EXISTENCE AND DECAY OF POLYMERIC FLOWS

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Abstract. We study the existence and decay of solutions to kinetic models of incompressible polymeric flow. We consider dumbbell type models in the case when the drag term is co-rotational and weak solutions are constructed via a Leray-type approximation. We analyze the decay when the space of elongations is bounded, and the spatial domain of the polymer is either a bounded domain $\Omega \subset \mathbb{R}^n$, $n = 2, 3$ or it is the whole space $\mathbb{R}^3$. The decay is first established for the L^2 norm of the probability density function ψ and then this decay is used to obtain L^2-decay of the velocity field u. Consideration is also given to solutions where the probability density function is radial in the admissible elongation vectors q. In this case, the velocity field u becomes a solution to the incompressible Navier–Stokes equations, and thus decay follows from known results for the Navier–Stokes equations.

1. Introduction and statement of results

This paper is concerned with questions related to the existence and decay of solutions to a system of nonlinear partial differential equations which arises from the kinetic theory of dilute polymer solutions. The fluid under consideration is viscous, incompressible, isothermal and Newtonian. It will be supposed that the fluid is either inside a bounded open set $\Omega \subset \mathbb{R}^d$, $d = 2$ or 3, with solid boundary $\partial\Omega$, or that the flow domain $\Omega = \mathbb{R}^3$; in the first case it will be assumed that the velocity field u satisfies the no-slip boundary condition $u = 0$ on $\partial\Omega$. The polymer chains are suspended in the solvent and are assumed not to interact with each other. The set of admissible elongation vectors is assumed to be bounded. We recall that a polymer is a substance composed of molecules with large molecular mass consisting of repeated structural units, or monomers, connected by covalent chemical bonds; the bonds are due to the sharing of electrons between atoms. The attraction-repulsion stability that is caused by the common electron is what characterizes the covalent bonding. The idea of covalent bonding between long chains of atoms was introduced in a ground-breaking and controversial paper by Hermann Staudinger in 1920 (Nobel Laureate in Chemistry, 1953).

The simplest model to account for non-interacting polymer chains is the so-called dumbbell model [1]. A dumbbell consists of two beads connected by an elastic spring. One can imagine that in this model the dumbbells represent the atoms, while the elastic spring gives the covalent bond. For descriptions of this model we refer the reader to [1, 3], for example.

In its original form, the model to be considered consists of the coupling of the incompressible Navier–Stokes equations, with a source term representing the divergence of the elastic *extra stress* tensor τ (i.e. the polymeric part of the Cauchy stress tensor), to a set of stochastic ordinary differential equations. This model is then restated as a fully deterministic set of equations, by replacing the stochastic ordinary differential equations with the associated Kolmogorov equation that describes the evolution of the associated probability density function. This leads to a coupled Navier–Stokes–Fokker–Planck system: the conservation of momentum and mass equations for the

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solvent have the form of the incompressible Navier–Stokes equations in which the elastic extra-stress tensor appears as a source term. The extra stress tensor, in turn, depends on the probability density function, which satisfies a Fokker–Planck equation as will be described below.

Thus we consider the following initial-boundary-value problem:

$$\frac{\partial}{\partial t}u + (u\cdot\nabla_x)u - \nu\Delta_x u + \nabla_x p = \nabla_x\cdot\tau, \text{ in } \Omega\times(0,T]\ , \tag{1.1}$$

$$\mathrm{div} u = 0, \text{ in } \Omega\times(0,T] \tag{1.2}$$

$$\frac{\partial}{\partial t}\psi + (u\cdot\nabla)\psi - \nabla_q\cdot(\sigma(u)q\psi) = \frac{1}{2\lambda}\nabla_q\cdot(\nabla_q\psi + U'q\psi) + \mu_o\Delta\psi, \quad \text{in } \Omega\times D\times(0,T] \tag{1.3}$$

$$\nabla_x\psi\cdot n = 0, \text{ in } \partial\Omega\times D\times(0,T], \qquad \psi = 0, \text{ in } \Omega\times\partial D\times(0,T] \tag{1.4}$$

$$\psi(x,q,t) = \psi_o(x,q,0)\ \forall\ (x,q) \text{ in } \Omega\times D\ . \tag{1.5}$$

where u is the velocity field, p is the pressure of the fluid, and $\nu\in\mathbb{R}_+$ is the viscosity coefficient. The probability density $\psi(x,q,t)$ represents the probability at time t of finding a dumbbell located 'between' x and $x+\mathrm{d}x$ having elongation 'between' q and $q+\mathrm{d}q$. The probability density ψ satisfies the Fokker–Planck equation, together with suitable boundary and initial conditions. For details we refer the reader to [3].

For an explanation of the appearance of derivation of the diffusive term in the "Fokker -Planck" equation we refer the reader to [2]. In particular in [2] in their derivation they show that there is a " x-dissipative centre-of-mass diffusion term $\mu_0\,\Delta_x\psi$ on the right-hand side of the Fokker–Planck equation (1.3). In standard derivations of bead-spring models the centre-of-mass diffusion term is routinely omitted, on the grounds that it is several orders of magnitude smaller than the other terms in the equation." ([2], page 7).

In our paper the term $\mu_o\Delta\psi$ yields the necessary regularity to be able to pass to the limit in our approximating solutions in order to construct our solutions .

The symmetric extra-stress tensor, $\tau(\psi) : (x,t)\in\mathbb{R}^{d+1}\mapsto\tau(\psi)(x,t)\in\mathbb{R}^{d\times d}$ is dependent on a probability density function $\psi : (x,q,t)\in\mathbb{R}^{2d+1}\mapsto\psi(x,q,t)\in\mathbb{R}$, defined as

$$\tau(\psi) = k\,\mu\,(\mathcal{C}(\psi) - \rho(\psi)\,\mathcal{I}). \tag{1.6}$$

Here k, $\mu\in\mathbb{R}_+$ are, respectively, the Boltzmann constant and the absolute temperature, $\mathcal{I}$ is the unit $d\times d$ tensor, and

$$\mathcal{C}(\psi)(x,t) = \int_D \psi(x,q,t)\,U'(\tfrac{1}{2}|q|^2)\,q\,q^\top\,dq \text{ and } \rho(\psi)(x,t) = \int_D \psi(x,q,t)\,dq. \tag{1.7}$$

In addition, the real-valued, continuous, nonnegative and strictly monotonic increasing function U, defined on a relatively open subset of $[0,\infty)$, is an elastic potential which gives the elastic force $\mathcal{F} : D\to\mathbb{R}^d$ on the springs defined by

$$\mathcal{F}(q) = U'(\tfrac{1}{2}|q|^2)\,q. \tag{1.8}$$

We will only consider the co-rotational case. It is well known that the co-rotational case is not justified physically. However it is used in the mathematical litereature, see [10]That is we suppose that the drag term is skew-symmetric:

$$\sigma(u) = -\sigma(u)^\perp \tag{1.9}$$

We shall assume that there are no body forces present: If there were, we would need to impose decay conditions on the forces. The extra stress tensor τ is defined as the second moment of ψ, the probability density function of the (random) conformation vector of the polymer molecules.

The Kolmogorov equation satisfied by ψ is a Fokker–Planck type second-order parabolic equation whose transport coefficients depend on the velocity field u.

On introducing the (normalized) Maxwellian

$$M(q) = \frac{e^{-U(\frac{1}{2}|q|^2)}}{\displaystyle\int_D e^{-U}\, dq},$$

we have

$$M\,\nabla_q M^{-1} = -M^{-1}\,\nabla_q M = U'\,q. \tag{1.10}$$

In addition, the following identities hold:

$$\nabla_q U = U'\,q, \qquad \nabla_q U' = U''\,q \qquad \text{and} \qquad \Delta_q U = U''\,|q|^2 + U'\,d. \tag{1.11}$$

Thus, the Fokker-Planck equation (1.2) can be rewritten as

$$\begin{aligned} \frac{\partial}{\partial t}\psi + (u\cdot\nabla)\psi - \nabla_q\cdot(\sigma(u)q\psi) &= \frac{1}{2\lambda}\nabla_q\cdot\left(M\nabla_q\left(\frac{\psi}{M}\right)\right) + \mu_o\Delta\psi, \\ &\text{in } \Omega\times D\times(0,T] \\ \nabla_x\psi\cdot n = 0, \text{ in } \partial\Omega\times D\times(0,T], \qquad & \psi = 0, \quad \text{in } \Omega\times\partial D\times(0,T] \\ \psi(x,q,0) &= \psi_o(x,q)\ \ \forall\ (x,q) \text{in } \Omega\times D \end{aligned} \tag{1.12}$$

In the sequel, as in [3], we will suppose that the potential U satisfies

Assumption: A1

$$\mathcal{M} = \int_D M[U']^2|q|^4 dq < \infty \tag{1.13}$$

We now recall two well known examples

1.1. **Two Examples. 1. FENE-type models.** A widely used model is the FENE (Finitely Extensible Nonlinear Elastic) model, where

$$D = B(0, b^{\frac{1}{2}}), \quad \text{and} \quad U(s) = -\frac{b}{2}\ln\left(1 - \frac{2\,s}{b}\right), \tag{1.14}$$

$$\text{and hence } \ e^{-U(\frac{1}{2}|q|^2)} = \left(1 - \frac{|q|^2}{b}\right)^{\frac{b}{2}}. \tag{1.15}$$

Here $B(0,s)$ is the bounded open ball of radius $s > 0$ in $\mathbb{R}^d$ centred at the origin, and $b > 0$ is an input parameter. Hence the elongation $|q|$ cannot exceed $b^{\frac{1}{2}}$.

2. Hookean dumbbells. Letting $b \to \infty$ in (1.15) leads to the so-called Hookean dumbbell model where

$$D = \mathbb{R}^d \qquad \text{and} \qquad U(s) = s, \qquad \text{and therefore} \qquad e^{-U(\frac{1}{2}|q|^2)} = e^{-\frac{1}{2}|q|^2}. \tag{1.16}$$

Remark 1.1. **We note that**

- Assumption A1 holds for the Fene models described below. For details see [3].
- Assuming A1 allows for potentials which are more general then the ones used in [9], since these are specific FENE models.
- As remarked before the diffusive term in the probability equation does appear in derivations of the FENE models and in papers such as [10], [11] is omitted, due to its small size compared to other terms in the equation. For details of the derivation please see [2].

The paper is organized as follows. After the introduction, section two describes notation and some weighted Poincaré inequalities, essential for the decay estimates. Section three deals with the construction of a weak solution, via a Leray aproximation. It is known that there in the literatute there are several existence theorems corresponding to solutions to similar systems. (See [3], [2], [8], [10]). The reason for obtaining a new existence theorem is that we need to be able to derive our decay estimates on approximating solutions which are sufficiently regular. This allows us to work formally and then pass to the limit. Section four deals with decay of the probability. We show that in the co-rotational case: If D is a bounded domain in $\mathbb{R}^n$ and, Ω is either a bounded domain or the whole space $\mathbb{R}^n$ then the energy of the probability density has exponential decay. This decay follows by using a weighted Poincaré inequality. In section five we obtain estimates for the velocity, using the decay estimates obtained in section three. Section six in the co-rotational case we show that: If D and Ω are bounded domains in $\mathbb{R}^n$, then the energy of the velocity has exponential decay. In section six we establish that the decay of energy of the velocity is algebraic in the co-rotational case for the bounded Domain D of elongations , and unbounded spatial domain Ω . The decay follows by by Fourier splitting analysis. Section seven studies the existence of probabilities when the data separates into a radial function in the elongations times a function that depends on space and time. This is done by reducing the problem to solving a Sturm-Liouville problem. In the next section we show that if the probability density is radial in the elongations the velocity satisfies a Navier-Stokes equation. We believe that such solutions will only exists in the co-rotational case, since we expect that nonco-rotational drag will not allow radial elongations to be sustained. We note that the co-rotational solutions constructed in section seven satisfy the radial condition.

Remark 1.2. **Final introductory remarks**

As pointed out above, all the work will be done in the co-rotational case. We recognize that this case is not well justified physically. However it is used in many mathematical works, to mention a few [10], [3].
There are many existence results for systems of the type described in this paper. In the stochastic direction, let us mention, Jourdain,Lelièvre and Le Bris, [8]. In two dimensions, for coupled Navier-Stokes with Fokker-Planck: Constantin, Fefferman and Zarnescu [5]. For FENE models we would like to mention the fundamental work of Lions and Masmoudi [10] and Masmoudi, [11]. Existence results can also be found in the papers of Barret, Suli and Schwab [3] and [2]. For well posedness we refer the reader to [7], and for local results to [15].
In this paper we have included an existence result for several reasons:

- It allows for potentials more general than FENE potentials.
- The proof is straightforward, using standart Galerkin methods.

The potentials we use, were first considered in [3], that is they satisfy Assumption A.1. For general FENE potentials $U = \frac{-k}{2} log[1 - \frac{|R|^2}{R_0|^2}],\ k > 0$, the decay proof in this paper can be applied to the approximations constructed by Lions-Masmoudi, [10]. We note that the approximating solutions constructed in [3], are discrete time approximations and we could not use them for our decay proof.

In the course of this paper we will work formally. To make our arguments rigourous we can apply the formal arguments to the approximations constructed and pass to the limit.

2. Preliminaries: Notation and weighted Poincaré inequalities

2.1. **Notation.** The following notation will be used

- The Fourier transform of $v \in \mathcal{S}(\mathbb{R}^n)$ is defined by

$$\widehat{v}(\xi) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} v(x)\, dx,$$

extended as usual to $\mathcal{S}'$.

- For a function $v : \mathbb{R}^n \to \mathbb{C}$ and a multi-index $\gamma = (\gamma_1, \gamma_2, ..., g_n)$,

$$D^\gamma v = \frac{\partial^{\gamma_1} .... \partial^{\gamma_n} v}{\partial x_1^{\gamma_1} .... \partial x_n^{\gamma_n}}$$

.

- If v depends also on time $D_t^j v = \frac{\partial^j v}{\partial t^j}$.
- For $s \in R$, $H^s(\Omega) = \{f \in \mathcal{S}', \ni (1 + |\xi|^2)^{s/2} \hat{f}(\xi) \in L^2\}$.
- $W^{k,p}(\Omega),\ k = 1, 2.., p = 1, 2... = \{u :$ generalized derivatives up to order k $\in L^P(\Omega)$.
- $W^{k,2}(\Omega) = H^k(\Omega)$.
- $H_o^1(\Omega) =$ closure of $C_o^\infty(\Omega)$ in the norm $||\nabla u||_2$.
- $H^{-1}(\Omega) =$ the dual space of $H_o^1(\Omega)$.
- $\mathcal{V} = C_o^\infty(\Omega) \cap \{u : \nabla \cdot u = 0\}$.
- $H =$ closure of $\mathcal{V} \in L^2(\Omega)$.
- $V =$ closure of $\mathcal{V} \in H_o^1(\Omega)$.
- $\mathcal{K}(\Omega \times D) = \mathcal{K} := \left\{\varphi : \int_{\Omega\times D} \left[\frac{|\varphi|^2}{M}\right] dq\, dx < \infty\right\}$.

We define

- $P_i(\Omega) = \{$ projection of Ω onto hyperplane $x_i = 0\}$.
- $AC(\Omega) = \{$ absolutely continuous functions$\}$.
- $D_i = D_i(\Omega) =$ diameter of Ω in the direction of the x_i axis.
- $W(\Omega) = \{w :$ set of function that are measurable, positive and finite a.e in $\Omega\}$.

2.2. **Weighted Poincaré inequalities.** We will need the following weighted Poincaré inequality [12], which works for weights depending on $n-1$ variables, where n is the dimension of the domain.

Theorem 2.1. [12] *Let $1 \le p < \infty$. Let Ω, Q be domains in $\mathbb{R}^n$, w, $v_j \in W(\Omega)$, $j = 1, ...n$. Let F be a regular one-to-one mapping from Q onto Ω with Jacobian $D_F(y)$. Let $D_i = D_i(Q) < \infty$ for some $i \in \{1, ..., n\}$ and let $w[F(y)]$ be independent of y_i. Suppose there exist positive constants c, C, C_j, $j = 1, ..., n$, and a measurable function $d_F : P_i(Q) \to R^+ \ni$ for $y = (y_i', y_i) \in Q$,*

$$c d_F(y_i') \le |D_F(y)| \le C d_F(y_i') \tag{2.17}$$

$$w[F(y)] \left|\frac{\partial F_j(y)}{\partial y_i}\right|^p \le C_j v_j |F(y)|^p \tag{2.18}$$

Then the inequality

$$\int_\Omega |u(x)|^p w(x) dx \le C_0^p \sum_{j=1}^n \int_\Omega \left|\frac{\partial u(x)}{\partial x_j}\right|^p v_j(x) dx \tag{2.19}$$

holds for every function $u = u(x)$ on $\Omega \ni u(F(y)) \in C_0^1(Q)$ with the constant

$$C_0^p = \frac{[D_i(Q)]^p}{p} N^{p-1} \frac{C}{c} max_{\{j=1,..n\}} C_j$$

.

Proof *See* [12] □

The above theorem can be streamlined so that the function u needs only to be AC. We now apply the last Theorem in the following case

Proposition 2.2. *Let $1 \leq p < \infty$. Let $D = B_R(0), D \subset \mathbb{R}^n$, $n = 2, 3$. Suppose $w(x) = w(|x|)$ is measurable, positive and finite. Then*

$$\int_D |u(y)|^p w(y) dy \leq C_p(R) \sum_{j=1}^{n} \int_D \left| \frac{\partial u}{\partial q_j} \right|^p w(y) dy. \tag{2.20}$$

where the function u is absolutely continuous and vanishes on the boundary.

Proof Let $w(x) = w(|x|)$be given.
1. Three dimensions

- Let $Q = (0, R) \times (0, \pi) \times (0, 2\pi)$, and, $D = B_R(0)$
- $q \in Q \to q = (r, \phi, \theta)$
- Define $F : Q \to D$ as the map $F(q) = (r \cos\theta \sin\phi, r \sin\theta \sin\phi, r \cos\phi)$, then $|D_F(r, \pi, \theta)| = |r^2 \sin\phi|$.
- Note that $|F(q)| = r$ and $w(F(q)) = w(|F(q)|) = w(r)$.
- Define $d_F(r, \phi) = |r^2 \sin\phi|$.
- Choose $C = c = 1,\ C_j = R^p,\ j = 1, ..., n$.
- Let $v_j(x) = w(x)$.

Using the above information it is easy to see that the inequalities (2.17) and (2.18) hold and hence we obtain (2.20).

Remark 2.3. The two dimensional case is similar only that one uses polar coordinates instead of spherical coordinates.

□

3. Approximating solutions and existence

In what follows we will suppose that the Maxwellian M is less or equal to one. If not so we would have to have a constant floating around. In this section we construct a sequence of approximating solutions to the polymeric equations. The construction is done in several steps

- Regularize equations via a Leray type regularization: The approximation equations are linear equations with C^∞ coefficients.
- Use Galerkin method to obtain a solution of the probability density approximation equation.
- Obtain exponential decay for the L^2 norm of the probability density approximation, independent of the approximation.
- Show that $\nabla \cdot (\tau) \in L^2((0, T), L^2(\Omega))$, with τ corresponding to the probability approximation.
- Use Galerkin method to obtain a solution of the velocity approximation equation.
- Obtain uniform estimates in L^2 in space and H^1 in space -time.
- Pass to the limit: obtain weak solutions to the polymer equations.
- The uniform estimates yield the L^2 decay of the solution to the polymer equations.

3.1. **The approximating equations.** Without loss of generality let $0 \in \Omega$. Let α be so that $B_0(\alpha) \subset \Omega$, define the mollifier ϕ by

$$\phi \in C_0^\infty(\Omega), \ \ \mathrm{supp}\phi \subset B_0(\alpha) \tag{3.21}$$

$$\int_\Omega \phi dx = 1, \ \ \phi_\delta(u(x)) = \delta^{-3}\phi(\frac{x}{\delta}) \star u \tag{3.22}$$

The idea is to construct an approximation of both the velocity and the probability equations. We first obtain the solution to the approximating equations for the probability. Then we use this probability in the tensor on the right hand side of the Navier-Stokes equations. We show that this stress tensor is in $L^2(0,T);V')$. Hence well known techniques for Navier-Stokes yield the existence of a weak solution to the corresponding approximation for the Navier-Stokes equation. This process will be applied alternatively to obtain approximations to the polymer equations in in $\Omega \times D \times (0,T]$

$$\frac{\partial}{\partial t}u_N + (u_N \cdot \nabla_x)u_N - \nu\Delta_x u_N + \nabla_x p_N = \nabla \cdot \tau(\psi_N), \, x \in \Omega, \, t > 0 \tag{3.23}$$

$$\mathrm{div}\ u_N = 0, \, x \in \Omega, \, t > 0 \tag{3.24}$$

$$\frac{\partial}{\partial t}\psi_N + (\phi_\delta(u_{N-1}) \cdot \nabla_x)\psi_N - \nabla_q \cdot (\sigma(\phi_\delta(u_{N-1})q\psi_N) \tag{3.25}$$

$$= \frac{1}{2\lambda}\nabla_q \cdot (\nabla_q \psi_N + U'(\frac{1}{2}|q|^2)q\psi_N) + \mu_o \Delta\psi_N, \, (x,q,t) \in \Omega \times D, \, t > 0$$

$$\psi_N = 0 \ \text{ on } \ \Omega \times \partial D \times (0,T], \ \ n \cdot \nabla_x \psi_N = 0 \ \text{ on } \ \partial\Omega \times D \times (0,T] \tag{3.26}$$

$$\psi_N(x,q,0) = \psi_o(x,q) \ \ \forall \ \ (x,q) \text{ in } \ \Omega \times D. \tag{3.27}$$

$$u_N(x,t) = 0 \ \text{ on } \ \partial\Omega \times (0,T], \ \ u_N(x,0) = u_o(x) \ \ \forall \ \ x \ \text{ in } \ \Omega. \tag{3.28}$$

where n is the outward normal and $\delta = \frac{1}{N}$. If $\Omega = \mathbb{R}^3$ we replace the boundary conditions by

$$\lim_{|x|\to\infty} u = 0, \quad \lim_{|x|\to\infty} \psi = 0,$$

Let $N = 1,2,3....$ When $N = 1$ to avoid confusion with the initial data, denote $u_{N-1} = U_0$ and suppose that $U_0 = 0$. In the sequel we assume

Using the approximations above the following global existence theorem for weak solutions can be established.

Theorem 3.1. *Let $n = 2,3$, $D = B_R(0) \subset \mathbb{R}^n$, and $\Omega \subset \mathbb{R}^n$ a bounded set. Let $u_o \in H(\Omega)$ and $\psi_o \in \mathcal{K}(\Omega \times D)$. Suppose ((1.13)) holds*

Then there exists a weak solution (u,p,ψ) of the co-rotational polymer equations with data (3.26), (3.27) and, (3.28) satisfying

$$u \in L^2(0,T;V) \cap L^\infty(0,T:H), \quad \psi \in L^\infty(\mathcal{K}(\Omega \times D)), \ \ T > 0$$

This solution will satisfy that $u(t) \to u_o$ weakly in H as $t \to 0$. and $\psi(t) \to \psi_o$ in H as $t \to 0$ for each $q \in D$.

If $\Omega = \mathbb{R}^3$ replace the appropriate boundary conditions by

$$\lim_{|x|\to\infty} u = 0, \quad \lim_{|x|\to\infty} \psi = 0,$$

and the conclusion of the theorem is the same.

The proof will follow by obtaining approximating solutions (u_n, ψ_n) that are bounded uniformly in $L^\infty((0,T);H) \cap L^2(0,T;V) \times L^\infty(\mathcal{K}(\Omega \times D))$. We first need the following auxiliary theorem:

First Step: The probability

Theorem 3.2. *Let D and Ω be as in Theorem 3.1, $\psi_o \in \mathcal{K}(\Omega \times D)$. Let $w \in C^\infty(\Omega \times R_+)$, and $div_x w = 0$. Then there exists a weak solution ψ of*

$$\frac{\partial}{\partial t}\psi + ((w) \cdot \nabla_x)\psi - \nabla_q \cdot (\sigma(w) q\psi) \tag{3.29}$$

$$= \frac{1}{2\lambda}\nabla_q \cdot (\nabla_q \psi + U'(\frac{1}{2}|q|^2) q\psi) + \mu_o \Delta\psi,\ (x, q, t) \in \Omega \times D,\ t > 0 \tag{3.30}$$

$$\psi = 0 \ \ on\ \ \Omega \times \partial D \times (0, T],\ \ n \cdot \nabla_x \psi = 0 \ \ on\ \ \partial\Omega \times D \times (0, T] \tag{3.31}$$

$$\psi(x, q, 0) = \psi_o(x, q) \ \ \forall \ \ (x, q) \ in \ \ \Omega \times D. \tag{3.32}$$

(3.33)

Proof we note that equation (3.29) is a linear second order evolution equation. The solution can be obtained by standard Galerkin method, [6]. We outline the steps here details can be found in the Appendix.

Step 1

By Galerkin method yields an approximation of the form: $\psi_m = \sum_{j,k=1}^{n} d_m^k(t) W_k$, We choose $\{W_k\}_{k=1}^\infty$ to be a basis of eigenvectors in H^1, which are orthogonal in H^1 and orthonormal in L^2 and corresponding eigenvalues Λ_k, satisfying for $(\Delta_q + \mu_0 \Delta_x) W_k = \Lambda_k W_k$. The W_k need to satisfy the boundary conditions imposed on ψ. The construction is explained in the Appendix.

Step 2

We show that

- ψ_m in $L^\infty((0, T), L^2(\Omega \times D))$
- $\Delta_x \psi_m$ in $L^2((0, T), L^2(\Omega \times D))$
- $\Delta_q \psi_m$ in $L^2((0, T), L^2(\Omega \times D))$

  and we obtain uniform bounds in m and time in the above spaces.

Step 3

The bounds above allow to extract a subsequence ψ_m that converges weakly to ψ which is a solution to (3.30). This solution will satisfy the appropriate boundary and initial data conditions.

□

Second Step: The velocity

In order to obtain he existence of the velocity u we need to show that $\nabla \cdot \tau(\psi_m) = \nabla \cdot \tau_N \in L^2([0, T), V')$. Then standart Navier-Stokes theory gives the existence. See [4, 14] We will show that the estimate on τ_m is independent of m. We first give a decay estimate for the L^2 norm of $\frac{\psi^2}{M}$ where ψ is a solution to (3.30). This done by getting uniform bounds for the ψ_m

Theorem 3.3. *Let D, Ω, ψ_0 be as in Theorem 3.2. Let ψ, be the solution to (3.30) constructed in Theorem 3.2, then for all $t > 0$*

$$\|\psi(t)\|_2^2 \leq \left\| \frac{\psi(t)}{\sqrt{M}} \right\|_2^2 \leq C(R) \left\| \frac{\psi_0}{\sqrt{M}} \right\|_2^2 \exp[-C_o t] \tag{3.34}$$

$$\frac{1}{2\lambda} \int_0^t \int_{\Omega \times D} M \left| \nabla_q (\frac{\psi}{M}) \right|^2 \, dq\, dx \leq \frac{1}{2\lambda} \left\| \frac{\psi_0}{\sqrt{M}} \right\|_2^2,\ t > 0 \tag{3.35}$$

$$\mu_0 \int_0^t \int_{\Omega \times D} |\nabla_x \psi|^2 \ dq\, dx \leq \mu_0 \left\| \frac{\psi_0}{\sqrt{M}} \right\|_2^2,\ t > 0 \tag{3.36}$$

Proof

We note first that the solutions of (3.30), where constructed for all $T > 0$. For the first part of inequality (3.34) use that $M(r) \leq 1$. For the second part we need the estimate of Lemma A.2 for the approximations ψ_m constructed via Galerkin

$$\frac{d}{dt}\int_{\Omega\times D}\left|\frac{\psi_m}{\sqrt{M}}\right|^2 dq\ dx = -\frac{1}{2\lambda}\int_{\Omega\times D} M\left|\nabla_q(\frac{\psi_m}{M})\right|^2\ dq\ dx - \mu_0\int_0^t\int_{\Omega\times D}|\nabla_x\psi|^2\ dq\ dx \tag{3.37}$$

Next we will use the weighted Poincaré inequality (2.20) with $p = 2$, $u = \psi_m$ and inthe case that $M(r) = 0$ on the boundary then weight $w(r) = [\phi_k(r)]^{\frac{1}{2}}[M(r)]^{-\frac{1}{2}}$, where $\phi_k(q) = \phi_k(r) \in C_0^\infty(D)$ is positive and and

$$\phi_k(r) = 1 \text{ for } |q| \le R - \frac{1}{k},\ \ \phi_k(r) = 0 \text{ for } R - \frac{1}{2k} \le |q| < R, \text{ and } \phi_k \le 1$$

(Note that we multiply by the cut-off function since $M = 0$ on boundary of D). Hence If $M \neq 0$ we can use the weight to be $w = M$. We only work out the case when M is zero on the boundary since the second case is easier and as such is omitted.

$$\int_{\Omega\times D}|\psi_m(x,q)|^2\phi_k(r)[M]^{-1}dxdq\int_\Omega \le C_2(R)\sum_{j=1}^3\int_D|\frac{\partial\psi_m}{\partial q_j}|^2\phi_k(r)[M]^{-1}dxdq.$$

Fatou's Lemma, the last inequality, $\lim_{k\to\infty}\phi_k = \phi$, with $\phi(q) = 1\ \forall q \in D$, and $\phi_k \le 1$ yield

$$\begin{aligned}\int_{\Omega\times D}|\psi_m(x,q)|^2[M]^{-1}dxdq &\le \liminf_{k\to\infty}\int_{\Omega\times D}|\psi_m(x,q)|^2\phi_k(r)[M]^{-1}dxdq \\ \le \liminf_{k\to\infty}\left[C_2(R)\int_\Omega\int_D|\nabla_q\psi_m|^2[M]^{-1}dxdq\right] &= C_2(R)\int_\Omega\int_D|\nabla_q\psi_m|^2[M]^{-1}dxdq.\end{aligned} \tag{3.38}$$

Combining (3.38) and (3.37) yields

$$\frac{d}{dt}\int_{\Omega\times D}\left|\frac{\psi_m}{\sqrt{M}}\right|^2 dq\ dx \le -C_2(R)\int_{\Omega\times D}|\psi_m(y)|^2[\sqrt{M}]^{-1}|^2dxdq$$

Hence it follows that

$$\|\frac{\psi_m(t)}{\sqrt{M}}\|_2^2 \le \|\frac{\psi_0}{\sqrt{M}}\|_2^2\exp[-C_o t] \tag{3.39}$$

From where (3.34) follows, since ψ_m and $\frac{\psi_m}{\sqrt{M}}$ converge in L^2 weak to ψ and $\frac{\psi}{\sqrt{M}}$ respectively.

Inequalities (3.35) and (3.36) follow by (3.37) and passing to the limit. For details see Theorem A.1 in the appendix. □

The next Proposition shows that $(\nabla\cdot\tau(\psi) \in L^2([0,T) : V')$ for all $T > 0$. This estimate will be necessary for the existence of the velocity.

Proposition 3.4. *Let ψ be the solution to (3.30), then $\nabla\cdot\tau(\psi) \in L^2([0,T) : V')$*

Proof Let $\mathcal{H} = \{v \in H^1 : \|v\|_{H^1} = 1\}$. We need to show that

$$I = \int_0^T\left|\sup_{\{v\in\mathcal{H}\}}\int_\Omega\nabla\cdot\tau v\ dx\right|^2 dt < \infty$$

since

$$\begin{aligned}\left|\int_\Omega\nabla(\cdot\tau(\psi))v\ dx\right|^2 &= \left|\int_\Omega\tau(\psi)\nabla v\ dx\right|^2 dt \le \\ \int_\Omega|\tau(\psi)|^2\ dx\int_\Omega|\nabla v|^2\ dx &\le \int_\Omega|\tau(\psi)|^2\ dx\end{aligned} \tag{3.40}$$

Hence we only need to estimate

$$\int_0^T \int_\Omega |\tau(\psi)|^2 \, dxdt = \int_0^T \int_\Omega |k\,\mu\,(\mathcal{C}(\psi) - \rho(\psi)\,\mathcal{I})|^2 dxdt \tag{3.41}$$

where we recall that

$$\mathcal{C}(\psi)(x,t) = \int_D \psi(x,q,t)\,U'(\tfrac{1}{2}|q|^2)\,q\,q^\top \, dq \text{ and } \rho(\psi)(x,t) = \int_D \psi(x,q,t)\,dq. \tag{3.42}$$

To bound (3.41) we proceed in the following two steps **Step1**

$$J_1 = \int_\Omega |\int_D \psi(x,q,t)\,U'(\tfrac{1}{2}|q|^2)\,q\,q^\top \, dq|^2 dx \le \int_{\Omega\times D} \frac{|\psi|^2}{M} dx\,dq \int_D M|U'|^2|q|^4 dq \le C_0 < \infty \tag{3.43}$$

The last inequality follows by assumption A1 and by the estimates in Theorem 3.3. **Step2**

$$J_2 = \int_\Omega |\int_D \psi(x,q,t) dq|^2 dx \le \int_{\Omega\times D} \frac{|\psi|^2}{M} dx\,dq \int_D M dq \le C_1 \int_{\Omega\times D} \frac{|\psi|^2}{M} dx\,dq. \tag{3.44}$$

The last inequality follows by the estimates in Theorem 3.3. Combining the estimates from J_1 and J_2 obtained in (3.43) and (3.44) and estimate and inequality (3.34)yields

$$I \le C \int_0^T \int_{\Omega\times D} \frac{|\psi|^2}{M} dx\,dq\,dt \le C(R) \int_0^T \|\frac{\psi_0}{\sqrt{M}}\|_2^2 \exp[-C_o t]\,dt \tag{3.45}$$

Hence

$$I \le \tilde{C} < \infty \tag{3.46}$$

Where $\tilde{C}$ is a constant depending on R and the bound in Assumption A1 only. Hence $\nabla\cdot\tau(\psi) \in L^2(0,T,V')$. The proof of the Proposition is now complete. □

The next Theorem is auxiliary on how to find the solutions of approximations to the velocity equations

Theorem 3.5. *Let Ω be as in Theorem 3.1, $\psi_o \in \mathcal{K}(\Omega\times D)$. Let ψ be the solution constructed in Theorem 3.2. Then for all $T > 0$ there exists a weak solution $u \in L^\infty(0,T;H) \cap L^([0,T),V)$ and u is weakly continuous from $[0,T]$ into H^1*

$$\frac{\partial}{\partial t} u + (u\cdot\nabla_x)u - \nu\Delta_x u + \nabla_x p = \nabla_x\cdot\tau(\psi),\ x\in\Omega,\ t>0 \tag{3.47}$$

$$div\,u = 0,\ x\in\Omega, t>0, \tag{3.48}$$

$$u(x,t) = 0 \ \ on\ \ \partial\Omega\times(0,T],\ \ u(x,0) = u_o(x)\ \ \forall\ \ x\ \ in\ \ \Omega. \tag{3.49}$$

Proof By Proposition 3.4 we have that our equation is a Navier-Stokes equation with a forcing term $f = (\nabla\cdot\tau(\psi) \in L^2([0,T):V')$ for all $T>0$. Hence can apply Theorem 3.1 and the solution to Problem 3.2 in Chapter 3 of [14], which gives the weak existence of a solution u as required in the Theorem. □

Proposition 3.6. *Under the same Hypothesis of Theorem 3.5 it follows that*

$$\|u(T)\|^2_{L^2(\Omega)} + \|\nabla u\|_{L^2([0,T);L^2(\Omega)} \le \|u_0\|_{L^2} + \tilde{C}(R). \tag{3.50}$$

Proof We will establish the above estimate formally. We note that the construction in [14] that we used for the existence of a solution in Theorem 3.5, was done using a Galerkin approximation. Hence to make our calculations rigorous, one has to apply them to the Galerkin approximation and pass to the limit. Multiply the equation by u and integrate in space and time and integrate by parts the forcing term and use Hölder inequality

$$\|u(t)\|^2_{L^2(\Omega)} + 2\|\nabla u\|_{L^2([0,T);L^2(\Omega))} \leq \|u_0\|_{L^2} + C\int_0^t \|\nabla u\|_{L^2(\Omega)}\|\tau(\psi)\|_{L^2(\Omega)}$$

Recall that by (3.46) $\int_0^t \|\tau(\psi)\|^2_{L^2(\Omega)} \leq \tilde{C}$.
Hence the RHS can be estimated by

$$\int_0^t \|\nabla u\|_{L^2(\Omega)}\|\tau(\psi)\|_{L^2(\Omega)} \leq \int_0^t \nabla u\|^2_{L^2(\Omega)} + C^2C(R)$$

The proposition follows by letting $\tilde{C}(R) = \tilde{C}C^2C(R)$. □

Remark 3.7. The above Corollary is also valid in the case that $\Omega = \mathbb{R}^n$,

Third Step: Proof of Theorem 3.1

Now we can proof Theorem 3.1

Proof The plan is to construct a sequence of solutions (u_N, p_N, ψ_N). We proceed inductively, starting with $U_0 = 0$, (recall that, to avoid confusions with the notation for the initial data we renamed u_{N-1} when $N = 1$ by U_0). We, now construct a solution to (3.25), with $N = 1$. This equation is linear, and the solution can be constructed using Theorem 3.2. This yields ψ_0. Now Proposition pr:vprima is used to yield u_1, a solution to (3.23). This process is repeated to obtain alternatively $\{psi_N$ and then u_N, ∂_N. That is we proceed inductively using Galerkin, starting with the probability equation using for the drag $w = w_N = \phi_\delta(u_{N-1})$.

Wlog we can suppose $M \leq 1$, (otherwise the bounds will include a bound for M). This construction combined with (3.34), and (3.50) yields

$$\|\psi_N(t)\|_2^2 \leq \left\|\frac{\psi_N(t)}{\sqrt{M}}\right\|_2^2 \leq C(R)\left\|\frac{\psi_0}{\sqrt{M}}\right\|_2^2 \exp[-C_o t] \tag{3.51}$$

$$\frac{1}{2\lambda}\int_0^t\int_{\Omega\times D} M\left|\nabla_q(\frac{\psi_N}{M})\right|^2\,dq\,dx \leq \left\|\frac{\psi_0}{\sqrt{M}}\right\|_2^2,\ t>0 \tag{3.52}$$

$$\mu_0\int_0^t\int_{\Omega\times D}|\nabla_x\psi_N|^2\,dq\,dx \leq \left\|\frac{\psi_0}{\sqrt{M}}\right\|_2^2,\ t>0 \tag{3.53}$$

and

$$\|u_N(t)\|_{L^2(\Omega)} + \nu\|\nabla u_N\|_{L^2([0,T);L^2(\Omega)} \leq \|u_0\|_{L^2} + \tilde{C}(R) \tag{3.54}$$

Hence we can extract subsequences which will converge as follows (as $N \to \infty$)

$$\nabla_x\psi_N \to \nabla\psi,\ \text{weak} \in L^2([0,T);L^2(\Omega\times D)) \tag{3.55}$$

$$\psi_N \to \psi,\ \text{strong} \in L^2([0,T);L^2(\Omega\times D)) \tag{3.56}$$

$$\frac{\psi_N}{M} \to \frac{\psi}{M},\ \text{weak } \in L^\infty([0,T);L^2(\Omega\times D)) \tag{3.57}$$

$$\sqrt{M}\nabla_q\left(\frac{\psi_m}{M}\right) \to \sqrt{M}\nabla_q\left(\frac{\psi}{M}\right), \text{ weak } \in L^2([0,T);L^2(\Omega\times D)) \tag{3.58}$$

$$u_N \to u \text{ strong} \in L^2([0,T);L^2(\Omega)) \tag{3.59}$$

$$\nabla_x u_N \to \nabla_x u \text{ weak } \in L^2([0,T);L^2(\Omega)) \tag{3.60}$$

$$\phi_{\frac{1}{N}}(u) \to u \text{ strong} \in L^\infty([0,T);L^2(\Omega)) \tag{3.61}$$

We have chosen $\delta = \frac{1}{N}$ for our mollifier. Hence the limiting (u,ψ) will be a weak solution. Notice that since we constructed u_N to be solutions to the Navier-Stokes equations with forcing terms $\nabla\cdot\tau(\psi_N)$ this created a sequence of pressures p_N. It is easy to find a uniform bound in N for $p_N \in L^2((0,T);L^2(\Omega))$. This bound might depend on T, but since it is independent of N, it will allow to extract a subsequence that converges in

$$p_N \to p \text{ weak } \in L^2([0,T);L^2(\Omega)) \tag{3.62}$$

This limit (u,p,ψ) is a solution to the polymer equations, and by construction satisfies the initial and boundary conditions since each of the approximating solutions did.

Remark 3.8. We note that all the same procedure can be applied to obtain weak solutions to the polymer equations, when $\Omega = \mathbb{R}^n, n = 2,3$, see Remark 3.1 and 3.5 in [14], provided the boundary conditions are changes as mentioned in the hypothesis of the Theorem.

□

4. Decay of the probability density in the co-rotational case

In this section we investigate the decay of a weigthed Sobolev norm for the probability density, in the case when the elongations vectors are in a bounded domain. More precisely we will require that the domain is a Ball centered at the origin. We also require the drag to be co-rotational.

Theorem 4.1. *Let $D = B_b(0)$ be an open ball in $\mathbb{R}^n$, $n = 2,3$ and Ω bounded in $\mathbb{R}^n, n = 2,3$. Let $(u.\psi)$, be a weak solution of of equations (1.1) , (1.2) , (1.3) satisfying the boundary and initial conditions of conditions of Theorem 3.1 and co-rotational drag. Then there exist constants C_o so that*

$$\int_{\Omega\times D} M\left|\frac{\psi(t)}{M}\right|^2\, dq\, dx \le C(R)\exp[-C_o t] \tag{4.63}$$

where $C_o = C_o(\psi_o, R)$

Proof By (3.34) it follows that for all the approximating probability densities

$$\left\|\frac{\psi_N(t)}{\sqrt{M}}\right\|_2^2 \le C(R)\left\|\frac{\psi_0}{\sqrt{M}}\right\|_2^2 \exp[-C_o t] \tag{4.64}$$

Since $\frac{\psi_N(t)}{\sqrt{M}} \to \frac{\psi(t)}{\sqrt{M}}$ weakly in L^2, it follows that the limit will satisfy (4.63). □

Corollary 4.2. Let $D = B_b(0)$ be an open ball in $\mathbb{R}^n$, $n = 2,3$ and $\Omega = \mathbb{R}^n$ Let $(u.\psi)$, be a weak solution of of equations (1.1) , (1.2) , (1.3) satisfying the conditions for unbounded solutions of Theorem 3.1and co-rotational drag. Then there exist constants C_o so that

$$\int_{\Omega\times D} M\left|\frac{\psi(t)}{M}\right|^2\, dq\, dx \le C(R)\exp[-C_o t] \tag{4.65}$$

where $C_o = C_o(\psi_o, R)$

Proof Works the same as in the bounded case. □

5. Bounds and decay for the energy of the velocity: co-rotational case

The next theorems establishes the decay of the energy of the velocities. We have two cases. If Ω is bounded the decay follows by the Poincaré inequality. If $\Omega = \mathbb{R}^n$ the method is based on Fourier splitting [13]

Bounded domain Ω

Theorem 5.1. *Let D be bounded M a radial Maxwelian as before, and the drag be co-rotational. Assume $U \in C^1$. Let Ω a bounded open set with Lipshitz-continuous boundary $\partial\Omega$. Let u be the solutions of (1.1), (1.2) with data $u(x,0) = u_0$, with boundary described by (1.4) and 1.5 . Where the initial data $u_0 \in H^1$ then*

$$\|u(t)\|_{L^2(\Omega)} \leq C \exp[-C_o t]$$

Where the constants depend on the data and on the size of the domains and the viscosity coefficient of the velocity.

Proof. We have the following inequality, for the approximating solutions. We do the computations formally, when applied to the approximations the result are rigorous. Multiply the velocity equation by u and integrate.

$$\frac{d}{dt}\left[\int_\Omega |u|^2\, dx\right] + \leq -2\nu \int_\Omega |\nabla u|^2\, dx + 2\int_\Omega u\nabla\cdot\tau(\psi)\, dx. \tag{5.66}$$

The RHS of (5.66) can be estimated by

$$\begin{aligned} 2\int_\Omega u\nabla\cdot\tau(\psi)\, dx &\leq \nu\int_\Omega |\nabla u|^2\, dx + \frac{C}{\nu}\int_\Omega |\tau(\psi)|^2\, dx \\ &\leq \nu\int_\Omega |\nabla u|^2\, dx + C(R)\exp[-C_0 t]\int_{\Omega\times D}\left|\frac{\psi_0}{\sqrt{M}}\right|^2\, dx\, dq. \end{aligned} \tag{5.67}$$

Combining the last two inequalities yields Where we used the estimates from (3.42), (3.43), (3.44). Applying for the approximations estimates (3.37) with(3.34), passing to the limit yields

$$\frac{d}{dt}\left[\int_\Omega |u|^2\, dx\right] + \leq -\nu\int_\Omega |\nabla u|^2\, dx + C(R)\exp[-C_0 t]\int_{\Omega\times D}\left|\frac{\psi_0}{\sqrt{M}}\right|^2\, dx\, dq. \tag{5.68}$$

$$\frac{1}{2}\frac{d}{dt}\left[\int_{\Omega\times D}\frac{|\psi|^2}{M}\, dqdx\right] \leq -\frac{1}{2\lambda}\int_{\mathbb{R}^n\times D}\frac{|\psi|^2}{M} dq\, dx. \tag{5.69}$$

Summing (5.68) and (5.69), (where we bound the RHS of (5.68) , by (5.67)) and use the weighted Poincaré inequality and Poincaré inequality yields

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\left[\int_\Omega |u|^2 + \int_{\Omega\times D}\frac{|\psi|^2}{M}\, dqdx\right] &\leq -\frac{\nu}{2}\int_\Omega |u|^2\, dx \\ &-\frac{1}{2\lambda}\int_{\mathbb{R}^n\times D}\frac{|\psi|^2}{M} dq\, dx + C(R)\exp[-C_0 t]. \end{aligned} \tag{5.70}$$

The last inequality yields

$$\frac{d}{dt}\left[\exp(C_1t)\left(\frac{1}{2}\int_\Omega |u|^2 + \int_{\Omega\times D}\frac{|\psi|^2}{M}\,dqdx\right)\right] \le C(R)\exp(C_1t)\exp[-C_0t] \tag{5.71}$$

Now integrate in time and the decay follows for the approximating solutions. Since the approximations converge in L^2 the decay follows for the solutions.

□

Unbounded case: $\Omega = \mathbb{R}^3$

Theorem 5.2. *Suppose the conditions of Theorem(5.1) i with $\Omega = \mathbb{R}^3$. Suppose the boundary conditions: $|u| \to 0$ and $|\psi| \to 0$ as $|x| \to \infty$. If in addition $\psi_0 \in L^1(\mathbb{R}^3 \times D)$ and $u_0 \in L^1(\mathbb{R}^3)$ then*

$$\|u(t)\|^2_{L^2(\mathbb{R}^3)} \le C(t+1)^{-\frac{1}{2}}$$

Remark 5.3.

Proof. The computations are formal. They are rigorous for the approximations. Passing to the limit will give the result for the solution. The formal computations can be done for any $n \ge 3$. We show formally that in n-dimensions

$$\|u(t)\|^2_{L^2(\mathbb{R}^n)} \le C(t+1)^{-\frac{n}{2}-1}$$

For the unbounded case we will have (5.68), (5.67) and (5.69) on the approximations. Estimating the RHS of (5.68) by (5.67) and summing to (5.69) yields formally

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\left[\int_\Omega |u|^2 + \int_{\Omega\times D}\frac{|\psi|^2}{M}\,dqdx\right] &\le -\frac{\nu}{2}\int_\Omega |\nabla u|^2\,dx \\ &-\frac{1}{2\lambda}\int_{\mathbb{R}^n\times D}\frac{|\psi|^2}{M}dq\,dx + C(R)\exp[-C_0t]\end{aligned} \tag{5.72}$$

We rewrite this equation by taking the Fourier Transform in the x variable. Using Plancherel Theorem yields

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\left[\int_{\mathbb{R}^n} |\widehat{u}|^2 ds + \int_{\mathbb{R}^n\times D}\frac{|\widehat{\psi}|^2}{M}\,dqdx\right] &\le -\frac{\nu}{2}\int_{\mathbb{R}^n} |\xi|^2|\widehat{u}|^2\,dx \\ &-\frac{C}{2\lambda}\int_{\mathbb{R}^n\times D}\frac{|\widehat{\psi}|^2}{M}dq\,dx + C(R)\exp[-C_0t]\end{aligned} \tag{5.73}$$

Let

$$S(t) = \left\{\xi : |\xi| \le \left(\frac{2n}{\nu(t+1)}\right)^{\frac{1}{2}}\right\}$$

and S^c the complement of S in $\mathbb{R}^n$. We split the domain of the integral of the diffusive term of the velocity to get

$$-\frac{\nu}{2}\int_{\mathbb{R}^n}|\xi|^2|\widehat{u}|^2\,dx \le -\frac{n}{t+1}\int_{S^c}|\widehat{u}|^2\,dx = -\frac{n}{t+1}\int_{\mathbb{R}^n}|\widehat{u}|^2\,dx + \frac{n}{t+1}\int_S |\widehat{u}|^2\,dx$$

Using this last inequality in (5.73) yields

$$\frac{d}{dt}\left[(t+1)^n\left(\int_\Omega |\widehat{u}|^2 dx + \int_{\mathbb{R}^n\times D}\frac{|\widehat{\psi}|^2}{M}\right)\right] \le (t+1)^{n-1}\int_{S(t)}|\widehat{u}|^2\, dx$$
$$-(t+1)^n\frac{C}{2\lambda}\int_{\Omega\times D}\frac{|\widehat{\psi}|^2}{M}\, dqdx + n(t+1)^{n-1}\int_{\Omega\times D}\frac{|\widehat{\psi}|^2}{M}\, dqdx + K_o\exp[-K_1 t] = J.$$

Where $K_o = \int_{\Omega\times D}\frac{|\widehat{\psi_0}|^2}{M}\, dqdx$. From Corollary 4.2 that the RHS of the last inequality can be bounded by

$$J \le (t+1)^{n-1}\int_{S(t)}|\widehat{u}|^2\, dx + [(t+1)^n + n(t+1)^{n-1}+1]K_o\exp[-K_1 t]$$

Hence we have

$$\frac{d}{dt}\left[(t+1)^n\left(\int_\Omega |\widehat{u}|^2 dx + \int_{\mathbb{R}^n\times D}\frac{|\widehat{\psi}|^2}{M}\right)\right] \le (t+1)^{n-1}\int_{S(t)}|\widehat{u}|^2\, dx$$
$$+[(t+1)^n + n(t+1)^{n-1}+1]K_o\exp[-K_1 t] \tag{5.74}$$

We now proceed to estimate the first term of the RHS of the last inequality. For this a bound for $|\widehat{u}|$ is necessary

ESTIMATES FOR $|\widehat{u}|$

Take the Fourier Transform of equation (1.1)

$$\frac{\partial}{\partial t}\widehat{u} + \widehat{(u\cdot\nabla_x)u} + \nu|\xi|^2\widehat{u} + \widehat{\nabla_x p} = \widehat{\nabla\cdot\tau},$$

Hence $\widehat{u}$ can be expressed as

$$\widehat{u_j} = \exp\left(-|\xi|^2 t\right)\widehat{u_j^0} - \int_0^t \exp\left(-|\xi|^2(t-s)\right)\left[\widehat{(u\cdot\nabla_x)u_j} - \sum_{k=0}^n i\xi_k\widehat{\tau}_i + \widehat{\partial_{x_j}p}\right]\, ds \tag{5.75}$$

Now we need the following bounds :

Bound for pressure term

For this take the div of the equation, then in Fourier space we have

$$|\xi|^2\widehat{p} = -\sum_{i,j=1}^3 |\xi_i i\xi_j\widehat{u_i u_j}| + |\xi|^2\int_D \widehat{\psi}U'' qq^T\, dq \tag{5.76}$$

We recall that by Remark(3.7) $\|u(t)\|_2 \le C(R, u_0, \psi_0$, and an easy computation (see [3]) shows that $\|\psi(t)\|_{L^1(\mathbb{R}^3\times D)} \le \|\psi_0\|_{L^1(\mathbb{R}^3\times D)}$. Hence it follows that

$$|\xi_k||\widehat{p}| \le |\xi|\sum_{i,j=1}^3 |\widehat{u_i u_j}| + C\int_D |\widehat{\psi}||\xi|\, dq \tag{5.77}$$
$$\le C_o(\|u_0\|_{L^2(\mathbb{R}^3)} + \frac{K_o}{K_1})|\xi| + \|\psi\|_{L^1(\mathbb{R}^3\times D)}|\xi|) \le C_o|\xi|$$

Bound for convective term

$$|\widehat{(u\cdot\nabla_x)u}| \le |\xi|\sum_i^3 |\widehat{u_i u_j}| \le C_o(\|u(t)\|_{L^2(\mathbb{R}^3)}|\xi|) + \le C_o|\xi| \tag{5.78}$$

Bound for tensor term τ

$$\widehat{\tau} = \int_D \widehat{\psi} dq + \int_D \widehat{\psi}(\xi, |q|, t) U'(\frac{1}{2}|q|^2) qq^T dq \leq C \|\psi_0\|_{L^1(\mathbb{R}^3 \times D)} \leq C \tag{5.79}$$

Combining inequalities (5.75) with (3.24) and (3.25) and (5.79) yields

$$|\widehat{u}| \leq |\widehat{u_0}| \exp{(-t|\xi|^2)} + C_o|\xi|^{-1} \tag{5.80}$$

Hence near zero and in particular for $\xi \in S(t)$ we have

$$|\widehat{u}| \leq \frac{C_o}{|\xi|}$$

Remark 5.4. For this last inequality we need that $u_o \in L^1$ so that $\widehat{u}_o \in l^\infty$

In $\mathbb{R}^3$ it follows that (the computation works for all $n \geq 3$)

$$\int_{S(t)} |\widehat{u(t)}|^2 d\xi \leq C \int_{S(t)} \frac{1}{|\xi|^2} \, d\xi = C(t+1)^{-\frac{n}{2}+1} \tag{5.81}$$

From the last estimate it follows that

$$(t+1)^{n-1} \int_{S(t)} |\widehat{u}|^2 \, dx \leq C(t+1)^{\frac{n}{2}} \tag{5.82}$$

Since $[(t+1)^n + n(t+1)^{n-1} + 1] K_o \exp[-K_1 t] \leq K \exp[-K_1 t]$, combining (5.74)with the last inequality yields

$$\frac{d}{dt}\left[(t+1)^n \left(\int_\Omega |\widehat{u}|^2 dx + \int_{\mathbb{R}^n \times D} \frac{|\widehat{\psi}|^2}{M} \right) \right] \leq C(t+1)^{\frac{n}{2}} + K \exp[-K_1 t]. \tag{5.83}$$

Let $C_o = \|u_0\|_{L^2(\mathbb{R}^n)} + \|\frac{|\psi_)|^2}{M}\|_{L^2(\mathbb{R}^n \times D)}$. Integrating in time in the interval $[0, t]$ gives using Plancherel

$$[(t+1)^n \left(\int_\Omega |u|^2 dx + \int_{\mathbb{R}^n \times D} \frac{|\psi|^2}{M} \right) \leq C_o + C(t+1)^{\frac{n}{2}+1} + K[\frac{1 - \exp[-K_1 t]}{K_1}]. \tag{5.84}$$

Dividing by $(t+1)^n$, yields

$$\|u(t)\|^2_{L^2(\Omega)} \leq C(t+1)^{-\frac{n}{2}+1}$$

This concludes the formal part of the theorem. Applying it to the Galerkin approximations yields the estimate in the limit.

$\square$

Remark 5.5. We believe that the correct decay rate should be $(t+1)^{-\frac{n}{2}}$. But since we have have no decay for the $\|\psi\|_1$ norm we cannot use a bootstrap of the type used for Navier-Stokes. In this case the above theorem would give also decay in the two dimensional case.

6. CO-ROTATIONAL CASE: WITH DATA THAT SEPARATES

In this section we analyze the probability density in the co-rotational case for data that separates. We will suppose that the elongations are restricted to the ball $D = D_{b^{1/2}}(0) \subset \mathbb{R}^n$. We look for solutions to the Fokker-Panck equations for the probability density of the form $\psi = N(r)f(x,t)$, where $N(r) = N(|q|^2)$, with initial data of the form $\psi(x,|q|,0) = N(r)f(x,0)$. An easy computation yields that the Fokker-Planck equations can be expressed as

$$N(r)f_t + N(r)u \cdot \nabla f = G(r)N(r)f \tag{6.85}$$

Where

$$G(r) = \frac{1}{2\lambda}\nabla_q \cdot \left(M\nabla_q(\frac{N(r)}{M(r)})\right)N^{-1}, \tag{6.86}$$

This follows since $\sigma(u)$ is co-rotational and hence the drag term $\nabla_q(\sigma(u)qN(|q|^2))f = 0$

Note that if $N(r) = \alpha M(r) \quad \Rightarrow \quad G(r) = 0$. This is a case already considered in [3]. Hence we will suppose that $N(r) \neq M(r)$, then (6.85) yields

$$f_t + u \cdot \nabla f - fG(r) = 0 \text{ and } G(r) = [f_t + u \cdot \nabla f]\, f^{-1}. \tag{6.87}$$

Hence $G(r) = C$. Define $v = f\exp[-G(r)t] = f\exp[-Ct]$. Then v satisfies

$$v_t + u \cdot \nabla v = 0$$

Hence

$$\|v(t)\|_2 \leq \|v(o)\|_2$$

Since $f = v\exp[Ct]$ it follows that

$$\|f(t)\|_2 \leq \|v(0)\|_2 \exp(Ct) \tag{6.88}$$

Thus if there exist N satisfying (6.85), then the solution ψ will have exponential decay if $C < 0$, no decay if $C = 0$ and increases exponentially if $C > 0$. For appropriate M below we will rule out the possibility of exponential growth. The next step will be to show that there exist N which are solutions to (6.85).

Theorem 6.1. *There $\exists$ $N(r) = N(|q|^2)$ such that if*

$$\psi_o(x,t) = \psi(x,|q|,0) = N(r)f(x,0) \quad \Rightarrow \quad \psi = N(r)f(x,t)$$

is a solution to the Fokker Planck equation in the form given by (6.85).

Proof. We note that from (6.86) we have since $G(R) = C$

$$CN = \frac{1}{2\lambda}\nabla_q \cdot \left(M\nabla_q(\frac{N(r)}{M(r)})\right), \tag{6.89}$$

Let $K(r) = \frac{N}{M}$ then the equation (6.89) becomes

$$K'' + K'[\frac{M'}{M} + \frac{3}{r} - 1] = CK, \quad r \in (0, b^{1/2}) \tag{6.90}$$

The problem will be to find K for appropriate C, satisfying (refeq:K). For such K define $N = KM$, then N will be a a solution to (6.85)

To solve equation (6.90) proceed as follows.

First add the boundary data

$$K'(0) = 0,\ K'(b^{1/2}) = 0 \tag{6.91}$$

Now define

$$v(r) = \frac{M'}{M} + \frac{3}{r} - 1 = -U'r + \frac{3}{r} - 1$$

The second term follows by the definition of the Maxwelian.

Hence (6.90) can be rewritten as a Sturm-Liouville problem

$$-\frac{d}{dr}\left[\exp\int_a^r (v(r))\, dr \frac{d}{dr}K\right] + \lambda(\exp\int_a^r (v(r))\, dr)K = 0 \tag{6.92}$$

$$K'(0) = 0,\ K'(b^{1/2}) = 0 \tag{6.93}$$

where we choose any fixed $a \in (0, b^{1/2})$.

By standart Sturm-Liouville theory, since $\int_a^r exp(v(r)) > 0$ and $v(r)$ is continuous and differentiable in $(0, b^{1/2})$, there exist a countable set of real eigenvalues

$$\lambda_1 < \lambda_2 < ....\lambda_n < ... < \infty.$$

and corresponding to each eigenvalue there is a unique eigenfunction $K_n(r)$ with $n-1$ zeros in $(0, b^{1/2})$.

Hence there exist the solutions $(K_k(r), \lambda_k)$, $\ k = 1, 2, 3.....$

□

Remark 6.2. Choosing $C = -\lambda_k$ for any positive eigenvalue, will yield a solution of (6.90), were $C < 0$.

In summary we have constructed the following solutions

Theorem 6.3. *Let $\psi_0 = N(r)f(x, 0)$ and $N(r) = N(|q|^2) \neq M(r)$. Let K be an eigen-solution of the Sturm-Liouville problem (6.92) with data (6.93). Let f satisfy*

$$f_t + u \cdot \nabla f + f\lambda = 0$$

with λ and eigenvalue of the Sturm-Liouville problem (6.92). Let $N = KM$, then

$$\psi = N(r)f(x, t)$$

is a solution to the co-rotational Fokker-Plank equation with data $\psi = N(r)f(x, 0)$.

Proof. Since K satisfies (6.92) , hence K satisfies (6.90) and equivalently it follows that for $N(r) = K(r)M(r)$

$$\frac{1}{2\lambda}\nabla_q \cdot \left(M\nabla_q(\frac{N(r)}{M(r)})\right) N^{-1} = -\lambda_k,$$

and hence $\psi = N(r)f(x, t)$ satisfies (6.85), That is it satisfies the co-rotational Fokker-Plank equation with data $\psi_0 = N(r)f(x, 0)$.

□

Corollary 6.4. Under the conditions of the above theorem it follows that

$$\|\psi_\lambda(t)\|_2 \leq C_o \exp(-\lambda t)$$

where $\psi_\lambda = N_\lambda(r)f$, $\ N_\lambda = K_\lambda M$ and K_λ is the solution to the Sturm-Liouville problemom corresponding to the eigenvalue λ. Here C_o depends on the norms of K_λ and M

Proof. Follows immediately from inequality (6.88)when we take $C = -\lambda$ with λ a positive eigenvalue of the Sturm-Liouville problem (6.92). □

7. Reduction to Navier-Stokes on bounded balls

In this section we show that equation (1.1) reduces to a Navier-Stokes equation provided the elongations of the probability density are radial. As we have shown in the last section in the co-rotational case there are such type of solutions.

Theorem 7.1. *Suppose that the solution to the Fokker-Plank equation is radial in the elongations, that is $\psi(x,q,t) = \psi(x,|q|,t)$, then the corresponding velocity satisfies a Navier-Stokes equation.*

Proof. To show that equation (1.1) converts into Navier-Stokes we compute the matrix $\tilde{C}$ and show that it is a diagonal matrix. Hence the matrix $\tilde{\tau}$ becomes diagonal and $\nabla_x \tilde{\tau} = \nabla_x P$. Thus if we replace in equation (1.1) the pressure p by the new pressure $p - P$ we have a Navier-Stokes equation.

Let $D = B(0,R)$ We only have to calculate the value of

$$J = \int_D \psi(x,|q|,t)U'(\frac{1}{2}|q|^2)qq^T dq$$

Passing to spherical coordinates with $|q| = r$. the last integral reduces to

$$\int_0^{\pi}\int_0^{2\pi}\int_0^R H(r)F(r,\theta,\phi)r^2 \sin\phi \, dr \, d\phi \, d\theta$$

where we have

$$H(r) = \psi(x,|q|,t)U'(\frac{1}{2}|q|^2) \quad \text{and} \quad F(r,\theta,\phi) = (F_{i,j})_{i,j}$$

where $F_{i,j} = q_i q_j$, where in spherical coordinates we have as usual

$$q_1 = r\cos\theta\sin\phi$$
$$q_2 = r\sin\theta\sin\phi$$
$$q_3 = r\cos\phi$$

Hence the integral J "reduces" to nine integrals of the form

$$J_{i.j} = \int_0^R H(r)r^4 \int_0^{\pi}\int_0^{2\pi} \tilde{q_i}\tilde{q_j}\sin\phi \, d\phi \, d\theta \, dr =$$
$$\int_0^R H(r)r^4 dr \int_0^{\pi}\int_0^{2\pi} N_{i,j} \, d\phi \, d\theta$$

with $N_{i,j} = \int_0^{\pi}\int_0^{2\pi} \tilde{q_i}\tilde{q_j}\sin\phi \, d\phi \, d\theta$ where

$$\tilde{q_1} = \cos\theta\sin\phi$$
$$\tilde{q_2} = \sin\theta\sin\phi$$
$$\tilde{q_3} = \cos\phi$$

An easy computation shows that

$$N_{1,1} = \int_0^{\pi}\int_0^{2\pi} (\cos\theta)^2(\sin\phi)^3 \, d\phi \, d\theta =$$
$$\int_0^{\pi}\int_0^{2\pi} \frac{1+\cos(2\theta)}{2}[1-(\cos\phi)^2]\sin\phi \, d\phi \, d\theta =$$
$$\pi(\int_0^{\pi} \sin\phi - (\cos\phi)^2 \, d\phi) = 2\pi - 2/3\pi = 4/3\pi$$

A similar simple computation shows that $N_{22} = 4/3\pi$. The computation of $N_{3,3}$ is even simpler

$$N_{3,3} = \int_0^{\pi}\int_0^{2\pi} (\cos\phi)^2 \sin\phi \, d\phi \, d\theta =$$
$$-2\pi \int_{-1}^{1} u^2 du = 4/3\pi$$

Because of symmetry we have $J_{i.j} = J_{j,i}$, $i, j = 1, 2, 3$. Again straightforward computation give

$$N_{1,2} = N_{2,1} = \int_0^{\pi}\int_0^{2\pi} (\cos\theta)\sin\theta(\sin\phi)^3 \, d\phi \, d\theta = 0$$

since $\int_0^{2\pi}(\cos\theta)\sin\theta d\theta = 0$ The other two integrals follow similarly

$$N_{1,3} = N_{3,1} = \int_0^{\pi}\int_0^{2\pi} (\cos\theta)\cos\phi(\sin\phi)^2 \, d\phi \, d\theta = 0$$

The above is zero since $\int_0^{2\pi}(\cos\theta) \, d\theta = 0$.

Finally

$$N_{2,3} = N_{3,2} = \int_0^{\pi}\int_0^{2\pi} \sin\theta(\sin\phi)^2 \, d\phi \, d\theta = 0$$

since $\int_0^{2\pi}\sin\theta \, d\theta = 0$

Hence it follows that the matrix $\tilde{C}$ is diagonal, and since the second term in $\tilde{\tau}$ also was diagonal, the right hand of (1.1) is a gradient and consequently (1.1) reduce to a Navier-Stokes system.

□

Remark 7.2. In this case the decay of the velocity is well known since we can use all the decay results of solutions to the Navier-Stokes equations.

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Appendix A. Existence

We now give the details for the existence Theorems. The first theorem we will establish is Theorem 3.2. For completeness we will state it again

Theorem A.1. *Let D and Ω be as in Theorem 3.1, $\psi_o \in \mathcal{K}(\Omega \times D)$. Let $w \in C^\infty(\Omega \times R_+)$, and $div_x w = 0$. Then there exists a weak solution ψ of*

$$\frac{\partial}{\partial t}\psi + ((w)\cdot\nabla_x)\psi - \nabla_q\cdot(\sigma((w)q\psi) \tag{A.94}$$
$$= \frac{1}{2\lambda}\nabla_q\cdot(\nabla_q\psi + U'(\frac{1}{2}|q|^2)q\psi) + \mu_o\Delta\psi,\ (x,q,t) \in \Omega\times D,\ t > 0$$

$$\psi = 0 \;\; on \;\; \Omega\times\partial D\times(0,T], \;\; n\cdot\nabla_x\psi = 0 \;\; on \;\; \partial\Omega\times D\times(0,T] \tag{A.95}$$

$$\psi(x,q,0) = \psi_o(x,q) \;\; \forall \;\; (x,q) \; in \; \Omega\times D. \tag{A.96}$$

Proof Let w_j be the a basis of eigenfunctions (with eigenvalues λ_j), corresponding to $\Delta_q\Phi_1(q) = \lambda_j\Phi(q)$ with zero Dirichlet boundary conditions. The w_j can be taken orthonormal in $H_0^1(D)$ and orthogonal in $L^2(D)$. Let v_j be the a basis of eigenfunctions (with eigenvalues γ_j) corresponding $\mu_0\Delta_x\Phi_2(x) = \gamma_j\Phi_2(x)$ with zero Neuman boundary conditions. The v_j can be taken orthonormal in $H^1(\Omega)$ and orthogonal in $L^2(\Omega)$. That is we can take as basis of eigenvectors $\{\Phi_{j,k} = w_j v_k\}_{k,j}$

for the system corresponding to $(\Delta_q+\mu_0\Delta_x)\Phi_1\Phi_2=\Lambda\Phi_1\Phi_2$ with eigenvalues $\Lambda=\Lambda_{j,k}=\lambda_j+\gamma_k$, We are looking for solutions via a Galerkin method, that is we approximate the solutions by

$$\psi_m=\sum_{j,k=1}^{n} d_m^{k,l}(t)w_k(q)v_j(x)$$

We can reorder eigen-functions and rename them as $\{W_k\}$, with eigenvalues Λ_k, hence

$$\psi_m=\sum_{j,k=1}^{n} d_m^{k}(t)W_k$$

We can set the problem to be solved in a weak formulation. Let

$$\begin{aligned}B[\psi,\phi;t]=\int_\Omega\sum_{i=1}^{3}\frac{1}{2\lambda}\psi_{q_i}\phi_{q_i}+\sum_{i=1}^{3}\mu_0\psi_{x_i}\phi_{x_i}\\+\sum_{i=1}^{3}a^i(\cdot,t)\psi\phi_{q_i}+\sum_{i=1}^{3}b^i(\cdot,t)\psi\phi_{x_i}\end{aligned}\tag{A.97}$$

Letting

$$a=\sigma((w)q+\frac{1}{2\lambda}U'(\frac{1}{2}|q|^2)q,\ \ b=w.$$

Hence the weak formulation of (A.94) is

$$(\psi_m',W_k)+B[\psi_m,W_k;t]=0,\ \text{for } 0\le t\le T,\ k=1,...m$$

The last equation can be solved in straightforward fashion, and as such we omit the details.([6]).

To obtain a weak solution we need some energy estimates for ψ_m. This will be done in the next Lemma and Corollary.

Lemma A.2. *Let ψ_m be defined as in the Theorem A.1 then*

$$\frac{d}{dt}\int_{\Omega\times D}\left|\frac{\psi_m}{\sqrt{M}}\right|^2 dq\,dx=-\int_{\Omega\times D}M\left|\nabla_q(\frac{\psi_m}{M})\right|^2 dq\,dx-\mu_0\int_{\Omega\times D}\left|\nabla_x\left(\frac{\psi_m}{\sqrt{M}}\right)\right|^2 dx\,dq\tag{A.98}$$

and

$$\begin{aligned}\int_{\Omega\times D}\left|\frac{\psi_m(t)}{\sqrt{M}}\right|^2+\mu_0\int_0^t\int_{\Omega\times D}\left|\nabla_x\left(\frac{\psi_m}{\sqrt{M}}\right)\right|^2 dx\,dq\,ds\\+\int_0^t\int_{\Omega\times D}\left|\sqrt{M}\nabla_q\left(\frac{\psi_m}{M}\right)\right|^2 dxdq\le\int_{\Omega\times D}\left|\frac{\psi_m(0)}{\sqrt{M}}\right|^2\end{aligned}\tag{A.99}$$

Proof Inequality (A.98) follows by writing the probability equation in Maxwelian form (1.12), using the weak formulation and multiply the equation by d_m^k, sum over $k=1,2..m$ and integrate over $\Omega\times D$ For details see [6], Chapter 7.

We note here that because the drag term is co-rotational after multiplying by d_m^k, summing over $k=1,2..m$ and integrating over $\Omega\times D$, the term vanishes. This is a simple application of te divergence theorem. For details see [3]

Inequality (A.99) follows integrating inequality (A.98) in tine over $[0,t]$

□

Remark A.3. Note that the bound obtained for the LHS of inequality (A.99) is uniform in time and m.

Corollary A.4. Under the hypothesis of the Theorem, it follows that for all $T > 0$

$$\mu_0 \int_{\Omega\times D} |\psi_m(t)|^2 + \int_0^t \int_{\Omega\times D} |\nabla_x \psi_m|^2 \; dx\, dq \leq \int_{\Omega\times D} \left|\frac{\psi_m(0)}{\sqrt{M}}\right|^2 \; dx\, dq \tag{A.100}$$

Proof Follows by inequality (A.99) , since $M \leq 1$.

□

We proceed with the last part of the Theorem, by the Lemma A.2 and Corollary A.4 it follows that we can extract a subsequence, which we call again $\{\psi_m\}$ such that for all$T > 0$ which converges in the following sense

$$\nabla_x \psi_m \to \nabla \psi, \text{ weak} \in L^2([0,T); L^2(\Omega \times D)) \tag{A.101}$$

$$\psi_m \to \psi, \text{ strong} \in L^\infty([0,T); L^2(\Omega \times D)) \tag{A.102}$$

$$\frac{\psi_m}{M} \to \frac{\psi}{M}, \text{ weak } \in L^\infty([0,T); L^2(\Omega \times D)) \tag{A.103}$$

$$\sqrt{M}\nabla_q \left(\frac{\psi_m}{M}\right) \to \Phi, \text{ weak } \in L^2([0,T); L^2(\Omega \times D)) \tag{A.104}$$

By (A.103) and (A.104) it follows that $\Phi = \sqrt{M}\nabla_q \left(\frac{\psi}{M}\right)$

We recall that $\nabla_q M = -MU'q$, hence

$$\sqrt{M}\nabla_q \left(\frac{\psi_m}{M}\right) = \frac{1}{\sqrt{M}} \left[\nabla_q \psi_m + U'q\psi_m\right]$$

Combining the last equality with (A.100) yields, since $M \leq 1$

$$\int_{\Omega\times D} |\nabla_q \psi_m + U'q\psi_m|^2 \; dxdq \leq \int_{\Omega\times D} \left|\frac{\psi_m(0)}{\sqrt{M}}\right|^2 \; dx\, dq \tag{A.105}$$

Hence we have that

$$\sqrt{M}\nabla_q \left(\frac{\psi_m}{M}\right) \to \Gamma \in L^2([0,T); L^2(\Omega \times D)) \tag{A.106}$$

Combining (A.102) and (A.106) yields that $\Gamma = \sqrt{M}\nabla_q \left(\frac{\psi}{M}\right)$ Hence combining (A.102) and (A.106) yields

$$\nabla_q \psi_m \to \nabla_q \psi \in L^2([0,T); L^2(\Omega \times D)) \tag{A.107}$$

By (A.101) and (A.102) and (A.107) it follows that we can pas to the limit of the subsequence ψ_m and the limit will satisfy equation (**??**). Due to the construction it follows that the limit will have the right boundary conditions. The initial data is obtained as usual by being able to select the $d_m^k(0) = \int_{\Omega\times D} \psi_0(x,q), W_k dxdq$.

This concludes the proof of the Theorem.

□

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