting Method. This next theorem will be used in the remaining decay proofs.

**Theorem 6.6.** *Let $\|\nabla^m w(0)\|_2 < \infty$. Given an energy inequality of the form*

$$\frac{1}{2}\frac{d}{dt}\|\nabla^m w\|_2^2 + \nu \|\nabla^{m+1} w\|_2^2 \le C(1+t)^\gamma$$

*and the bound*

$$|\hat{w}(\xi, t)| \le C(1+t)^\beta$$

*which holds for $|\xi|^2 < \frac{d}{\nu(1+t)}$, we can deduce the asymptotic behavior*

$$\|\nabla^m w\|_2^2 \le C \left[ (1+t)^{-m-n/2+2\beta} + (1+t)^{\gamma+1} \right]$$

*Proof.* We proceed directly with Fourier Splitting. Apply Plancherel's Theorem and break up the integral on the LHS.

$$\frac{1}{2}\frac{d}{dt}\|\xi^k \hat{w}\|_2^2 + \nu\rho^2\|\xi^k\hat{w}\|_2^2 \le \nu\rho^{2k+2}\int_{B(\rho)} \hat{w}^2\, d\xi + C(1+t)^\gamma$$

Choose, for some large $d$,

$$\rho^2 = \frac{d}{\nu(1+t)}$$

Then, using the assumption for the bound on $\hat{w}$ and performing the integration on the RHS we have

$$\frac{d}{dt}((1+t)^d\|\xi^m\hat{w}\|_2^2) \le C\left[(1+t)^{-m-1+d+2\beta-n/2} + (1+t)^{\gamma+d}\right]$$

Integration in time and another application of the Plancherel Theorem now finishes the proof. $\square$

As a first application of the above theorem, we will compute the decay rate for all spacial derivatives for solutions of the VCHE.

**Theorem 6.7.** *Let $v$ be the solution of the VCHE (1.1) constructed in Theorem 4.3 with $\Omega = \mathbb{R}^n$, corresponding to $v_0 \in H_\sigma^K \cap L^1(\mathbb{R}^n)$. These solutions satisfy the following decay rate for all $m \le K$*

$$\|\nabla^m v\|_2^2 \le C(t+1)^{-m-n/2}$$

*Proof.* The cases $m = 0, 1$ are Theorems 6.5 and 6.3 respectively. To prove the remaining cases, we first find an inequality in a form suitable for Theorem 6.6, then using inductive arguments establish decay. Having previously established regularity of solutions, we proceed formally. Let $M \le K$ and multiply the VCHE (1.1) by $\triangle^M v$ and integrate by parts to find

$$\frac{d}{dt}\|\nabla^M v\|_2^2 + \nu\|\nabla^{M+1}v\|_2^2 \le I_{M,0} + J_{M,0}$$

where

$$I_{M,0} = \sum_{m=0}^{M}\binom{M}{m} < \nabla^m u\cdot\nabla^{M+1}v, \nabla^{M-m}v >$$

$$J_{M,0} = \sum_{m=0}^{M-1}\binom{M-1}{m} < \nabla^{M+1}v\cdot\nabla^{m+1}u, \nabla^{M-m}v >$$

Using the Holder inequality, the Sobolev inequality, Corollary 4.4, and the Cauchy inequality we find

$$\begin{aligned} I_{M,0} &= C\sum_{m=0}^{M}\|\nabla^m u\|_n\|\nabla^{M-m}v\|_{\frac{2n}{n-2}}\|\nabla^{M+1}v\|_2 \\ &\le C\|v\|_2^2\|\nabla^{M+1}v\|_2^2 + C\|\nabla v\|_2^2\|\nabla^M v\|_2^2 \\ &+ C\sum_{m=2}^{M}\|\nabla^{m-1}v\|_2^2\|\nabla^{M+1-m}v\|_2^2 + \frac{\nu}{4}\|\nabla^{M+1}v\|_2^2 \end{aligned}$$

We treat the other term in a similar way.

$$\begin{aligned} J_{M,0} &\leq C \sum_{m=0}^{M-1} \|\nabla^{M+1} v\|_2 \|\nabla^{m+1} u\|_n \|\nabla^{M-m} v\|_{\frac{2n}{n-2}} \\ &\leq C\|v\|_2^2 \|\nabla^{M+1} v\|_2^2 + C\|\nabla v\|_2^2 \|\nabla^M v\|_2^2 \\ &+ C \sum_{m=2}^{M-1} \|\nabla^{m-1} v\|_2^2 \|\nabla^{M+1-m} v\|_2^2 + \frac{\nu}{4}\|\nabla^{M+1} v\|_2^2 \end{aligned}$$

Together, this leaves

$$\begin{aligned} \frac{d}{dt}\|\nabla^M v\|_2^2 + \frac{\nu}{2}\|\nabla^{M+1} v\|_2^2 \leq C\|v\|_2^2 \|\nabla^{M+1} v\|_2^2 & \\ + C\|\nabla v\|_2^2 \|\nabla^M v\|_2^2 & \\ + C \sum_{m=2}^{M} \|\nabla^{m-1} v\|_2^2 \|\nabla^{M+1-m} v\|_2^2 & \end{aligned} \tag{6.5}$$

The remaining part of this proof will proceed by induction where the base case is Theorem 6.5 and 6.3. We assume that the decay

$$\|\nabla^m v\|_2^2 \leq C(t+1)^{-m-n/2}$$

holds for all $m < M$ and will show that it holds for $m = M$. Apply the inductive assumption to (6.5) to find

$$\begin{aligned} \frac{d}{dt}\|\nabla^M v\|_2^2 + \frac{\nu}{4}\|\nabla^{M+1} v\|_2^2 &\leq C(1+t)^{-n/2}\|\nabla^{M+1} v\|_2^2 \\ &+ C(1+t)^{-1-n/2}\|\nabla^M v\|_2^2 + C(1+t)^{-M-n/2} \end{aligned} \tag{6.6}$$

For $t$ large enough, we have $C(1+t)^{-n/2} \leq \nu/4$. Then, subtracting the first term on the RHS, (6.6) becomes

$$\begin{aligned} \frac{d}{dt}\|\nabla^M v\|_2^2 + \frac{\nu}{4}\|\nabla^{M+1} v\|_2^2 &\leq +C(1+t)^{-1-n/2}\|\nabla^M v\|_2^2 \\ &+ C(1+t)^{-M-n/2} \end{aligned}$$

The next step is to apply the bound $\|\nabla^M v\|_2^2 \leq C$ (Theorem 4.3) with Theorem 6.6 to obtain the decay rate

$$\|\nabla^M v\|_2^2 \leq C(1+t)^{-n/2}$$

Continuing with a bootstrap argument, placing this new bound into (6.6) and again using Theorem 6.6 the optimal decay rate is obtained and the proof is complete. □

The next goal is to extend the decay results to time derivatives of the solution. To begin, we will compute a frequency bound for the spacial derivatives of solutions to the VCHE. This next lemma will be used inductively with Theorem 6.6 to compute decay rates for the $L^2$ norm of all time derivatives.

**Lemma 6.8.** *Let $v$ be the solution of the VCHE (1.1) constructed in Theorem 4.3 with $\Omega = \mathbb{R}^n$, corresponding to $v_0 \in H^1_\sigma \cap L^1(\mathbb{R}^n)$. For $P \geq 1$, if*

$$\|\partial_t^p \nabla^m v\|_2^2 \leq C(1+t)^{-2p-m-n/2}$$

*for all $p < P$ and $m = 0, 1$, then*

$$|\partial_t^P \hat{v}(\xi)| \leq C(1+t)^{-P}$$

*for* $|\xi|^2 \leq \frac{d}{\nu(1+t)}$.

*Proof.*

$$\mathcal{F}(v) = e^{-t|\xi|^2}\mathcal{F}(v_0) + \int_0^t e^{-(t-s)|\xi|^2}\Psi(\xi,s)\,ds$$

where

$$\Psi(\xi,t) = \xi\mathcal{F}(p) + \sum_j \xi_j\mathcal{F}(u_jv) + \sum_j \mathcal{F}(v_j\nabla u_j)$$

Using the chain rule

$$\frac{d}{dt}\int_0^t f(t,s)\,ds = f(t,t) + \int_0^t \frac{\partial f(t,s)}{\partial t}\,ds$$

shows

$$\partial_t^P\mathcal{F}(v) = (-1)^P|\xi|^{2P}e^{-t|\xi|^2}\mathcal{F}(v_0) + \sum_{p=0}^{P-1}(-|\xi|^2)^{P-1-p}\partial_t^p\Psi(\xi,t)$$
$$+ \int_0^t (-|\xi|^2)^P e^{-(t-s)|\xi|^2}\Psi(\xi,s)\,ds$$

We bound $\Psi$ similar to Lemma 6.1, but using the assumption.

$$\partial_t^p\Psi(\xi,t) = \partial_t^p A + \partial_t^p B + \partial_t^p C$$

$$\begin{aligned}
|\partial_t^p A| &= |\partial_t^p \sum_j \xi_j\mathcal{F}(u_jv)| \\
&\leq \sum_{l=0}^p C|\xi|\|\partial_t^l v\|_2\|\partial_t^{p-l}v\|_2 \\
&\leq C(1+t)^{-p-n/2-1/2}
\end{aligned}$$

$$\begin{aligned}
|\partial_t^p B| &= |\partial_t^p \sum_j \mathcal{F}(v_j\nabla u_j^T)| \\
&\leq \sum_{l=0}^p C\|\partial_t^l v\|_2\|\partial_t^{p-l}\nabla v\|_2 \\
&\leq C(1+t)^{-p-n/2-1/2}
\end{aligned}$$

$$\begin{aligned}
|\partial_t^p C| &= |\partial_t^p \xi\mathcal{F}(\pi)| \\
&\leq |\partial_t^p A| + |\partial_t^p B| \\
&\leq C(1+t)^{-p-n/2-1/2}
\end{aligned}$$

Using the bound $|\hat{v}| \leq C$ (Corollary 6.4) and $|\xi| < \frac{d}{\sqrt{\nu(1+t)}}$ we have

$$|\partial_t^P\hat{v}| \leq C(1+t)^{-P} \tag{6.7}$$

$\square$

Note that the conclusion for $P = 0$ is true by Corollary 6.4.

**Theorem 6.9.** *Let $v$ be the solution of the VCHE (1.1) constructed in Theorem 4.3 with $\Omega = \mathbb{R}^n$, corresponding to $v_0 \in H^K_\sigma \cap L^1(\mathbb{R}^n)$. These solutions satisfy the following decay rate for all $m + 2p \le K$*

$$\|\partial_t^p \nabla^m v\|_2^2 \le C(t+1)^{-2p-m-n/2}$$

*Proof.* This proof follows closely the proof of Theorem 6.7, we first find an inequality in a form suitable for Theorem 6.6, then using inductive arguments we establish decay. Choose $P$ and $M$ such that $2P + M \le K$, then apply $\partial_t^P$ to the VCHE (1.1), multiply by $\partial_t^P \triangle^M v$ and integrate by parts to find

$$\frac{d}{dt}\|\partial_t^P \nabla^M v\|_2^2 + \nu\|\partial_t^P \nabla^{M+1} v\|_2^2 \le I_{M,P} + J_{M,P}$$

where

$$I_{M,P} = \sum_{p=0}^{P}\sum_{m=0}^{M}\binom{P}{p}\binom{M}{m} < \partial_t^p \nabla^m u \cdot \partial_t^P \nabla^{M+1} v, \nabla^{M-m}\partial_t^{P-p} v >$$

$$J_{M,P} = \sum_{p=0}^{P}\sum_{m=0}^{M-1}\binom{P}{p}\binom{M-1}{m} < \partial_t^P \nabla^{M+1} v \cdot \nabla \partial_t^p \nabla^m u, \partial_t^{P-p}\nabla^{M-m} v >$$

or, in the case $M = 0$,

$$J_{0,P} = \sum_{p=0}^{P}\binom{P}{p} < \partial_t^P v \cdot \nabla \partial_t^p u, \partial_t^{P-p} v >$$

As before, use the Holder inequality, the Sobolev inequality, Corollary 4.4, and the Cauchy inequality we find for $M > 0$

$$\begin{aligned} I_{M,P} &= C\sum_{p=0}^{P}\sum_{m=0}^{M}\|\partial_t^p \nabla^m u\|_n \|\partial_t^{P-p}\nabla^{M-m} v\|_{\frac{2n}{n-2}}\|\partial_t^P \nabla^{M+1} v\|_2 \\ &\le C\sum_{p=0}^{P}\|\partial_t^p v\|_2^2 \|\partial_t^{P-p}\nabla^{M+1} v\|_2^2 + C\sum_{p=0}^{P}\|\partial_t^p \nabla v\|_2^2\|\partial_t^{P-p}\nabla^M v\|_2^2 \\ &+ C\sum_{p=0}^{P}\sum_{m=2}^{M}\|\partial_t^p \nabla^{m-1} v\|_2^2\|\partial_t^{P-p}\nabla^{M+1-m} v\|_2^2 + \frac{\nu}{4}\|\partial_t^P \nabla^{M+1} v\|_2^2 \end{aligned}$$

Similarly for the second term if $M > 0$,

$$\begin{aligned} J_{M,P} &\le C\sum_{p=0}^{P}\sum_{m=0}^{M-1}\|\partial_t^P \nabla^{M+1} v\|_2 \|\partial_t^p \nabla^{m+1} u\|_n \|\partial_t^{P-p}\nabla^{M-m} v\|_{\frac{2n}{n-2}} \\ &\le C\sum_{p=0}^{P}\|\partial_t^p v\|_2^2 \|\partial_t^{P-p}\nabla^{M+1} v\|_2^2 + C\sum_{p=0}^{P}\|\partial_t^p \nabla v\|_2^2\|\partial_t^{P-p}\nabla^M v\|_2^2 \\ &+ C\sum_{p=0}^{P}\sum_{m=2}^{M-1}\|\partial_t^p \nabla^{m-1} v\|_2^2\|\partial_t^{P-p}\nabla^{M+1-m} v\|_2^2 + \frac{\nu}{4}\|\partial_t^P \nabla^{M+1} v\|_2^2 \end{aligned}$$

In the case $M = 0$ these estimates become

$$I_{0,P} + J_{0,P} \le C\sum_{p=0}^{P}\|\partial_t^p v\|_2^2\|\partial_t^{P-p}\nabla v\|_2^2 + \frac{\nu}{4}\|\partial_t^P \nabla v\|_2^2$$

Combining these two estimates we have shown in the case $M > 0$

(6.8)
$$\begin{aligned}\frac{d}{dt}\|\partial_t^P \nabla^M v\|_2^2 + \frac{\nu}{2}\|\partial_t^P \nabla^{M+1} v\|_2^2 &\leq C\sum_{p=0}^{P} \|\partial_t^p v\|_2^2 \|\partial_t^{P-p}\nabla^{M+1} v\|_2^2 \\ &+ C\sum_{p=0}^{P} \|\partial_t^p \nabla v\|_2^2 \|\partial_t^{P-p}\nabla^{M} v\|_2^2 \\ &+ C\sum_{p=0}^{P}\sum_{m=2}^{M} \|\partial_t^p \nabla^{m-1} v\|_2^2 \|\partial_t^{P-p}\nabla^{M+1-m} v\|_2^2\end{aligned}$$

and in the case $M = 0$ that

$$\frac{d}{dt}\|\partial_t^P v\|_2^2 + \frac{\nu}{2}\|\partial_t^P \nabla v\|_2^2 \leq C\sum_{p=0}^{P} \|\partial_t^p v\|_2^2 \|\partial_t^{P-p}\nabla v\|_2^2 \tag{6.9}$$

We now begin the inductive part of our argument where the base case is Theorem 6.7. Pick $P \leq K/2$ and assume the decay

$$\|\partial_t^p \nabla^m v\|_2^2 \leq C(t+1)^{-2p-m-n/2} \tag{6.10}$$

holds for all $p < P$ and $m$ such that $2p + m \leq K$. We will show that the decay holds for $p = P$ with $m$ such that $2P + m \leq K$. To prove the inductive claim, first it will be shown the the decay rate holds for $p = P$ and $m = 0$ using (6.9). Then, using (6.8) it will be shown that the decay rate holds for the remaining values of $m$ using another inductive argument.

To establish the decay for $p = P$ and $m = 0$, apply the inductive assumption to (6.9) to find

$$\frac{d}{dt}\|\partial_t^P v\|_2^2 + \frac{\nu}{2}\|\partial_t^P \nabla v\|_2^2 \leq C(1+t)^{-n/2}\|\partial_t^P \nabla v\|_2^2 + C(1+t)^{-2P-1-n}$$

Again, take $t$ large enough so that $C(1+t)^{-n/2} \leq \nu/4$ and move the first term on the RHS to the left side

$$\frac{d}{dt}\|\partial_t^P v\|_2^2 + \frac{\nu}{2}\|\partial_t^P \nabla v\|_2^2 \leq C(1+t)^{-2P-1-n}$$

Now, an application of Theorem 6.6 with Lemma 6.8 establishes the decay (6.10) for $p = P$ and $m = 0$. This is the base case for the next inductive argument. Assume the decay (6.10) holds for $m \leq M + 1$ when $p < P$, and $m < M$ when $p = P$, we will show that this implies the decay holds for $m = M$ and $p = P$. Proving this inductive claim will finish the proof. Begin by applying the inductive assumption to (6.8) to see

$$\begin{aligned}\frac{d}{dt}\|\partial_t^P \nabla^M v\|_2^2 + \frac{\nu}{2}\|\partial_t^P \nabla^{M+1} v\|_2^2 &\leq C(1+t)^{-n/2}\|\partial_t^P \nabla^{M+1} v\|_2^2 \\ &+ C(1+t)^{-n/2-1}\|\partial_t^P \nabla^{M} v\|_2^2 \\ &+ C(1+t)^{-2p-M-n}\end{aligned}$$

Take $t$ large so that $C(1+t)^{-n/2} \leq \nu/4$ and move the first term on the RHS to the LHS. Then apply Theorem 6.6 with Lemma 6.8 to establish the decay rate

$$\|\partial_t^p \nabla^m v\|_2^2 \leq C(t+1)^{-n/2} \tag{6.11}$$

Another bootstrap argument gives the optimal decay and finishes the proof. □

## 7. Convergence of the VCHE to the NSE in the Whole Space

A solution $u$ of the Helmholtz equation corresponding to $v$ will approach $v$ weakly as the filter parameter tends to zero. Indeed, fix $v \in L^p(\mathbb{R}^n)$ and let $\alpha$ be a sequence tending to zero. By the above theorem, for each $\alpha$ there is a weak solution $u_\alpha \in W^{1,p}(\mathbb{R}^n)$ of the Helmholtz equation such that

$$\int_{\mathbb{R}^n} u_\alpha \cdot \phi \, dx + \alpha^2 \int_{\mathbb{R}^n} \nabla u_\alpha \cdot \nabla \phi \, dx = \int_{\mathbb{R}^n} v \cdot \phi \, dx$$

The functions $u_\alpha$ are bounded in $L^p(\mathbb{R}^n)$ independent of $\alpha$, so there exists a subsequence $\alpha_i$ and a weak limit in $L^p(\mathbb{R}^n)$. Also, for $1/p + 1/q = 1$

$$\alpha^2 \int_{\mathbb{R}^n} \nabla u \cdot \nabla \phi \, dx \le \alpha^2 \|\nabla u\|_p \|\nabla \phi\|_q \le C(n) \alpha^{1/2} \|v\|_p \|\nabla \phi\|_q$$

which approaches zero as $\alpha \to 0$. This proves that $u_\alpha \rightharpoonup v$ in $L^p(\mathbb{R}^n)$. We can actually do better then this if $v$ is sufficiently differentiable.

To begin we state a theorem concerning the Helmholtz equation in all of space, the theorem is standard elliptic theory and no proof is given. This theorem can be proved using elliptic estimates and interpolation or if one multiplies the Helmholtz equation by $e^{\tau/\alpha^2}$ and divides by $\alpha^2$ it can be thought of as the heat equation and the bounds follow from estimates on the heat kernel.

**Theorem 7.1.** *Given $v \in L^p(\mathbb{R}^n)$, $p \in (1, \infty)$, there exists a $u \in W^{1,p}(\mathbb{R}^n)$ that is a weak solution to the Helmholtz equation $u - \alpha^2 \triangle u = v$. Moreover, this function satisfies*

$$\begin{aligned}
\|u\|_p &\le \|v\|_p \\
\|u\|_q &\le \frac{C(n,p,q)}{\alpha^{1+\gamma}} \|v\|_p \; for \; \gamma = \frac{n}{2}(\frac{1}{p} - \frac{1}{q}) < 1 \\
\|\nabla u\|_q &\le \frac{C(n,p,q)}{\alpha^{3/2+\gamma}} \|v\|_p \; for \; \gamma = \frac{n}{2}(\frac{1}{p} - \frac{1}{q}) < \frac{1}{2}
\end{aligned}$$

*If $n(2/p - 1) < 1$ then the solution is unique.*

**Theorem 7.2.** *Let $v \in W^{1,p}(\mathbb{R}^n)$ and let $u$ be the corresponding solution to the Helmholtz equation $u - \alpha^2 \triangle u = v$. Then*

$$\|u - v\|_q \le C(n,p,q) \alpha^{1/2-\gamma} \|\nabla v\|_p \; for \; \gamma = \frac{n}{2}(\frac{1}{p} - \frac{1}{q}) < \frac{1}{2}$$

*If $\alpha$ is a sequence tending to zero and $u_\alpha$ are solutions the Helmholtz equation, then $u_\alpha \to v$ strongly in $L^q(\mathbb{R}^n)$ for $1/p - 1/q < 1/n$.*

*Proof.* If $u$ and $v$ satisfy the Helmholtz equation, then

$$\|u - v\|_q \le \alpha^2 \|\triangle u\|_q \tag{7.1}$$

The Helmholtz equation is linear, so the derivatives of the functions obey the relation

$$\nabla u - \alpha^2 \triangle \nabla u = \nabla v$$

Applying Theorem 7.1 to this PDE with the restriction on $\gamma$ allows the bound

$$\|\triangle u\|_q \le \frac{C(n,p,q)}{\alpha^{3/2+\gamma}} \|\nabla v\|_p$$

Together with (7.1), this shows

$$\|u - v\|_q \leq C(n, p, q)\alpha^{1/2-\gamma}\|\nabla v\|_p$$

The second statement is an immediate consequence of this and the fact that $u_\alpha \rightharpoonup v$ in $L^p(\mathbb{R}^n)$. ☐

In [9], [10], the authors show how the solutions of the VCHE approach a solution of the NSE weakly when the parameter in the filter tends to zero ($\alpha \to 0$). We will show how solutions to the VCHE approach solutions to the NSE strongly as $\alpha \to 0$ when the solution to the NSE is known to be regular. The proof requires estimates on the solution of the VCHE which are independent of $\alpha$, but in regions of time where the NSE is known to be regular by some functional analytic arguments, the passive bound on the filter make this assumption reasonable.

For example, solutions of the Navier-Stokes equation obey the Prodi Inequality [19]

$$\frac{d}{dt}\|\nabla u\|_2^2 \leq C\|\nabla u\|_2^{2n}$$

This can be used to prove existence of a strong solution in some time interval $[0, T]$. The Prodi inequality is proved through energy estimates, using the passive bound for the filter in Theorem 7.1 and following the same energy arguments allows the same bound for solutions of the VCHE. This bound will be independent of $\alpha$, so we can apply the following theorem to conclude that in some closed interval $[0, T]$ the solution of the VCHE approaches a solution to the NSE strongly.

**Theorem 7.3.** *Let $v$ be the solution of the VCHE (1.1) constructed in Theorem 4.3 with $\Omega = \mathbb{R}^n$, corresponding to $w_0 \in H^1_\sigma(\mathbb{R}^n)$. If Let $w$ be the solution the NSE with initial conditions $w_0$. In any time interval $[0, T]$ where a solution to the NSE is known to be regular, if there exists a bound*

$$\sup_{t\in[0,T]} (\|v_\alpha\|_l + \|\nabla v_\alpha\|_l) < C$$

*which is independent of $\alpha$, then $v_\alpha$ approaches $w$ strongly in $L^\infty([0, T], L^q(\mathbb{R}^n))$ as $\alpha \to 0$, where $q = \frac{2l}{l-2}$.*

*Proof.* We begin with a mild form of the solutions to both problems. Since we are working in a time domain with known regularity, these are the unique solutions. Let $w$ be the solution to the NSE and $v$ the solution the VCHE with initial conditions $w_0$. If $\mathbb{P}$ is the Leray projector onto the divergence free subspace of $L^2$ and $\Phi$ is the heat kernel, then

$$w(t) = \Phi(t) * w_0 - \int_0^t \Phi(t-s) * \mathbb{P}\left[w \cdot \nabla w\right](s)\, ds$$
$$v(t) = \Phi(t) * w_0 - \int_0^t \Phi(t-s) * \mathbb{P}\left[u \cdot \nabla v + \sum u_j \nabla v_j\right](s)\, ds$$

By adding and subtracting cross terms we see

$$\begin{aligned} w(t) - v(t) = -\int_0^t \Phi(t-s) * \mathbb{P}\left[(w-u)\cdot\nabla w + u\cdot\nabla(w-v)\right] \\ + \mathbb{P}\left[u_j\nabla(v_j - w_j) + (u_j - w_j)\nabla w_j\right](s)\, ds \end{aligned}$$

We bound the first term in the integrand using Young's inequality and the definition of the projector

$$\begin{aligned}\|\Phi(t-s)*\mathbb{P}\left[(w-u)\cdot\nabla w\right](s)\|_q &\le \|\Phi(t-s)\|_p\|(w-u)\cdot\nabla w\|_2\\ &\le \|\Phi(t-s)\|_p\|w-u\|_q\|\nabla w\|_l\end{aligned}$$

where $1/q+1=1/p+1/2$ and $1/2=1/q+1/l$. Then using Theorem 7.2 with $\gamma=(1/2-1/q)n/2<1/2$ we obtain

$$\|\Phi(t-s)*\mathbb{P}\left[(w-u)\cdot\nabla w\right](s)\|_q \le \|\Phi(t-s)\|_p\|\nabla w\|_l\left(\|w-v\|_q + C\alpha^{1/2-\gamma}\|\nabla v\|_2\right)$$

The fourth term can be bounded in essentially the same way. We approach the second term in a slightly different way, first by passing the derivative to the heat kernel. These functions are smooth functions of the whole space so the projector will commute with the derivative.

$$\|\Phi(t-s)*\mathbb{P}\left[u\cdot\nabla(w-v)\right]\|_q = \|\nabla\Phi(t-s)*\mathbb{P}\left[u\cdot(w-v)\right]\|_q$$

Then by using Young's inequality and the definition of the projector

$$\|\Phi(t-s)*\mathbb{P}\left[u\cdot\nabla(w-v)\right]\|_q \le \|\nabla\Phi(t-s)\|_p\|u\|_l\|w-v\|_q$$

To bound the third term, start with the product rule and again pass the derivative the heat kernel

$$\begin{aligned}\|\Phi(t-s)*\mathbb{P}\left[u_j\nabla(v_j-w_j)\right]\|_q &= \|\nabla\Phi(t-s)*\mathbb{P}\left[u_j(w_j-v_j)\right]\|_q\\ &\quad + \|\Phi(t-s)*\mathbb{P}\left[\nabla u_j(w_j-v_j)\right]\|_q\end{aligned}$$

Then, using Young's inequality and again the definition of the projector

$$\begin{aligned}&\|\Phi(t-s)*\mathbb{P}\left[u_j\nabla(v_j-w_j)\right]\|_q\\ &\qquad\le \left(\|\nabla\Phi(t-s)\|_p\|u_j\|_l + \|\Phi(t-s)\|_p\|\nabla u_j\|_l\right)\|w_j-v_j\|_q\end{aligned}$$

Putting all of these bounds together and estimating the heat kernel yields

$$\begin{aligned}\|v-w\|_q &\le A\alpha^{1/2-\gamma} + B\int_0^t \frac{1}{(t-s)^\delta}\|v-w\|_q(s)\,ds\\ A &= C\int_0^t \|\nabla v\|_2\,ds\\ B &= C\sup_{s\in[0,T]}\left(\|\nabla w\|_l + \|u\|_l + \|\nabla u\|_l\right)\end{aligned}$$

Here, $\delta=1/2+(1-1/p)n/2<1$ by the assumption $l>n$. Application of the Gronwall inequality finishes the proof. For example, the modified Gronwall inequality by M. E. Schonbek and T. Schonbek [25] now shows

$$\begin{aligned}\|v-w\|_q &\le A\alpha^{1/2-\gamma}\Upsilon(B\Gamma(1-\delta)t^\delta)\\ \Upsilon(z) &= \sum_{n=0}^{\infty}\frac{z}{\Gamma(n(1-\delta)+1)}\end{aligned}$$

See also [1]. Letting $\alpha\to 0$ we see that $v\to w$ strongly in $L^q(\mathbb{R}^3)$. $\square$

## 8. Appendix

Here we construct a weak solution to the VCHE. Due to the close relation between the VCHE and the Navier-Stokes equation, our proof is similar to known existence proofs for the NSE. See, for example, [1], [2], [7], [13], [15],[27]. First, we construct solutions on any bounded $\Omega$ with smooth boundary using the Galerkin method, this is where the Stokes operator is known to be compact thanks to the Poincaré Inequality. Special care is taken to use only inequalities which do not depend on the size of $\Omega$ so that we can use these solutions to prove existence of a weak solution in unbounded domains. The only step that requires $\Omega$ bounded is in the compact inclusion used to obtain the strong convergence necessary to pass limits through the non-linear term. This problem is overcome by working in the compact support of the test functions.

To begin we note a standard but useful elliptic estimate.

*Remark* 8.1. Let $2 \leq n \leq 4$ and $\Omega \subset \mathbb{R}^n$ be an open set with smooth boundary. If $u \in H^2_\sigma$, and $v \in L^2_\sigma$ satisfy the Helmholtz equation

$$u - \alpha^2 \triangle u = v$$

on $\Omega$, then

$$\begin{aligned} \|u\|_n &\leq C\|v\|_2 \\ \|\nabla u\|_n &\leq C\|v\|_2 \end{aligned}$$

where the constants $C$ depend only on $\alpha$, $\Omega$, and $n$.

The stationary Stokes equation

$$\begin{aligned} \triangle u + \nabla p &= v \\ u|_{\partial\Omega} &= 0 \end{aligned}$$

is known to have a solution $u \in H^1_\sigma(\Omega)$ for each $v \in L^2_\sigma(\Omega)$ when $\Omega$ is an open bounded set. Let $\Lambda : L^2_\sigma(\Omega) \to H^1_\sigma(\Omega)$ be the operator defined by solving this PDE. Composing this with the compact inclusion $H^1_\sigma(\Omega) \to L^2_\sigma(\Omega)$ gives a compact and self-adjoint operator. Owing to spectral theory, it has a countable number of positive eigenvalues $\lambda_i$, and associated smooth, divergence-free eigenfunctions $\omega_i$ which form a basis for $L^2_\sigma(\Omega)$. These functions satisfy the relation

$$-\triangle \omega_i = \lambda_i \omega_i$$

**Lemma 8.2.** *Let $\Omega \subset \mathbb{R}^n$ be an open bounded set and $\{\omega_j\}_{j=1}^\infty$ an orthonormal basis of $L^2_\sigma(\Omega)$ where each $\omega_j$ is an eigenfunction of the Stokes Operator on $\Omega$. The associated eigenvalues are all positive real numbers and the eigenvectors are smooth and approach zero on the boundary. Let $H_m = span\{\omega_1, ..., \omega_m\}$ and let $P_m$ be the orthogonal projection $P_m : L^2_\sigma(\Omega) \to H_m$. Given $v_0 \in C_0^\infty(\Omega)$, for each $m$ there is an approximate solution*

$$v_m = \sum_{j=1}^m g_{jm}(t)\omega_j$$

*and*

$$u_m = \sum_{j=1}^m \frac{g_{jm}(t)}{1 + \alpha^2 \lambda_j}\omega_j$$

*These approximate solutions are defined on some time interval* $[0, T_m]$ *and satisfy the following relations*

$$< \partial_t v_m, \omega_i > + < u_m \cdot \nabla v_m, \omega_i > - < \omega_i \cdot \nabla v_m, u_m > = \nu < \triangle v_m, \omega_i > \tag{8.1}$$
$$v_m(0) = P_m v_0$$

*Proof.* To determine the scalars $g_{im}$ we construct a system of $m$ ODE's.

$$\frac{dg_{im}}{dt} + \nu\lambda_i g_{im} + \sum_{j,k=1}^{m} \frac{g_{jm}g_{km}}{1+\alpha^2\lambda_k}(< \omega_k \cdot \nabla\omega_j, \omega_i > - < \omega_i \cdot \nabla\omega_j, \omega_k >) = 0$$

Local existence of solutions to ODE's give existence of solutions $g_{im}$, which are defined for some time interval $[0, T_m]$. □

The bounds in the next lemma will prove that $T_m$ can be bounded independent of $m$, and in fact $T_m = \infty$ for all $m$.

**Lemma 8.3.** *For* $2 \le n \le 4$*, the approximate solutions constructed in Lemma 8.2 have the following bounds, which do not depend on* $T$*,* $\Omega$ *or* $m$*.*

$$\|v_m\|_{L^\infty([0,T];L^2_\sigma(\Omega))} + \|\nabla v_m\|_{L^2([0,T];L^2_\sigma(\Omega))} \le C(n, \alpha, \nu, \|v_0\|_2)$$
$$\|\partial_t v_m\|_{L^2([0,T];(H^1_\sigma)'(\Omega))} \le C(n, \alpha, \nu, \|v_0\|_2)$$

*Proof.* Similar to formal multiplication of the VCHE (1.1) by $u$, multiply (8.1) by $\frac{1}{1+\alpha^2\lambda_i} g_{im}$, sum, then apply Lemma 3.1 to see

$$\|u_m\|_2^2 + \alpha^2\|\nabla u_m\|_2^2 + 2\nu\int_0^T \|\nabla u_m\|_2^2\, dt + 2\alpha^2\nu\int_0^T \|\triangle u_m\|_2^2\, dt = \|u_0\|_2^2 + \alpha^2\|\nabla u_0\|_2^2 \tag{8.2}$$

This is the first bound in the theorem. Using 8.1 we can deduce

$$\|u_m\|_n^2 + \int_0^\infty \|\nabla u_m\|_n^2\, dt < C(n, \alpha, \nu, \|u_0\|_2, \|\nabla u_0\|_2) \tag{8.3}$$

To bound the derivative start with (8.1). Any $\phi \in H^1_\sigma$ can be written as a sum of the $\omega_i$ so each approximate solution satisfies

$$< \partial_t v_m, \phi > + < u_m \cdot \nabla v_m, \phi > - < \phi \cdot \nabla v_m, u_m >= \nu < \triangle v_m, \phi >$$

After integration by parts and applying the Hölder inequality with the Gagliardo-Nirenberg-Sobolev inequality we find

$$| < \partial_t v_m, \phi > | \le C\|u_m\|_n\|\nabla v_m\|_2\|\nabla\phi\|_2 + C\|\nabla v_m\|_2\|\nabla\phi\|_2$$

We can now conclude

$$\|\partial_t v_m\|_{(H^1_\sigma)'} \le C(\|u_m\|_n\|\nabla v_m\|_2 + \|\nabla v_m\|_2)$$

This, together with (8.2) and (8.3), proves the second bound in the theorem. □

**Theorem 8.4.** *Let* $\Omega \in \mathbb{R}^n$*,* $2 \le n \le 4$ *be a bounded set with smooth boundary and* $v_0 \in C_0^\infty(\Omega)$*. Then, there exists a weak solution to the VCHE (1.1) in the sense of Definition (4.1).*

*Proof.* Thanks to Lemmas 8.2 and 8.3 we only need to prove the convergence of the approximate solutions. Lemma 8.3 shows how the sequence $v_m$ remains bounded, so using a possible subsequence and the Banach-Alaoglu theorem, there exists a function

$$v \in L^\infty([0,T]; L^2_\sigma(\Omega)) \cap L^2([0,T]; H^1_\sigma(\Omega))$$
$$\partial_t v \in L^2([0,T]; (H^1_\sigma)'(\Omega))$$

such that

$$v_m \rightharpoonup v \textit{ in } L^\infty([0,T]; L^2_\sigma(\Omega)) \textit{ weak}* \tag{8.4}$$

$$v_m \rightharpoonup v \textit{ in } L^2([0,T]; H^1_\sigma(\Omega)) \textit{ weakly} \tag{8.5}$$

We will now show that $v$ is a weak solution to the VCHE (1.1).

By the construction of our approximate solutions and integration by parts, we know for any basis vector $\omega_j \in L^2_\sigma(\Omega)$ and any smooth scalar function of time $\phi_j(t)$ such that $\phi_j(T) = 0$,

$$\int_0^T < v_m, \phi' \omega_j > \, ds + \int_0^T < u \cdot \nabla v, \phi \omega_j > \, ds + \int_0^T < \phi \omega_j \cdot \nabla u, v > \, ds$$
$$+ \int_0^T < \nabla v, \nabla \phi \omega_j > \, ds = < v_m(0), \phi(0)\omega_j >$$

The convergence (8.4) and (8.5) implies

$$\int_0^t < v_m, \phi_j' \omega_j > \, ds \to \int_0^t < v, \phi_j' \omega_j > \, ds$$
$$\int_0^t < \nabla v_m, \phi_j \nabla \omega_j > \, ds \to \int_0^t < \nabla v, \phi_j \nabla \omega_j > \, ds$$

Also,

$$< v_m(0), \phi_j(0)\omega_j > = < P_m(v_0), \phi_j(0)\omega_j > \to < v_0, \phi_j(0)\omega_j > \tag{8.6}$$

Passing through the non-linear terms will require strong convergence, so we use the fact that the bounds in Lemma 8.3 imply (see [7], Lemma 8.2) the existence of a possible subsequence $v_m$ such that

$$v_m \to v \textit{ in } L^2([0,T]; L^2(\Omega)) \textit{ strongly} \tag{8.7}$$

By Theorem 7.1, there is a function $u$ which satisfies

$$u - \alpha^2 \triangle u = v$$

Recall the construction of $u_m$ in Lemma 8.2 to see

$$\|u_m - u\|_2^2 + \alpha^2 \|\nabla(u_m - u)\|_2^2 + \alpha^4 \|\nabla^2(u_m - u)\|_2^2 = \|v_m - v\|_2^2$$

In particular, applying the Gagliardo-Nirenberg-Sobolev Inequality shows

$$\|u_m - u\|_n^2 \leq C \|v_m - v\|_2^2$$

This, with the strong convergence (8.7), shows how $u_m$ approaches $u$ strongly.

We can now prove the convergence of the non-linear terms

$$\int_0^T < u_m \cdot \nabla v_m, \phi_j \omega_j > \, ds \to \int_0^T < u \cdot \nabla v, \phi_j \omega_j > \, ds$$
$$\int_0^t < \phi_j \omega_j \cdot \nabla v_m, u_m > \, ds \to \int_0^t < \phi_j \omega_j \cdot \nabla v, u > \, ds$$

Indeed, adding and subtracting the cross terms, then using the Hölder Inequality, the Gagliardo-Nirenberg-Sobolev Inequality

$$| < u_m \cdot \nabla v_m, \phi \omega_j > - < u \cdot \nabla v, \phi_j \omega_j > | \le A_1 + B_1$$

$$\begin{aligned} A_1 &= | < (u_m - u) \cdot \nabla v_m, \phi_j \omega_j > | \\ &\le \|u_m - u\|_n \|\nabla v_m\|_2 \|\phi_j \omega_j\|_{\frac{2n}{n-2}} \\ &\le \|v_m - v\|_2 \|\nabla v_m\|_2 \|\phi_j \nabla \omega_j\|_2 \end{aligned}$$

Due to the strong convergence (8.7) the bound in Lemma 8.3, and the Hölder inequality, we see that

$$\int_0^T A_1 \, ds \to 0$$

Similarly,

$$\begin{aligned} B_1 &= | < u \cdot \nabla (v_m - v), \phi_j \omega_j > | \\ &= | < u \cdot \nabla \phi_j \omega_j, (v_m - v) > | \\ &\le \|u\|_n \|\phi_j \nabla \omega_j\|_{\frac{2n}{n-2}} \|v_m - v\|_2 \\ &\le C \|v\|_2 \|\phi_j \nabla \omega_j\|_{\frac{2n}{n-2}} \|v_m - v\|_2 \end{aligned}$$

Again, owing to (8.7), Lemma 8.3, and the Hölder inequality,

$$\int_0^T B_1 \, ds \to 0$$

This shows

$$\int_0^T < u_m \cdot \nabla v_m, \phi_j \omega_j > \, ds \to \int_0^T < u \cdot \nabla v, \phi_j \omega_j > \, ds$$

The remaining non-linear term is handled in a similar way

$$| < \phi_j \omega_j \cdot \nabla v_m, u_m > - < \phi_j \omega_j \cdot \nabla v, u > | \le A_2 + B_2$$

$$\begin{aligned} A_2 &= | < \phi_j \omega_j \cdot \nabla v_m, (u_m - u) > | \\ &\le \|\phi_j \omega_j\|_{\frac{2n}{n-2}} \|\nabla v_m\|_2 \|u_m - u\|_n \\ &\le C \|\phi_j \nabla \omega_j\|_2 \|\nabla v_m\|_2 \|v_m - v\|_2 \end{aligned}$$

$$\begin{aligned} B_2 &= | < \phi_j \omega_j \cdot \nabla (v_m - v), u > | \\ &= | < \phi_j \omega_j \cdot \nabla u, v_m - v > | \\ &\le \|\phi_j \omega_j\|_{\frac{2n}{n-2}} \|v_m - v\|_2 \|\nabla u\|_n \\ &\le C \|\phi_j \nabla \omega_j\|_2 \|v_m - v\|_2 \|v\|_2 \end{aligned}$$

Applying (8.7) with Lemma 8.3 and the Hölder inequality shows

$$\int_0^T < \phi_j\omega_j \cdot \nabla v_m, u_m > ds \to \int_0^T < \phi_j\omega_j \cdot \nabla v, u > ds$$

Since the $\omega_j$ are dense in $L^2_\sigma$ and $\phi_j$ is an arbitrary smooth function, we have completed the proof. □

*Corollary* 8.5. The conclusions of Theorem 8.4 hold with the relaxed hypothesis $v_0 \in L^2_\sigma(\Omega)$.

*Proof.* Note that all of the bounds attained in Lemma 8.3 and used in the proof of the previous theorem depend only on the $L^2$ norm of the initial data. Let $v_0^i \in C_0^\infty(\Omega)$ be a sequence of functions approaching $v_0$ strongly in $H_0^1$ such that

$$\|v_0^i\|_{H_0^1} \leq \|v_0\|_{H_0^1}$$

Such a sequence can be constructed using standard mollifiers and cutoff functions. Considering each $v_0^i$ as initial data, Theorem 8.4 and its corollary give the existence of a weak solution $v^i$ in the sense of Definition 4.1. Applying (8.3), we see that these weak solutions satisfy the bounds

$$\begin{aligned} \|v^i\|_{L^\infty([0,T];L^2_\sigma(\Omega))} + \|\nabla v^i\|_{L^2([0,T];L^2_\sigma(\Omega))} &\leq C(n, \alpha, \nu, \|v_0\|_2) \\ \|\partial_t v^i\|_{L^2([0,T];(H^1_\sigma)'(\Omega))} &\leq C(n, \alpha, \nu, \|v_0\|_2) \end{aligned}$$

and for each $\phi \in H^1_\sigma$ the relation

(8.8)

$$\begin{aligned} \int_0^T < v^i, \partial_t\phi > ds + \int_0^T < u^i \cdot \nabla v^i, \phi > ds \\ + \int_0^T < \phi \cdot \nabla u^i, v^i > ds + \int_0^T < \nabla v^i, \nabla\phi > ds =< v_0, \phi > \end{aligned}$$

As before, using the Banach-Alaoglu Theorem and extracting a possible subsequence implies that there exists a function

$$\begin{aligned} v &\in L^\infty([0,T]; L^2_\sigma(\Omega)) \cap L^2([0,T]; H^1_\sigma(\Omega)) \\ \partial_t v &\in L^2([0,T]; (H^1_\sigma)'(\Omega)) \end{aligned}$$

such that

$$\begin{aligned} &v^i \rightharpoonup v \text{ in } L^\infty([0,T]; L^2_\sigma(\Omega)) \text{ weak}* \\ &v^i \rightharpoonup v \text{ in } L^2([0,T]; H^1_\sigma(\Omega)) \text{ weakly} \end{aligned}$$

Passing the limits through (8.8) follows by the same steps as in the proof of the previous theorem. □

**Theorem 8.6.** *Let $v_0 \in L^2_\sigma(\mathbb{R}^n)$ Then, there exists a weak solution in the sense of Definition 4.2, with initial data $v_0$ in the whole space $\mathbb{R}^n$, $2 \leq n \leq 4$.*

*Proof.* Let $R_i$ be a sequence tending to infinity and $\chi_{R_i}$ a smooth cutoff function which is equal to 1 inside the ball of radius $R_i - \epsilon$ and zero on the boundary of the ball with radius $R_i$. The corollary to Theorem 8.4 now gives existence of a weak solution $v^{R_i}$ on the ball of radius $R_i$ with initial conditions $v_0\chi_{R_i}$. Extend $v^{R_i}$ to all of $\mathbb{R}^n$ by setting it equal to zero outside the ball of radius $R_i$. All of the bounds in Lemma 8.3 were found independent of the size of $\Omega$, so here they hold independent of $R_i$. Using the Banach-Alaoglu Theorem we have the existence of a function

$$\begin{aligned} v &\in L^\infty([0,T]; L^2_\sigma(\mathbb{R}^n)) \cap L^2([0,T]; H^1_\sigma(\mathbb{R}^n)) \\ \partial_t v &\in L^2([0,T]; (H^1_\sigma)'(\mathbb{R}^n)) \end{aligned}$$

such that

$$v^{R_i} \rightharpoonup v \ in \ L^\infty([0,T]; L^2_\sigma(\mathbb{R}^n)) \ weak* \tag{8.9}$$

$$v^{R_i} \rightharpoonup v \ in \ L^2([0,T]; H^1_\sigma(\mathbb{R}^n)) \ weakly \tag{8.10}$$

There exists an orthogonal basis $\{\phi_i\}$ for $L^2([0,T];(\mathbb{R}^n))$ where each function in the basis is smooth and has compact support in space. For $R_i$ larger then the support of $\phi$, Theorem 8.4 with it's corollary show

$$\int_0^T < v^{R_i}, \partial_t \phi > \, ds + \int_0^T < u^{R_i} \cdot \nabla v^{R_i}, \phi > \, ds + \int_0^T < \phi \cdot \nabla u^{R_i}, v^{R_i} > \, ds \\ + \int_0^T < \nabla v^{R_i}, \nabla \phi > \, ds =< v_0, \phi >$$

The limit $m \to \infty$ can be passed through the linear terms just as before. In the support of each basis function $\phi_j$, we have the strong convergence to pass the limit through the non-linear terms. A diagonal argument shows this convergence holds as $R_i \to \infty$. □

In the above existence theorems, the pressure term can be found by either taking the divergence of the VCHE and solving the Poisson equation, or using a famous result of de Rham. See, for example, [27].

**Theorem 8.7.** *The solution to the VCHE constructed in Theorems 8.4 and 8.6, with initial data in $v_0 \in H^K_\sigma$, satisfies the bounds*

$$\|\nabla^M v\|_2^2 + \int_0^t \|\nabla^{M+1} v\|_2^2 \leq C(n, \alpha, \nu, \|v\|_{H^K_0}) \tag{8.11}$$

*for all $M \leq K$.*

*Proof.* We will do the calculations formally and note that these bounds can be applied to the approximate solutions constructed in Theorem 8.2. This proof proceeds by induction. The inductive assumption is that the following bound holds for all $m < M$.

$$\|\nabla^m v\|_2^2 + \int_0^T \|\nabla^{m+1} v\|_2^2 \, dt \leq C$$

The base case ($m = 0$) is true by Lemma 8.3, we will now show that it holds for $m = M$ which will prove the theorem. First note, using 8.1, that the inductive assumption implies

$$\|\nabla^m u\|_n^2 + \int_0^T \|\nabla^{m+1} u\|_n^2 \, dt \leq C$$

Multiply the VCHE (1.1) by $\triangle^M v$ and integrate by parts to find

$$\frac{1}{2} \frac{d}{dt} \|\nabla^M v\|_2^2 + \nu \|\nabla^{M+1} v\|_2^2 \leq I_M + J_M \tag{8.12}$$

$$I_M = \sum_{m=0}^{M} \binom{M}{m} < \nabla^m u \cdot \nabla\nabla^{M-m} v, \nabla^M v >$$

$$J_M = \sum_{m=0}^{M} \binom{M}{m} < \nabla^M v \cdot \nabla\nabla^{m} u, \nabla^{M-m} v >$$

The two integrals on the RHS are estimated essentially the same way. The key difference is that in the first one we use the relation $< u \cdot \nabla v, v >= 0$ while in the second we can place an extra derivative on $u$.

With application of $< u, \nabla v, v >= 0$ the first bound becomes

$$I_M = \sum_{m=1}^{M} \binom{M}{m} < \nabla^m u \cdot \nabla\nabla^{M-m} v, \nabla^M v >$$

Hölder's inequality, the Sobolev inequality, and Cauchy's inequality show

$$\begin{aligned} I_M &\le C \sum_{m=1}^{M} \|\nabla^m u\|_n \|\nabla^{M+1-m} v\|_2 \|\nabla^M v\|_{\frac{2n}{n-2}} \\ &\le C \sum_{m=1}^{M} \|\nabla^m u\|_n^2 \|\nabla^{M+1-m} v\|_2^2 + \frac{\nu}{4}\|\nabla^{M+1} v\|_2^2 \end{aligned}$$

Similarly for the second term

$$\begin{aligned} J_M &\le C \sum_{m=0}^{M} \|\nabla^M v\|_{\frac{2n}{n-2}} \|\nabla^{m+1} u\|_n \|\nabla^{M-m} v\|_2 \\ &\le \frac{\nu}{4}\|\nabla^{M+1} v\|_2^2 + C \sum_{m=0}^{M} \|\nabla^{m+1} u\|_n^2 \|\nabla^{M-m} v\|_2^2 \end{aligned}$$

Equation (8.12) now becomes

$$\frac{d}{dt}\|\nabla^M v\|_2^2 \le C \sum_{m=0}^{M} \|\nabla^{m+1} u\|_n^2 \|\nabla^{M-m} v\|_2^2$$

The Gronwall inequality with application of the inductive assumption finish the proof. □

**Theorem 8.8.** *The solution to the VCHE constructed in Theorems 8.4 and 8.6, with initial data in* $v_0 \in H^K_\sigma$*, satisfies the bounds*

$$\|\partial_t^p \nabla^m v\|_2^2 + \int_0^t \|\partial_t^p \nabla^{m+1} v\|_2^2 \le C(n, \alpha, \nu, \|v\|_{H^K_0}) \tag{8.13}$$

*for all* $M + 2P \le K$*.*

*Proof.* To prove this, we will bound the time derivatives of the solution in terms of the space derivatives, then use the previous theorem to establish regularity. We will do the calculations formally and note that these bounds can be applied to the approximate solutions constructed in Theorem 8.2.

Apply $\partial_t^P \nabla^M$ to the solution of the VCHE, from this we have the inequality

$$\|\partial_t^{P+1} \nabla^M v\|_2^2 \le C(\|\partial_t^P \nabla^{M+2} v\|_2^2 + \|\partial_t^P \nabla^M (u \cdot \nabla v)\|_2^2 + \|\partial_t^P \nabla^M (v \cdot \nabla u^T)\|_2^2)$$

Using the Gagliardo-Nirenberg-Sobolev inequality and 8.1 we can bound the first term on the right hand side as

$$\begin{aligned}\|\partial_t^P \nabla^M (u\cdot\nabla v)\|_2^2 &= \sum_{p=0}^{P}\sum_{m=0}^{M}\binom{P}{p}\binom{M}{m}\|\partial_t^p\nabla^m u\|_n^2\|\partial_t^{P-p}\nabla^{M+1-m}v\|_{\frac{2n}{n-2}}^2\\ &\le C\sum_{p=0}^{P}\sum_{m=0}^{M}\|\partial_t^p\nabla^m v\|_2^2\|\partial_t^{P-p}\nabla^{M+2-m}v\|_2^2\end{aligned}$$

Similarly for the second term,

$$\begin{aligned}\|\partial_t^P \nabla^M (v\cdot\nabla u^T)\|_2^2 &= \sum_{p=0}^{P}\sum_{m=0}^{M}\binom{P}{p}\binom{M}{m}\|\partial_t^p\nabla^{m+1} u\|_n^2\|\partial_t^{P-p}\nabla^{M-m}v\|_{\frac{2n}{n-2}}^2\\ &\le C\sum_{p=0}^{P}\sum_{m=0}^{M}\|\partial_t^p\nabla^{m+1} v\|_2^2\|\partial_t^{P-p}\nabla^{M+1-m}v\|_2^2\end{aligned}$$

We can now deduce that

$$\|\partial_t^{P+1}\nabla^M v\|_2^2 \le C\|\partial_t^P v\|_{H_0^{M+2}}^2$$

This implies, for all $M$, $P$, such that $M+2P\le K$

$$\|\partial_t^P\nabla^M v\|_2^2 \le C\|v\|_{H_0^K}^2$$

Appealing to Theorem 8.7 finishes the proof. □

The previous theorem demonstrates how the norms $\|v\|_{H^m}$ and $\|v\|_{H^{m+1}}$ can be bounded in terms of $\|v_0\|_{H^m}$. Since the PDE is parabolic we can expect regularity from interior estimates but the bounds will not depend explicitly on the initial conditions.

**Theorem 8.9.** *The solution to the VCHE constructed in Theorems 8.4 and 8.6 is unique.*

*Proof.* Let $v$ and $w$ be two solutions to the VCHE 1.1 with the same initial conditions. Let $u$ and $\omega$ be the corresponding "filtered" velocities. The difference solves the PDE

$$(v-w)_t - \nu\triangle(v-w) + \nabla p + u\cdot\nabla v - \omega\cdot\nabla w + v\cdot\nabla u^T - w\cdot\nabla\omega^T = 0$$

with zero initial conditions. Multiplying this relation by $v-w$ and integrating by parts leaves

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|v-w\|_2^2 + \nu\|\nabla(v-w)\|_2^2 =&< (u-\omega)\cdot\nabla w, (v-w) >\\ &+ < (v-w)\cdot\nabla(u-\omega), v >\\ &+ < (v-w)\cdot\nabla\omega, v-w >\end{aligned}$$

Using Hölder's inequality, the Gagliardo-Nirenberg-Sobolev inequality, Cauchy's inequality, and (8.3), estimate the RHS

$$\begin{aligned}
< (u-\omega)\cdot\nabla w, (v-w) > &\leq \|u-\omega\|_n \|\nabla w\|_2 \|v-w\|_{\frac{2n}{n-2}} \\
&\leq C\|v-w\|_2^2 \|\nabla w\|_2^2 + \frac{\nu}{4}\|\nabla(v-w)\|_2^2 \\
< (v-w)\cdot\nabla(u-\omega), v > &\leq \|v-w\|_2 \|\nabla(u-\omega)\|_n \|v\|_{\frac{2n}{n-2}} \\
&\leq C\|v-w\|_2^2 \|\nabla v\|_2^2 + \frac{\nu}{8}\|\nabla(v-w)\|_2^2 \\
< (v-w)\cdot\nabla\omega, v-w > &\leq \|v-w\|_2 \|\nabla\omega\|_n \|v-w\|_{\frac{2n}{n-2}} \\
&\leq C\|v-w\|_2^2 \|w\|_2^2 + \frac{\nu}{8}\|\nabla(v-w)\|_2^2
\end{aligned}$$

After using the bounds in Lemma 8.3 we have

$$\frac{1}{2}\frac{d}{dt}\|v-w\|_2^2 + \frac{\nu}{2}\|\nabla(v-w)\|_2^2 \leq C\|v-w\|_2^2$$

By assumption $\|v_0 - w_0\|_2 = 0$, so $\|v-w\|_2 = 0$ for all $t \in [0,T]$. □

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