Problems Related to Solutions to the Navier-Stokes Equations

Maria Elena Schonbek

Sofia Kowaleskaya Gastprofessorin
Universität Kaiserslautern
Department of Mathematics
University of California, Santa Cruz

Preface

These notes cover various aspects of the behaviour of solutions to the Navier-Stokes equations with special emphasis on their algebraic decay in several Sobolev spaces.

The notes represent lectures I gave during the spring quarter of 1998 while visiting the University of Kaiserslautern as Guest Professor Sofia Kowalevskaya. I am very grateful to Professor H. Neunzert for his invitation and to the many people in the department for their kind hospitality .

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1 Introduction.

Many interesting problems arise from the study of the behavior of fluids. From a theoretical point of view Fluid Dynamics works with a well defined set of equations for which it is expected to get a clear description of the solutions. Unfortunately, in general this is not easy even if the many experiments performed in the field seem to indicate which path to follow. Some of the basic questions are still either partially or widely open. For example we would like to have a better understanding on :

1. Questions for both bounded and unbounded domains on regularity, uniqueness, long time behavior of the solutions.

2. How well do solutions to the fluid equations fit to the real flow.

Depending on the type of data most of the answers to these questions are knonw, when we work in two dimensions. For solutions in three dimensions, in general, we have only partial answers.

In this set of notes we will discuss questions on existence, regularity and long time behavior. The notes will be based on several of my papers [S1], [S2], [S3], [S4], [SS], [SW], [MNPS], and complemented with material from the work of [L], [CKN], [Se], [Pl], [W], [K]. For general theory we will also use results from [La], [S], [M], [P], [C].

We start with some historical background.See [La] for a more detailed discussion. For many years most of the work was done on the so called potential, ideal, incompressible flows. We recall that:

Potential Flow : inviscid, irrotational flow.

Ideal Fluid : fluids such that the stress across its surface is given by $p(x,t)\cdot\vec{n}$, where $p(x,t)$ is a function called pressure and $\vec{n}$ is the normal vector with respect to the surface.

Incompressible Fluid : the volume of any subregion of the flow is constant in time. Incompressibility can be shown to be equivalent to the condition that divergence of velocity is zero. Briefly, a flow is incompressible if

$$0=\frac{d}{dt}\int_{W_t} dV=\frac{d}{dt}\int_W J\ dV=\int_W \frac{\partial}{\partial t}J\ dV=\int_W div\ u\ J\ dV=\int_{W_t} div\ u\ dV$$

for any moving subregions W_t , where $W_0=W$. Note that we used $\frac{d}{dt}J(t)=J(x,t)$, where u is the velocity of the flow and $J(x,t)$ is the Jacobian of the flow map $\Phi_t=\Phi(x,t)=$ trajectory followed by a particle which is at point x at time $t=0$.

Working with potential, incompressible, ideal flows had some problems. One such problems was due to the Euler-D'Alembert paradox:

"Total force acting on an object located on a potential flow was zero".

To resolve this paradox a new set of viscous equations was introduced: The Navier-Stokes equations

$$\begin{aligned} u_t + (u \cdot \nabla)u + \nabla p &= \nu \Delta u + f \\ div\ u &= 0 \\ u(x,0) &= g(x)\ \in \mathbf{X} \end{aligned}$$

The equations above are given in $\mathbf{R}^n$. The space $\mathbf{X}$ will be specified below. If we work on bounded domains or exterior domains a boundary condition needs to be added. We note that the Navier-Stokes equations also have problems.

<u>$Example 1$</u> : Poisselle flow (Paradox 1).

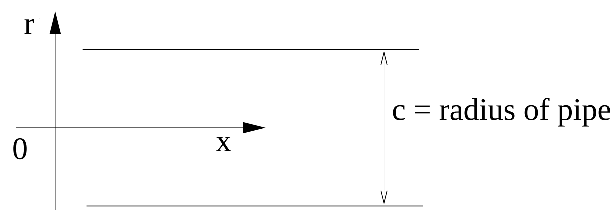

Figure 1: Infinitely long pipe, symmetric with respect to its axis.

This flow has solution for all $R \equiv$ Reynolds number. Note that if $v = v(x, r, \theta)$ in cylindrical coordinates, such that

$$\begin{cases} v_x = & (c^2 - r^2)a \\ v_r = & v_\theta = 0 \end{cases}$$

is a solution, where a is an arbitrary constant.
On the other hand in a real model the Poisselle flow has the following behavior:
There exists R_0 such that

1. For $R \le R_0$ nice flow is observed.

2. For $R \ge R_0$ there is turbulence.

$\underline{Example 2}$: Coriolis flow.

Theoretically solutions to the Coriolis flow exist for all Reynolds numbes R. On the other hand in real flow solution is only observed for small R. For large R the flow is no longer symmetric. The Coriolis flow is described by figure 2.

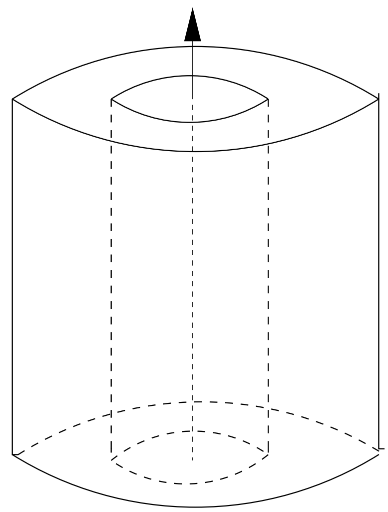

Figure 2: Flow between two rotating coaxial cylinders, which is invariant with respect to rotations along axis and translations along it.

One way in which the above paradox could be solved is to accept that the solutions to Navier-Stokes eqautions are not necessarly unique.
In this series of lectures we will focus on solutions to Navier-Stokes equations in the whole space $\mathbf{R}^n$, $n \geq 2$. We will address questions on existence, regularity and long time behavior of the solutions. The plan of the course is the following :

1. Solutions to the Navier-Stokes $u \in C([0,T], W^{1,2}(\mathbf{R}^3))$ are regular.
2. Existence and regularity for solutions in 2 and 3 dimensions: What is known in each case.
3. We show by Fourier methods that a solution $u \in H^1$ of Navier-Stokes equations with data $u_0 \in L^1 \cap H^1$ belongs to L^∞. Regularity follows easily.
4. Some regularity results : Caffarelli-Kohn and Nirenberg, Serrin and Kato.

5. Caffarelli - Kohn and Nirenberg :Cconstruction of weak solution.

6. Decay of solution in several Sobolev norms.

   a. L^2 decay (Leray' s conjecture).
   b. H^m decay.
   c. L^p decay.
   d. Decay of time derivatives in above norms.
   e. Moment decay.
   f. Pointwise decay.

7. Self-similar solutions. Leray' s conjecture. Non existence results. Cannone' s solution in Besov spaces.

The following notation will be used

$$L^p = \left\{ f \ : \ (\int_{\mathbf{R}^n} | f |^p \ dx)^{1/p} < \infty \right\}$$

$$H^k = W^{k,2} = \left\{ f \ : \ D^s f \in L^2, s \leq k \right\}$$

$$D^k f = \sum_{|\alpha|=k} \frac{\partial^\alpha f}{\partial^{\alpha_1} x_1 \cdots \partial^{\alpha_n} x_n}, \qquad \alpha = (\alpha_1, \cdots, \alpha_n), \ \alpha > 0, \qquad | \alpha | = \sum_{i=1}^n \alpha_i$$

In particular

$$H^0 = L^2$$

$$\partial_i = \frac{\partial}{\partial x_i} = D_i = \nabla_i$$

2 First Estimates.

We recall first that a function $u \in L^\infty((0,T), L^2) \cap L^2((0,T), V)$ with $\dfrac{\partial u}{\partial t} \in L^{4/3}((0,T), V^*)$ is a weak solution to Navier-Stokes if

$$< \frac{\partial u}{\partial t}, \varphi >_{V^*,V} + \int_{\mathbf{R}^3} u_k(t) \frac{\partial u}{\partial x_k}(t) \varphi \ dx + \nu (\nabla u(t), \nabla \varphi) = 0 \tag{1}$$

for all $\varphi \in \mathcal{D}(\mathbf{R}^3)$, satisfying $div \ \varphi = 0$ a.e. $(0,T)$.
Here

$$V = \left\{ u \ : \ u \in W^{1,2}, div \ u = 0 \right\}$$

$$V^* = \Big\{ F \ : \ F : V \to \mathbf{R} \quad \text{are bounded linear functionals} \Big\}$$

Note if $u \in C[t_1,t_2], W^{1,2}(\mathbf{R}^3)$ for $t_2 < T$, $\dfrac{\partial u}{\partial t} \in L^2((t_1,T),V^*)$ and hence (1) is valid for all $\varphi \in V$ and we can use u as a test function in (1).

We first show that:

Lemma 2.1 *Assume u is a solution to Navier-Stokes and*

$$u \in C([t_1,t_2],W^{1,2}) \tag{2}$$

Then

$$u \in L^2([0,T],W^{1,2}) \ , \quad \frac{\partial u}{\partial t} \in L^2([0,T],L^2) \ , \quad \nabla p \in L^2([0,T],L^2) \tag{3}$$

PROOF :

Let $I = (0,T)$. Denote (for $z(t) \in W^{1,2}(\mathbf{R}^3)$)

$$\Delta_r^h z(t) \equiv \frac{z(t,x+he^r) - z(t,x)}{h} \ , \quad r = 1,2,3, \tag{4}$$

where e^r are the unit vectors $(1,0,0),\ (0,1,0),\ (0,0,1)$. From the weak formulation (1) we get

$$< \frac{\partial}{\partial t}\Delta_r^h u(t), \varphi >_{V^*,V} + \nu(\Delta_r^h \nabla u(t), \nabla\varphi) +$$

$$+\frac{1}{h}\int_{\mathbf{R}^3} \Big((u_k \frac{\partial u_i}{\partial x_k})(t,x+he^r) - (u_k\frac{\partial u_i}{\partial x_k})(t,x) \Big) \varphi_i(x)\ dx = 0 \tag{5}$$

Taking $\Delta_r^h u(t)$ in (5) instead of φ , we obtain (for simplicity we write $\Delta_r^h u$ instead of $\Delta_r^h u(t)$)

$$\frac{1}{2}\frac{d}{dt}\|\Delta_r^h u\|_2^2 + \nu\|\Delta_r^h \nabla u\|_2^2 = -\int_{\mathbf{R}^3} \Delta_r^h u_k \frac{\partial u_i(t,x+he^r)}{\partial x_k} \Delta_r^h u_i\ dx \equiv Y,$$

where we used the fact that

$$\int_{\mathbf{R}^3} u_k \frac{\partial(\Delta_r^h u_i)}{\partial x_k}\Delta_r^h u_i\ dx = \frac{1}{2}\int_{\mathbf{R}^3} u_k \frac{\partial\|\Delta_r^h u_i\|^2}{\partial x_k}\ dx = 0,$$

which can be checked after integration by parts since u satisfies (4) and $C(\overline{I};\mathcal{D}(\mathbf{R}^3))$ is dense in $C(\overline{I};W^{1,2}(\mathbf{R}^3))$.

Further, using the interpolation inequality $\|z\|_4 \le \|z\|_2^{1/4}\|z\|_6^{3/4}$, the continuous embedding of $W^{1,2}(\mathbf{R}^3)$ into $L^6(\mathbf{R}^3)$, and the Young inequality, we obtain

$$\begin{aligned} |Y| &\le \|\Delta_r^h u\|_4^2\|\nabla u\|_2 \le \|\Delta_r^h u\|_2^{1/2}\|\nabla u\|_2\|\Delta_r^h u\|_6^{3/2} \le \\ &\le \|\nabla\Delta_r^h u\|_2^{3/2}\|\Delta_r^h u\|_2^{1/2}\|\nabla u\|_2 \le \frac{\nu}{2}\|\Delta_r^h\nabla u\|_2^2 + c\|\nabla u\|_2^4\|\Delta_r^h u\|_2^2 \ . \end{aligned}$$

Hence

$$\frac{d}{dt}\|\Delta_r^h u\|_2^2 + \nu\|\Delta_r^h\nabla u\|_2^2 \le c\|\nabla u\|_2^4\|\Delta_r^h u\|_2^2 \ .$$

Since $\|\nabla u\|_2^4 \in L^1(I)$ due to the hypothesis, we obtain by the Gronwall inequality

$$u \in L^2(I; L^2(\mathbf{R}^3)) \ . \tag{6}$$

Analogously, applying the difference quotient method with respect to t , we get from (1) (formally by testing by $\dfrac{\partial u}{\partial t}$)

$$\frac{\partial u}{\partial t} \in L^2(I; L^2(\mathbf{R}^3)) \ . \tag{7}$$

Hence, using Navier-Stokes, (6), and (7) we have

$$\nabla p \in L^2(I; L^2(\mathbf{R}^3)) \ . \tag{8}$$

Hiking up the regularity of the solution can be done formally. We show later on that weak solutions to Navier-Stokes can be constructed as limits of linearization. That is let $u = u_k,\ v = u_{k+1}$ and consider

$$\begin{aligned} v_t + (u\cdot\nabla)v + p_{k+1} &= \nu\Delta v \\ div\ v &= 0 \ . \end{aligned}$$

Then if

$$\|u_k\|_{H^m} \le C(m, \text{data}, T) = C_0$$

and u is the weak L^2-limit of u_k , then

$$\|u\|_{H^m} \le C_0$$

Here is why if

$$\|u_k\|_{H^m} \le C_0$$

then there exists a subsequence of $D^m u_k$ such that

$$D^m u_{k,j} \rightharpoonup y \in L^2 \quad \forall\, \varphi \in C_0^\infty \quad \text{(convergence is weak in } L^2\text{)}$$

and thus $\varphi \in L^2$:

$$< D^m u_{k,j}, \varphi > \longrightarrow < y, \varphi >, \quad < D^m u_{k,j}, \varphi > \longrightarrow < u_{k,j}, D^m \varphi > = \text{weak m-derivative of } u$$

Moreover, letting $\varphi = y$ it follows that $| < D^m u_{k,j}, y > | \le \|D^m u_{k,j}\|_2 \|y\|_2 \le C_0 \|y\|_2$. Taking the limits it follows that $\|y\|_2^2 \le C_0 \|y\|_2$ and thus $\|y\|_2 \le C_0$. Hence the result follows.

Moreover, notice that since $H^s \hookrightarrow C^k$ where $s > k + 1/2$, i.e. in our case if we want C^2 we only need $s > 1 + 1/2$, $s = 4$ suffices.

Let $u_0 \in H^k,\ div\ u = 0$.

Lemma 2.2 *Let* $u \in C((0,T); W^{1,2})$ *be a solution to Navier-Stokes. Then* $u \in W^{k,2}$.

PROOF :

The proof follows by first obtaining the result for approximating solutions and then passing to the limit.

3 Higher Order Estimates, via Fourier Transform

Estimate of higher norms can be obtained easier by using the Fourier transform. For this we first show the following auxiliary estimate.

Lemma 3.1 *Let* u *be a solution to Navier -Stokes such that* $u \in C([0,\infty), H^1)$. *Then if* $\hat{u_o} \in L^1 \Rightarrow \hat{u}(t) \in L^1$ *for* $t \ge 0$.

PROOF :
Remark if $u_0 \in H^1$, then it is known that for t large enough $u \in C([T_0,\infty), H^1)$ so the result is valid for $t > T_0$.
Step 1: Take the Fourier transform of the equation

$$\frac{d}{dt}\widehat{u}(\xi,t) + \widehat{u \cdot \nabla u} + \widehat{\nabla p} = -|\xi|^2 \widehat{u}.$$

Thus:

$$\widehat{u}(\xi,t) = e^{-|\xi|^2 t}\widehat{u_0}(\xi) - \int_0^t e^{-|\xi|^2(t-s)}(\widehat{u \cdot \nabla u} - \widehat{\nabla p}) \tag{9}$$

Note that taking the divergence of the Navier-Stokes equations yields

$$\Delta p = -\sum \partial_i \partial_j u_i u_j .$$

Thus the Fourier transform yields

$$-|\xi|^2 \widehat{p} = -\sum \xi_i \xi_j \widehat{u_i u_j}$$

and thus

$$|\widehat{\nabla_k p}| = |\xi_k \widehat{p}| \leq \frac{\sum \xi_i \xi_j \xi_k \widehat{u_i u_j}}{|\xi|^2} \leq c|u \cdot \nabla u|$$

Thus from (9) by integration and last inequality it follows that

$$\int_{\mathbf{R}^3} |\widehat{u}| d\xi \leq \int_{\mathbf{R}^3} |e^{-|\xi|^2 t} \widehat{u_0}| + c \int_{\mathbf{R}^3} \int_0^t e^{-|\xi|^2 t} |\widehat{u \cdot \nabla u}| \ ds$$

Applying Hölder inequality to the integrals on the right hand side

$$\|\widehat{u}\|_{L^1} \leq \|u_0\|_1 + c \int_0^t \left(\int_{\mathbf{R}^3} e^{-2|\xi|^2 (t-s)} \right)^{1/2} \|\widehat{u \cdot \nabla u}\|_2 \ ds$$

Note that we changed the order of integration. By Young's inequality and properties of the Fourier transform

$$\|\widehat{u \cdot \nabla u}\|_2 = \|\widehat{u} * \widehat{\nabla u}\|_2 \leq \|\widehat{u}\|_{L^1} \ \|\nabla u\|_{L^2}$$

Thus

$$\|\widehat{u}\|_{L^1} \leq \|\widehat{u_0}\|_{L^1} + c \int_0^t \frac{1}{(t-s)^{3/4}} \|\widehat{u}\|_1 \ \|\nabla u\|_2 \ ds \tag{10}$$

Here we used

$$\left(\int_{\mathbf{R}^3} e^{-2|\xi|^2 (t-s)} \ ds \right)^{1/2} \leq \frac{c}{(t-s)^{3/4}}.$$

By (9), (10) and standard fixed point arguments it follows that $\|\widehat{u}\|_{L_1}$ is bounded at least for small t . By an extension to Gronwall inequality we have

Lemma 3.2 *Let* $\phi(t)$ *satisfy* $\phi(t) \geq 0$ *and*

$$\phi(t) \leq A_n + B_n \int_o^t \frac{1}{(t-s)^{n/4}} \phi(s) \ ds \qquad t \in [0, T_0]$$

where $n = 2$ *or* 3 . *Then* $\phi(t) \leq 2 A_n exp \ \epsilon^{(1-n/4)T_0}$ *and* $\epsilon^{1-n/4} B_n = \frac{1}{2}(1 - n/4)$.

For a proof see Estimates for the pressure and the Fourier Transform for solutions and derivatives to the Navier-Stokes equations [S1].

In more general fashion we can get the

Lemma 3.3 *If* $2 \le \ n \le 5$ *and* $D^2 u \in C_0([T_0 + \frac{3\epsilon}{4}, \infty)\ L^2(\mathbf{R}^2))$ *, then* $\widehat{u}(t) \in L^1$ *for* $t > T_0 + \frac{3\epsilon}{4}$ *.*

Proof :

Let $t_m = T_0 + \varepsilon(1 - 2^{-m})$. We may represent the solution with the help of the Fourier transformation by

$$\widehat{u_i}(t + t_m) = (\delta_i j - \xi_i \xi_j |\xi|^{-2})\Big(e^{-t|\xi|^2}\widehat{u_j}(t_m) - \int_0^t e^{-(t-s)|\xi|^2} i\xi_k \widehat{u_j u_k}(s + t_m)\ ds\Big). \qquad (11)$$

As

$$\|(1 + |\xi|^2)(\widehat{u_j u_k})\|_2 = \|(I + \Delta)u_j u_k\|_2 \le C(\|u\|_2 + \|\nabla^2 u\|_2)\ \|u\|_\infty \le C_\varepsilon(t) \quad \text{for } t \ge t_2$$

we get

$$\begin{aligned} \|\widehat{u}(t + t_2)\|_1 \ &\le Ct^{-n/2} + c \int_0^t \Big(\int_0^\infty \frac{e^{-2(t-s)r^2} r^{n+1}}{(1 + r^2)^2}\ dr\Big)\ ds \\ &\le Ct^{-n/2} + c \int_0^t (t - s)^{-(n-2)/4}\ ds < \infty \end{aligned}$$

for $t > 0$ due to $n \le 5$.

From where we get

Lemma 3.4 *If* $2 \le \ n \le 5$ *, under the hypothesis of the last lemma* $D^\alpha u, D^\alpha u_t \in L^2(\mathbf{R}^2)$ *for all multi-indices* α *, provided* $t > T_0$ *.*

Proof :

The plan is to estimate (by induction) $\|\xi^m \widehat{u}\ \|_2 = \|D^m u\|_2$.
First note $|\xi|^q |(\widehat{u_j u_k})(\xi)| \le 2c_q(|\xi|^q |\widehat{u}(\xi)| * |\widehat{u}(\xi)|)$ due to $|\xi|^q \le (|\xi - \mu|^q + |\mu|^q)$. Hence the estimate for convolutions implies

$$\|\ |\xi|^q \widehat{u_j u_k}\ \|_2 \le 2c_q \|\widehat{u}\|_1\ \|\ |\xi|^q \widehat{u}\ \|_2$$

Assuming by induction that $\| \, |\xi|^{m-1/2} |\widehat{u}(t+t_m)\|_2 \leq C(s, m-1/2) < \infty$ for $s \geq t_m$, we may multiply (11) by $|\xi|^m$ to get the bound

$$\| \, |\xi|^m \widehat{u}(s)\|_2 \leq 2c_m \; t^{-1/2} + c_m \int_0^t (t-s)^{-3/4} \; ds \leq C(t,m) < \infty$$

for $t > 0$.

Thus $\| \, |\xi|^m \widehat{u}(t+t_m)\|_2$ is finite for all m and $t > 0$. The same reasoning applies after differentiating (11) with respect to t , which proves the claim.

From the above results we see that if $u \in C([0,\infty), W^{1,2})$ we have a simple proof of regularity (ie. $u(x,t) \in W^{k,2}$) if
$u \in C([0,\infty[, W^{1,2})$, $\hat{u_o} \in L^1$ and $u_o \in H^1(\mathbf{R}^n), n \leq 5$

In the next section we will discuss under which conditions the solution will be bounded in $H^1 = W^{1,2}$ and L^∞ (Note $\|u\|_\infty \leq \|\widehat{u}\|_{L^1}$).

4 Regularity.

It is well known that solutions of the Navier-Stokes equations are regular for a short period of time if we start with sufficiently smooth data. The time period for $n \geq 3$ will depend on the norms of the data. For $n = 2$ there is global regularity.

Based on the results of the last section we want to show that $u \in C([0,T), H^1)$, for some $T = T(data)$ for n dimensional solutions $n \geq 3$, and $T = \infty$ if $n = 2$. More precisely the plan now is to show

1. Regularity for all time if $n = 2$.
2. Regularity for $n = 3$ for $t < T_0$ where $\int_{\mathbf{R}^3} |u_0|^2 \int_0^t \int_{\mathbf{R}^3} |\nabla u|^2 < 1$ if $t < T_0$.

For completeness we recall a result of Leray [L]

Definition 4.1 *u is a regular solution in the sense of Leray in $(0,T)$ if u, u_t, u_{x_i}, $u_{x_i x_j}$ are continuous with respect to (x_1, x_2, x_3, t) and $\|u(t)\|_{L^2}$, $\|\nabla u\|_{L^2}$ are bounded by continuous functions of t .*

Theorem 4.1 *(Leray)*
If $u_i(x,t)$ is a regular solution to the Navier-Stokes equations for $0 < t < T$, then all its partial derivatives exist and are uniformly bounded functions in $(0,T)$.

PROOF :

By induction.

Remark 4.1 *(Main remark)*
Solution can be expressed as

$$u(x,t) = K * u_o - \frac{\partial}{\partial x_i} \int_{t_0}^{t} \int_{\mathbf{R}^n} T_{ij}(x-y, t-s)\ u_i\ u_j\ dy\ ds \tag{12}$$

where $K(x,t) = \dfrac{1}{(4\pi t)^{3/2}}\ exp\Big(\dfrac{-|x|^2}{4s}\Big)$ *is the heat kernel and* T *is a sum of the heat kernel and the Riesz operators (these will be defined later).*

The proof consists in obtaining from (12) a representation of the derivatives. Using bounds of the heat kernel and Riesz transform the L^∞ and L^2 bounds of derivatives will follow.

Theorem 4.2 *Let* $u_0 \in H^1(\mathbf{R}^3)$ $div\ u = 0$ *. Let* $u(x,t)$ *be a solution to the Navier-Stokes equations with data* u_0 *. Then*

1. *If* $n = 2,\ u(x,t) \in H^1$ *for all* $t > 0$ *.*
2. *If* $n = 3,\ \ u(x,t) \in H^1$ *for* $t < T_0,\ T_0 = T_0(|u_0|_{L^1}, |\nabla u_0|_{L^1})$ *.*

PROOF :

We use Prodi's inequality

$$\frac{d}{dt} \int_{\mathbf{R}^n} |\nabla u|^2\ dx \leq \Big(\int_{\mathbf{R}^n} |\nabla u|^2\ dx \Big)^n$$

To obtain this result one can use energy estimates. These results of (2.1) give essentially the result for $n = 3$. With Prodi's inequality in hand let $\varphi = \int_{\mathbf{R}^n} |\nabla u|^2\ dx$. Thus for $n = 2$ we obtain for $\frac{d}{dt}\varphi \leq \varphi^2$. Thus $\frac{d\varphi}{\varphi} \leq \varphi\ dt$ which yields after integration

$$ln\ \varphi(t) - ln\ \varphi_0 \leq \int_0^t \varphi\ dt = \int_0^t \int_{\mathbf{R}^2} |\nabla u|^2\ dx\ ds \leq C_0.$$

Thus

$$\int_{\mathbf{R}^2} |\nabla u|^2 \leq \int_{\mathbf{R}^2} |\nabla u_0|^2\ expC_0 \tag{13}$$

When we work in dimensions hugher than 2 we work with Leray-Hopf solutions i.e. solutions which satisfy the energy inequality

$$\int_{\mathbf{R}^n} |u|^2 + \int_0^t \int_{\mathbf{R}^n} |\nabla u|^2 \leq \int_{\mathbf{R}^n} |u_0|^2 \tag{14}$$

Here $n \geq 2$. Such solutions can be found by the constructions of [CKN] or Faedo-Galerkin methods [T].

Remark 4.2 *The computations for bounds of the H^1 norm in 2 dimensions will give regularity in 3 dimensionsprovided that* $\int_0^t \left(\int_{\mathbf{R}^3} |\nabla u|^2 \right)^2 ds < \infty$ *. This was already remarked in Leray 1934 paper [L].*

The above derivation applied to 3 dimensions will yield only local regularity. In this case we have

$$\frac{d}{dt}\varphi \leq \varphi^3 \tag{15}$$

Thus we have

$$\frac{1}{\varphi^2} d\varphi \leq \varphi \; dt$$

Thus

$$\frac{1}{\varphi_0} - \frac{1}{\varphi} \leq \int_0^t \varphi \; ds = \int_0^t \int_{\mathbf{R}^3} |\nabla u|^2 \; dx \; ds$$

$$\varphi(t) \leq \left(\frac{1}{\varphi_0} - \int_o^t \int_{\mathbf{R}^3} |\nabla u|^2 \right)^{-1} = \frac{\varphi_0}{1 - \int_o^t \int_{\mathbf{R}^3} |\nabla u|^2 dx dt \; \int_{\mathbf{R}^3} |\nabla u_0|^2} dx.$$

Thus we have that $\int_{\mathbf{R}^3} |\nabla u|^2 dx$ is bounded provided that $\int_{\mathbf{R}^3} |\nabla u_0|^2 dx \int_0^t \int_{\mathbf{R}^3} |\nabla u|^2 \; dx \; ds <$ 1 , i.e. provided $t < T_0$ where $\int_{\mathbf{R}^3} |\nabla u_0|^2 dx \int_0^{T_0} \int_{\mathbf{R}^3} |\nabla u|^2 \; dx \; ds = 1$.

Corollary 4.1 *Suppose* u_0 *is such that* $\int_{\mathbf{R}^3} |u_0|^2 dx \int_{\mathbf{R}^3} |\nabla u_0|^2 dx < 1$ *. Then* $u \in C([0,\infty), H^1)$.

PROOF :

Follows by last lemma noting that

$$\int_0^t \int_{\mathbf{R}^3} |\nabla u_0|^2 \ dx \ ds \leq \int_{\mathbf{R}^3} |u_0|^2 dx$$

from the energy estimate (14).

5 Serrin's Result.

In this section we just recall some regularity results due to Serrin and Kato.
We first recall the definition of weak solutions to the Navier-Stkes eqaution

Definition 5.1 *u is a weak solution if for all $\phi \in C_0^\infty(\mathbf{R}^n \times [0,t])$, div $\phi = 0$*

$$\int_{\mathbf{R}^n} (u, \phi_t) \ ds + \int_{\mathbf{R}^n} (u, \Delta\phi) \ ds + \int_{\mathbf{R}^n} (u, u \cdot \nabla\phi) \ ds = 0$$

Next we describe Serrin's results [Se]

Theorem 5.1 *(Serrin)*
If u is a weak solution to the Navier-Stokes equations in an open region (space-time), with $u \in L^{2,\infty}$, $\omega = curl\ u \in L^{s,s'}$ where $\frac{n}{s} + \frac{2}{s'} < 1$, then $u \in C^\infty$ and all derivatives are bounded in compact regions.

PROOF :
Here

$$L^{p,q}(\mathbf{R}^n) = \Big\{ u : \Big(\int_0^t \Big(\int_{\mathbf{R}^n} |u|^p \ dx \Big)^{q/p} dt \Big)^{1/q} < \infty \Big\}$$

For a proof see [Se]

The above theorem was extended by Takahashi to $\frac{n}{s} + \frac{2}{s'} = 1$.

In Serrin theorem it is essential that the data in H^1 . Otherwise it is possibel to construct a solution which is not in H^1 .

<u>Serrin's " Non H^1 Example"</u>

1. Let $a(t)$ be an integrable function.
2. Let ψ be harmonic, i.e. $\Delta\psi = 0$.

3. Define $u = (u_1, u_2, u_3)$ by $u_j(x,t) = a(t)\partial_j\psi$.

Then u does not belong to H^1 and is a weak solution to Navier-Stokes. To see that we compute

$$\mathrm{I} = \int_0^t \sum_j \int_{\mathbf{R}^n} a(t)\ \partial_j\psi\ \Phi_t^j = \int_0^t \sum \int_{\mathbf{R}^n} a(t)\ \psi\ div\ \Phi_t = 0$$

$$\mathrm{II} = \int_0^t \int_{\mathbf{R}^n} a(t)\partial_j\psi \sum_i \partial_{ii}\Phi_j = \int_0^t a(t) \int_{\mathbf{R}^n} \partial_j \Delta\psi\ \Phi_j = 0$$

$$\mathrm{III} = \int_0^t \int_{\mathbf{R}^n} a(t)\partial_j\psi \sum_i a(t)\ \partial_i\psi\ \partial_i\Phi_j =$$

$$-\int_0^t a(t)^2 \int_{\mathbf{R}^n} \sum_i \partial_i\partial_j\psi\ \partial_i\psi\Phi_j - \int_0^t a(t)^2 \int_{\mathbf{R}^n} \partial_j\psi \sum_i \partial_{ii}\psi\Phi_j =$$

$$\int_0^t a(t)^2 \int_{\mathbf{R}^n} \sum_i \partial_j \left[\frac{\partial_i\psi}{2}\right]^2 \Phi_j = \int_0^t a(t)^2 \int_{\mathbf{R}^n} \sum_i \left[\frac{\partial_i\psi}{2}\right]^2\ \partial_j\Phi_j = 0$$

Since we sum over j , we get $\sum \partial_j\Phi_j = div\ \Phi = 0$. Summing

$$\mathrm{I} + \mathrm{II} + \mathrm{III} = 0$$

yields that $a(t)\ \partial_j\psi$ is a weak solution. Note that this solution can actually be in C^∞ solution.

We now recall Kato's result. We state here Cannone's [C] version Kato's famous result.

Theorem 5.2 *Let q be fixed, $3 < q \leq 6$ and $|alpha = 1 - \frac{3}{q}$. There exists an absolute constant $\delta > 0$ such that for all initial data $v_O \in L^3(\mathbf{R}^3) with$ $\|v_O\|_3 < \delta$ and $\nabla \cdot v_o = 0$ then there exist an unique global "mild" solution (ie solution to corresponding integral equation) v(x.t) of the Navier-Stokes equations such that*

$$v(x,t) \in C([0,\infty); L^3(\mathbf{R}^3) \tag{16}$$

$$t^\alpha/2v(x,t) \in C([0,\infty); L^q(\mathbf{R}^3) \tag{17}$$

$$lim_{t\to 0} t^\alpha/2\|v(x,t)\|_q = 0 \tag{18}$$

Proof: See Kato [K] , [C].

6 Singular Set : Hausdorff and Parabolic Hausdorff Measures

As we have seen in previous sections it is still an open question if solutions to the Navier-Stokes equations in more than two spatial dimensions are regular. For this reason the question arose on how to estimate the measure of the possible singular set [Sc], [CKN].

Definition 6.1 *The singular set S is defined as*

$$S = \Big\{(x,t) : u \textit{ is a solution to Navier-Stokes}, u \notin L^{\infty}_{loc} \textit{ in any neighbourhood of } (x,t)\Big\}$$

Definition 6.2 *The regular points for Navier-Stokes equations are the elements of* S^c *.*

The measures in which the singular set S was estimated were Hausdorff measures or parabolic measures [CKN].

In the next few pages we present a short introduction to Hausdorff measures. This part of the notes is a summary of results presented in [M], [F]. We recall first some useful definitions.

Definition 6.3 *(diameter of a set)*
For all $S \subset \mathbf{R}^n$ *we define the diameter of* S *as*

$$diam\ S = sup\{|x-y| : x, y \in S\}$$

Definition 6.4 *Measure or outer measure in* $\mathbf{R}^n$ *:*

i) nonnegative function $\mu : \mathcal{P}(\mathbf{R}^n) \longrightarrow \mathbf{R}_+$ *, where* $\mathcal{P}(\mathbf{R}^n) = \{A : A \textit{ subset of } \mathbf{R}^n\}$ *, and values of* $\mathbf{R}_+$ *include infinity.*

ii) μ *is countably sub-additive, i.e.*

$$\mu(A) \leq \sum_i \mu(A_i) \qquad \textit{for } A = \bigcup_i A_i.$$

Definition 6.5 *Measurable sets:*

$$\mathcal{M} = \{A : A \subset \mathbf{R}^n,\ \mu(A \cap E) + \mu(A^c \cap E) = \mu(E),\ \forall\ E \subset \mathbf{R}^n\}$$

Definition 6.6 σ*-algebra:*
Family of sets closed under complementation, countable unions and intersections.

Definition 6.7 *Borel set:*
The smallest σ*-algebra containing all open sets.*

Definition 6.8 *Borel regular measure:*
If all Borel sets are measurable and every subset of $\mathbf{R}^n$ *is contained in a Borel set of the same measure.*

Remark 6.1 *If* $A = \bigcup_i A_i$, $A_i \cap A_j = \emptyset$, $i \neq j$ *, then* $\mu(A) = \sum_i \mu(A_i)$.

Notation: α_m the Lebesgue's measure of the m-dimensional closed unit ball.

Definition 6.9 *Lebesgue's measure:*
Unique Borel regular translation invariant measure in $\mathbf{R}^n$ *, such that* $\mu([0,1]^n) = 1$.

Definition 6.10

$$\mathcal{H}^m_\delta(A) = \inf_{\substack{A \subset \bigcup S_j \\ diam\ (S_j) \leq \delta}} \sum \alpha_m \left[\frac{diam\ (S_j)}{2}\right]^m$$

Definition 6.11 *m-dimensional Hausdorff measure:*

$$\mathcal{H}^m(A) = \lim_{\delta \to 0} \mathcal{H}^m_\delta$$

Definition 6.12 *k-dimensional parabolic measure* $\mathcal{P}^k$ *:*
Define

$$\mathcal{P}^k(X) = \lim_{\delta \to 0^+} \mathcal{P}^k_\delta(X)$$

where

$$\mathcal{P}^k_\delta(X) = inf\left\{\sum \tau_i^k : X \subset \bigcup Q_{\tau_i},\ \tau_i \leq \delta\right\}$$

$$Q_{\tau_i} = \{(y,\tau) : |x-y| \leq \tau_i,\ |t-\tau| \leq {\tau_i}^2\}$$

Example 1: (see [M])
Let $I = [0,1] \subset \mathbf{R}^1$. Show that $\mathcal{H}^1(I) = 1$.

Covering by n intervals of length $1/n$, show that $\mathcal{H}^1(I) \leq 1$. Suppose $\mathcal{H}^1(I) < 1$. Then there is a covering $\{S_j\}$ of I with

$$\sum diam\ S_j < 1.$$

By slightly increasing each $diam\ S_j$ if necessary, we may assume that the S_j are open intervals (a_j, b_j) . Since I is compact, we may assume that none contains another. Finally we may assume that $a_1 < a_2 < \cdots < a_n$ and hence $b_j > a_{j+1}$. Now

$$\sum_{j=1}^n diam\ S_j = \sum_{j=1}^n (b_j - a_j) \geq \sum_{j=1}^{n-1} (a_{j+1} - a_j) + (b_n - a_n) = b_n - a_1 > 1,$$

the desired contradiction.

Definition 6.13 *Hausdorff dimension:*
Let A be a nonempty set.

$$dim_{\mathcal{H}}(A) = inf\{m \geq 0 : \mathcal{H}^m(A) < \infty\} = inf\{m \geq 0 : \mathcal{H}^m(A) = 0\} =$$
$$= sup\{m \geq 0 : \mathcal{H}^m(A) > 0\} = sup\{m \geq 0 : \mathcal{H}^m(A) = \infty\}.$$

The equivalence of the definitions follows by
Example 2: (see [M])
Let $A \neq \emptyset$, $A \subset \mathbf{R}^n$, $0 \leq m \leq k$, $\mathcal{H}^m(A) < \infty$. Then $\mathcal{H}^k(A) = 0$.

For each $\delta > 0$, there is a cover $\{S_j(\delta)\}$ of A with $diam\ S_j(\delta) \leq \delta$ and

$$\sum \alpha_m \left(\frac{diam\ S_j(\delta)}{2}\right)^m \leq \mathcal{H}^m(A) < \infty$$

Consequently,

$$\lim \sum \alpha_k \left(\frac{diam\ S_j(\delta)}{2}\right)^k \leq \frac{\alpha_k}{\alpha_m}\mathcal{H}^m(A) \lim \delta^{k-m} = 0$$

Therefore $\mathcal{H}^k(A) = 0$.

From here it follows that for a fixed set A , there exists a nonnegative number d such that

$$\mathcal{H}^m(A) = \begin{cases} \infty & \text{if } 0 \leq m < d \\ 0 & \text{if } d < m < \infty \end{cases}$$

All four definitions of the Hausdorff measure of A yield d . Incidentally, $\mathcal{H}^d(A)$ could be anything : 0 , ∞ , or any positive real number, depending on what A is.

We also note that one can show

Lemma 6.1 $\mathcal{H}^n = \mathcal{L}^n$ *on* $\mathbf{R}^n$ *.*

PROOF :
See [M].

Hausdorff Measure of Singular Set.

Theorem 6.1 *Scheffer [Sc]*
Let u be a weak solution to the Navier-Stokes in $\Omega \times \mathbf{R}_+$, $\Omega \subset \mathbf{R}^3$ equations with zero external force. Let S be its singular set. Then

$$\mathcal{H}^{3/2}(S) \leq \infty$$

$$\mathcal{H}^1(S \cap \Omega \times \{t\}) \leq \infty \quad \textit{uniformly in } t.$$

We now proceed to recall the estimate on parabolic Hausdorff measures due to Caffarelli Kohn and Nirenberg. For this we need some auxiliary definitions.

Definition 6.14 *Suitable Weak Solution:*
(u, p) is called a suitable weak solution of Navier-Stokes equations on an open set $D \subset \mathbf{R}^3 \times \mathbf{R}_+$ with force f if

1. *u , p , f are measurable on D and $f \in L^q(D)$, $q > 5/2$, $\nabla \cdot f = 0$.*
2. *$p \in L^{5/4}(D)$.*
3. *For some E_0 , E_1*

   i) $\displaystyle\int_{D_t} |u(x,t)|^2 \, dx \leq E_0 \ , \quad D_t = D \cap [\mathbf{R}^3 \times \{t\}] \ .$

   ii) $\displaystyle\int \int_D |\nabla u|^2 \, dx \leq E_1 \ .$

4. *Generalized energy inequality (GEI):*
   For all $\Phi \in C_0^\infty(D)$, $\Phi \geq 0$

$$2 \int \int_D |\nabla u|^2 \ \Phi \ dx \ ds \leq \int \int_D |u|^2 \ (\Phi_t + \Delta \Phi) \ dx \ ds +$$
$$\int \int_D (|u|^2 + 2p) \ u \cdot \nabla \Phi + 2(u \cdot f)\Phi \ dx \ ds$$

Note GEI is obtained formally by multiplying Navier-Stokes equations by $u \cdot \Phi$ and integrating over D . We remark that it is easy to show if $u \in L^2(\mathbf{R}^3) \cap H^1(\mathbf{R}^3)$, $u \in L^{10/3}(D)$ and hence $p \in L^{5/3}(D)$ ([CKN]). This follows by Sobolev inequality

$$\int_{B_r} |u|^q \ dx \leq C(\int_{B_r} |\nabla u|^2 \ dx)^a \ (\int_{B_r} |u|^2 \ dx)^{q/2-a} + \frac{C}{r^{2a}} \ (\int_{B_r} |u|^2 \ dx)^{q/2}$$

where C is a constant independent of r , B_r the ball of radius r , $2 \leq q \leq 6$, $a = 3/4(q-2)$. If u has mean zero or $B_r = \mathbf{R}^3$, then the second term in right hand side can be omitted. In our case let $q = 10/3$, $a = 1$ and integrate in time the last inequality

$$\int \int_D |u|^{10/3} \ dx \ ds \leq C \ E_0^{2/3} \ E_1.$$

Now recall (taking divergence of the Navier-Stokes equations)

$$\begin{cases} \Delta p = & -\sum \partial_i \partial_j u_i u_j \\ \nu \cdot \nabla p = & \sum \nu_i \ \Delta u_i \quad on \quad \partial\Omega \times (0,t) \end{cases}$$

If $\Omega \equiv \mathbf{R}^3$ then

$$P = \sum R_i R_j (u_i u_j)$$

where R_i are Riezs operators, i.e.

Definition 6.15 *Riesz Operator:*

$$R_i \widehat{f}(\xi) = \frac{\xi_i}{|\xi|} \widehat{f}$$

These operators are Calderon Zigmund operators. See [S]. Hence we have they are bounded in L^q

$$\int_{\mathbf{R}^n} \int_D |p|^q \, dxdt = \sum \int \int_D |R_i R_j (u_i u_j)|^q dxdt \leq \sum C \int \int |u_i u_j|^q dxdt \leq C \int \int_D |u|^{2q} dxdt$$

Thus for $q > 1$, in particular $q = 5/3$

$$\int \int_D |p|^{5/3} dxdt \leq C \int \int_D |u|^{10/3} dxdt \leq C(E_0, E_1)$$

Now we state the C.K.N. result on parabolic Hausdorff measure for singular set for suitable weak solutions to the Navier-Stokes equations.

Theorem 6.2 *[CKN]*
For any suitable weak solution to the Navier-Stokes equations on open space time set,

$$\mathcal{P}^1(S) = 0.$$

7 Construction of a Weak Solution.

The construction we use was presented in [CKN]. We include this construction for completness. There are several such constructions, see for example [SWW] and [L].
We use is the following notation :

$H_0^1(\Omega)$ closure of $C_0^\infty(\mathbf{R}^3, \Omega)$ in $\left(\int_{\mathbf{R}^3} |\nabla u|^2 \right)^{1/2}$;

$H^{-1}(\Omega)$ dual of $H_0^1(\Omega)$;

$\mathcal{V} = C_0^\infty(\mathbf{R}^3, \Omega) \cap \{u : div \ u = 0\}$;

V closure of $\mathcal{V}$ in $H_0^1(\Omega)$;

H closure of ν in $L^2(\Omega)$;
V' dual of V ;
$D = \Omega \times (0,T)$;
$E_0(u) = ess\ sup_{0<t<T} \int |u|^2$;
$E_1(u) = \int_0^T \int_\Omega |\nabla u|^2$.

We will work in general with $\Omega = \mathbf{R}^3$. The conditions on f and u_o are as follows

$$\text{(F.1)} \qquad f \in L^2(0,T;H^{-1}(\mathbf{R}^3)) \qquad \nabla \cdot f = 0.$$

$$u_0 \in H^1 \cap W^{2/5}_{5/4}(\Omega).$$

These conditions imply that $\int_{\mathbf{R}^3} (u \cdot f)$ is defined if $u \in H^1_0(\mathbf{R}^3)$.

For future use we also recall the definitions

$$W^s_p = \{f \in S' : \|f\|^s_p < \infty\}$$

where

$$\|f\|^s_p = \|\mathcal{I}^s f\|_p$$
$$\mathcal{I}^s f = \mathcal{F}^{-1}[(1+|\cdot|^2)^{5/2}\mathcal{F}(f)]$$
$$\mathcal{F}(\mathcal{I}^s f) = (1+|\xi|^2)^{5/2}\widehat{f}$$

Hence if $f \in W^s_p$, then $\|\mathcal{F}^{-1}[(1+|\xi|^2)^{5/2}\widehat{f}]\|_p < \infty$. The main theorem is then

Theorem 7.1 *(See [CKN])*
For $\Omega = \mathbf{R}^3$, f and u_0 satisfy $(F.1)$ and $(u_0, 1)$, respectively. Then there exists a weak solution (u,p) to the Navier-Stokes equations on D with force f satisfying

$$u \in L^2(0,T;V) \cap L^\infty(0,T;H), \tag{19}$$

$$u(t) \longrightarrow u_0 \ \textit{weakly in } H \textit{ as } t \longrightarrow 0, \tag{20}$$

$$\begin{array}{c} p \in L^{5/3}(D) \ \textit{in case } \Omega = \mathbf{R}^3, \\ \nabla p \in L^{5/4}(D) \ \textit{in case } \Omega \textit{ is bounded,} \end{array} \tag{21}$$

if $\Phi \in C^\infty(\overline{D})$, $\Phi \geq 0$, and $\Phi = 0$ near $\partial\Omega \times (0,T)$, then for $0 < t < T$,

$$\begin{aligned} \int_{\Omega \times \{t\}} |u|^2\Phi + 2\int_0^t \int_\Omega |\nabla u|^2 \Phi \leq \int_\Omega |u_0|^2\Phi(x,0) + \int_0^t \int_\Omega |u|^2(\Phi_t + \Delta\Phi) \\ \int_0^t \int_\Omega (|u|^2 + 2p) u \cdot \nabla\Phi + 2\int_0^t \int_\Omega (f \cdot u)\Phi. \end{aligned} \tag{22}$$

The method of construction will be based on retarded mollification. That is
We find a solution of

$$u_t + \psi_\delta(u) \cdot \nabla u - \Delta u + \nabla p = f \text{on } D \tag{23}$$

(here $u = u_N$, $p = p_N$, $N > 0$, $\delta = T/N$). Here $\psi_\delta(u)$ denotes the retarded mollification, i.e. a smooth function whose values at t depend only on values of u at times $t - \delta$. One will solve on strips $\mathbf{R}^3 \times (m\delta, (m+1)\delta)$, $0 \le m \le N-1$. One gets estimates on the respective u_N which are independent of m . This allows us to pass to the limit. Since $\psi_\delta(u) \longrightarrow u$, it can be shown that u satisfies the Navier-Stokes equations.

Before sketching the proof we need some auxiliary lemmas (see [CKN]).

Lemma 7.1 *Suppose* $f \in L^2(0,T;V')$, $u \in L^2(0,T;V)$, p *is a distribution, and*

$$u_t - \Delta u + \nabla p = f \tag{24}$$

in the sense of distributions on D . *Then*

$$u_t \in L^2(0,T;V') \tag{25}$$

$$\frac{d}{dt}\int_\Omega |u|^2 = 2\int_\Omega (u_t, u) \tag{26}$$

in the sense of distributions on $(0,T)$ *, and*

$$u \in C([0,T], H) \tag{27}$$

after modification on a set of measure zero. Solutions of (24) are unique in the space $L^2(0,T;V)$ *for given data* $u_0 \in H$.

Assertion (25) follows by (24) and the hypothesis on u and f . Part (26), (27) follows by obtaining (26), (27) for smooth approximations and passing to the limit.

The plan now is to construct smooth solutions for the linearizations. For this a Faedo Galerkin method is used. Following [CKN]

Lemma 7.2 *Suppose* $f \in L^2(0,T;V')$, $u_0 \in H$ *, and* $w \in C^\infty(\overline{D};\mathbf{R}^3)$ *with* $\nabla \cdot w = 0$. *Then there exists a unique function* u *and a distribution* p *such that*

$$u \in C([0,T], H) \cap L^2(0,T;V), \tag{28}$$

$$u_t + w \cdot \nabla u - \Delta u + \nabla p = f \tag{29}$$

in the sense of distributions on D *, and*

$$u(0) = u_0. \tag{30}$$

PROOF :
We assert first the existence of a function u satisfying

$$u \in L^2(0,T;V) \cap L^\infty(0,T;H), \tag{31}$$

for each $v \in V$

$$\frac{d}{dt}\int_\Omega (u,v) + \int_\Omega (w\cdot\nabla u, v) + \int_\Omega (\nabla u, \nabla v) - \int_\Omega (f,v) = 0 \tag{32}$$

in the sense of distributions on $(0,T)$, and $u(0) = u_0$. It makes sense to impose the initial condition, because (31) and $\nabla \cdot w = 0$ imply $w \cdot \nabla u \in L^2(0,T;V')$; whence by Lemma 1.1, Chapter III of [35], $u_t \in L^2(0,T;V')$.

Existence follows as stated above by a Faedo-Galerkin method. We follow for this part Temam's book [T]. That is we had for a sequence $\{u_m\}$ satisfying

$$u_m = \sum_{i=1}^m g_{im}(t)\ \omega_i$$

where $\{\omega_i\}$ is a set which is linearly independent and generates V . Since V is separable such a sequence exists. Now define

$$(u,v) = \int_{\mathbf{R}^3} u\ v\ dx$$
$$((u,v)) = \int_{\mathbf{R}^3} \nabla u\ \nabla v\ dx.$$

Then recalling that V was the closure of $C_0^\infty(\mathbf{R}^3)$ divergence free functions in $H_0^1(\Omega)$, it follows that multiplying the Navier-Stokes equations by u and integrating yields

$$(u_m^{'},\omega_j) + \nu((u_m,\omega_j)) = < f, \omega_j > \tag{33}$$
$$u_m(0) = u_{m0} \tag{34}$$

where u_{0m} is the orthogonal projection of u_0 in H the space spanned by $\{\omega_i\}_{i=1}^m$. The functions g_{im} are scalar functions defined on $[0,T]$. Rewriting (33) yields

$$\sum_{i=1}^m (\omega_i,\omega_j)\ g_{im}^{'}(t) + \nu \sum_{i=1}^m ((\omega_i,\omega_j))\ g_{im}(t) = < f(t), \omega_j > \qquad j = 1,\dots,m$$

Since $\{\omega_i\}$ are independent, the matrix $W_m = (\omega_i,\omega_j)_{1\le i,j\le m}$ is non singular. Thus we can invert W_m to get

$$g_{im}^{'}(t) + \sum_{i=1}^{m} \alpha_{ij}\ g_{im}(t) = \sum_{i=1}^{m} \beta_{ij}\ < f(t), \omega_j > \qquad 1 \leq i \leq m. \tag{35}$$

The data condition is the same as

$$g_{im}(0) = \text{i-th component of } u_{0m} \text{ i.e. from (34)} \tag{36}$$

(35), (36) is an ODE system which defines the g_{im} uniquely in $[0, T]$.

Note that $t \longrightarrow < f(t), \omega >$ are scalar, square integrable functions. Hence so are the g_{im} . Thus for each m

$$u_m \in L^2(0, T; V), \qquad\qquad u_m^{'} \in L^2(0, T; V').$$

From here we can obtain appriori estimates which are independent of m . Thus one can pass to the limit on subsequences and obtain similar estimates for the limiting functions. See Teman [T] to show that

1) $u_m^{'} \longrightarrow u_m \in L^2(0, T; V) \cap L^\infty(0, T; H)$

2) u is a distributional solution to the Navier-Stokes equations.

The steps now in the [CKN] proof are as follows

Lemma 7.3 *Let* Ω *,* u_0 *, and* f *satisfy* $(u_0.1)$ *and* $(F.1)$ *, and let* $w \in C^\infty(\overline{D}, \mathbf{R}^3)$ *with* $\nabla \cdot w$ *. Let* (u, p) *be the solution of (29), (29). Then, for every* $\Phi \in C^\infty(\overline{D})$ *with* $\Phi = 0$ *near* $\partial\Omega \times (0, T)$ *, and for every* t *,* $0 < t \leq T$

$$\begin{aligned}\int_{\Omega \times \{t\}} |u|^2 \Phi + \int\int_D |\nabla u_0|^2 \Phi = \ & \int_\Omega |u_0|^2 \Phi(x, 0) + \int\int_D |u|^2 (\Phi_t + \Delta\Phi) + \\ & \int\int_D (|\nabla u|^2 w + 2pu) \cdot \nabla \Phi_t + 2 \int\int_D (u \cdot f)\Phi\end{aligned}$$

Proof :
See [CKN].

This last lemma just insures that the solution satisfies an approximating GEI. Following [CKN] we proceed to obtain the weak solution via the retarded mollifier. Let $\psi(x, t) \in C^\infty$

$$\psi \leq 0 \ \text{ and } \ \int_{-\infty}^{\infty} \int_{\mathbf{R}^3} \psi \ dx \ dt = 1, \tag{37}$$

$$\text{supp } \subset \{(x, t) : |x|^2 < t, \ 1 < t < 2\}. \tag{38}$$

For $u \in L^2(0, T; V)$, let $\tilde{u} : \mathbf{R}^3 \times \mathbf{R} \to \mathbf{R}^3$ be

$$\tilde{u} = \begin{cases} u(x,t) & \text{if } (x,t) \in D, \\ 0 & otherwise. \end{cases} \tag{39}$$

We set

$$\psi_\delta(u)(x,t) = \delta^{-4} \int_{-\infty}^{\infty} \int_{\mathbf{R}^3} \psi\left(\frac{y}{\delta}, \frac{\tau}{\delta}\right) \tilde{u}(x-y, t-\tau)\, dy\, d\tau. \tag{40}$$

The values of $\psi_\delta(u)$ at time t clearly depend only on the values of u at times $\tau \in (t-2\delta, t-\delta)$. With this definition in hand we return to the original theorem, where we look at u_N solution to

$$u_N + (\psi_\delta(u) \cdot \nabla)\, u_N + \nabla p_N - \Delta u_N = f$$
$$\text{div } u_N = 0.$$

One can show that

$$\begin{aligned} u_N \longrightarrow u_* \quad & \text{strongly in } L^2(D) \\ & \text{weakly in } L^2(0,T;V) \\ & \text{weak star in } L^\infty(0,T;H) \\ p_N \longrightarrow p_* \quad & \text{weakly in } L^{5/3}(D) \end{aligned}$$

Since u_N was bounded in $L^{10/3}(D)$, by interpolation $u_N \longrightarrow u_*$ in L^s , $2 \le s < 10/3$. Moreover, by definition of $\psi_\delta(u)$

$$\psi_\delta(u_N) \longrightarrow u_* \qquad \text{strongly in} \qquad L^s,\ 2 \le s < 10/3.$$

From the last observations it follows easily that u_* is a weak solution to the Navier-Stokes equations. For details see [CKN].

8 Decay of Solutions. Part 1 : Non-uniformity in L^2.

The plan now is to show decay of weak solutions in several Sobolev norms and weighted spaces : L^2 , H^m , moment spaces.
The problem of decay will be considered for solutions with large data. Note that since the solution eventually becomes smooth showing decay in H^m will mean that either we start with small data in an approprite Sobolev space or we wait long enough so that solution becomes small and hence smooth.
The irst question we want to address is how fast does the solution decay in $L^2(\mathbf{R}^n)$, $n \ge 2$. This question for, $n = 3$. was the closing remark in Leray's 1934 pioneering paper. It was first answered for small smooth data. In this case the main idea was to invert the linear part. Kato was the first to give an answer on decay of solutions with small data in L^p spaces, showing that the solutions decay at the same rate as solutions to the heat equation, provided the data is small. Specifically, Kato established:

Theorem 8.1 *[K]*
Let $a \in PL^m$ *. Then there is* $T > 0$ *and a unique solution* u *such that*

$$t^{(1-m/q)/2} u \in BC([0,T); PL^q) \quad \text{for } m \le q \le \infty,$$
$$t^{1-m/2q} \partial u \in BC([0,T); PL^q) \quad \text{for } m \le q < \infty,$$

both with values zero at $t = 0$ *except for* $q = m$ *in the first formula, in which* $u(0) = a$ *. Moreover,* u *has the additional property*

$$u \in L^r(0, T_1; PL^q) \qquad \text{with } 1/r = (1 - m/q)/2, \qquad m < q < m^2(m-2),$$

with some $0 < T_1 \le T$ *.*

Theorem 8.2 *There is* $\lambda > 0$ *such that if* $\|a\|_m \le \lambda$ *, then the solution* u *in Theorem* 8.1 *is global, i,e, we may take* $T = T_1 = \infty$ *. In particular,* $\|u(t)\|_q$ *decays like* $t^{-(1-m/q)/2}$ *as* $t \longrightarrow \infty$ *, including* $q = \infty$ *, and* $\|\partial u(t)\|_q$ *decays like* $t^{-(1-m/2q)}$ *, including* $q = m$ *.*

Theorem 8.3 *In Theorem* 8.2*, we have*

$$T^{-1} \int_0^T \|u(t)\|_m \, dt \longrightarrow 0 \qquad \text{as } T \longrightarrow \infty.$$

PROOF :
See [K].

Remark 8.1 *Here* $PL^m = PL^m(\mathbf{R}^m; \mathbf{R}^m)$ *, and* PL^p *is the subspace of* $L^p(\mathbf{R}^m; \mathbf{R}^m)$ *with divergence zero.*

The decay of solutions with large data in L^2 without rate in L^2 can be found in Masuda's paper [MAS]. The questions we will address here are:

1. Optimal algebraic decay of solutions to Navier-Stokes equations if data is large and belongs to $L^2 \cap \mathbf{X}$. $\mathbf{X}$ will be specified below.

2. Why solutions to the Navier-Stokes equations with data only in L^2 can only be expected to decay only to zero without any rate.

Other questions are related to decay in presence of external forces. We note that if $f(x,t)$ is such that $\|f(\cdot,t)\|_{\mathbf{X}}$ decays very fast (where $\mathbf{X} = L^2$ or $\mathbf{X} = H^m$), then the solution decay rate will not change.
The problem is if f decays slowly. At this point there are still open questions, i.e.

1. If f decays very slowly, will the decay of the solution be governed by the decay of f ?

2. If $f = f(x)$ is a function of x alone, what can be said about the decay? The expectation is that
$$\|u - u_S\|_{\mathbf{X}} \longrightarrow 0$$
where u_S is the solution of the stationary equation.

3. If $f = f(t)$, what happens with the solution as $t \longrightarrow \infty$?

We start now by studying the decay of solutions when $u_0 \in L^2 \cap \mathbf{X}$, where $\mathbf{X}$ will be specified below.

We note first that even at the heat equation level solutions with data only in L^2 cannot decay at a uniform algebraic rate. Thus we cannot expect a uniform rate for the solutions to the Navier-Stokes equations with data in L^2 alone.

Specifically for each sphere of radius β in L^2 there is a point on the sphere so that the corresponding solution decays arbitrarily slowly. In other words for each sphere of radius β and each time T , there exists data u_0 , with $\|u_0\|_{L^2} = \beta$ such that
$$\frac{\|u(T)\|_{L^2}}{\|u_0\|_{L^2}} \geq 1 - \epsilon.$$

Proposition 8.1 *There exists no functions $G(t, \beta)$ with the following two properties. If u is a solution to heat equation or to Navier-Stokes equations with data in $L^2(\mathbf{R}^n)$, $n = 2$, or 3 , then*

(i) $|u(\cdot, t)|_{L^2} \leq G(t, \|u_0\|_{L^2})$

(ii) $\lim_{t \to \infty} G(t, \beta) = 0$ *for all* $\beta \geq 0$.

Proof :
Heuristically follows from the conservation law of kinetic energy
$$\frac{d}{dt} \int_{\mathbf{R}^n} |u|^2 \, dx = - \int_{\mathbf{R}^n} |\nabla u|^2 \, dx$$
which can be interpreted by saying that the closer the solution is to a constant the smaller the gradient will be. Hence the smaller the rate of decay.

The proof of the above theorem for solutions to the heat equation is as follows:
Let $u_0^\alpha = \alpha^{n/2} \, u(\alpha x, \alpha^2 t)$. Then
$$\|u_0^\alpha\|_{L^2}^2 = \alpha^n \int |u(\alpha x, \alpha^2 t)|^2 \, dx = \int u(y, 0) \, dy = \|u_0\|_{L^2}^2$$

Let $u_\alpha(x,t) = \alpha^{n/2}\ u(\alpha x, \alpha^2 t)$, the solution to the heat equation with data u_0^α . Note that by Plancherel

$$|u|_{L^2}^2 = \alpha^n \int_{\mathbf{R}^n} |u(\alpha x, \alpha^2 t)|^2\ dx = \int_{\mathbf{R}^n} |u(y, \alpha^2 t)|^2\ dy =$$
$$\int_{\mathbf{R}^n} |\widehat{u}(\xi, \alpha^2 t)|^2\ d\xi = \int_{\mathbf{R}^n} e^{-2|\xi|^2\alpha^2 t} |\widehat{u}_0(\xi)|^2\ d\xi$$

Since by Lebesgue's dominated convergence theorem

$$\lim_{\alpha\to 0} \int_{\mathbf{R}^n} e^{-2|\xi|^2\alpha^2 t} |\widehat{u}_0(\xi)|^2\ d\xi = \int_{\mathbf{R}^n} |\widehat{u}_0(\xi)|^2\ d\xi$$

we have

$$\lim_{\alpha\to 0} \frac{|u_\alpha(\cdot, t)|_{L^2}^2}{|u(\cdot, 0)|_{L^2}^2} = 1$$

and thus *(ii)* cannot hold.

To obtain a similar result for solutions to the Navier-Stokes equations we need to consider initial data with divergence zero. Let $u_0^\alpha = \alpha^{n/2}\ u_0(\alpha x)$. Then from before we know

1. $\|u_0^\alpha\|_2^2 = \|u_0\|_2^2$
2. $\dfrac{\|v^\alpha\|_{L^2}^2}{\|u_0\|_{L^2}^2} \longrightarrow 1$ if v is solution to heat equation with data u_0 .

Hence, since u satisfies the integral equation (in Fourier space)

$$\widehat{u}_\alpha(\xi, t) = e^{-|\xi|^2 t}\ \widehat{u}_\alpha^0(\xi) - \int_o^t e^{-|\xi|^2(t-s)}\ \widehat{H}_\alpha(s)\ ds \tag{41}$$

where

$$|\widehat{H}_\alpha(s)|_\infty = |\sum \xi_i\ \widehat{u_j^\alpha u_i^\alpha} + \xi_j \widehat{p}^\alpha| \le |\xi|\ \|u^\alpha\|^2.$$

Suppose by contradiction that

$$\frac{\|u^\alpha(t)\|_{L^2}^2}{\|u_0^\alpha\|_{L^2}^2} \longrightarrow 0 \qquad \text{as } \alpha \longrightarrow 0 \tag{42}$$

We are going to suppose that we are working with a smooth solution to Navier-Stokes. This is true for all $t > 0$ if $n = 2$. If $n = 3$ we either start with small data or we restart our problem when t is sufficiently large. We note that by (41)

$$\frac{\|u_\alpha(\cdot, t)\|_{L^2}}{\|u_0\|_{L^2}} \le \frac{\|e^{-|s|^2 t}\ u_0^\alpha\|_{L^2}}{\|u_0\|_{L^2}} - \frac{\displaystyle\int_0^t \|e^{-|\xi|^2(t-s)} \widehat{H}_\alpha\|_{L^2} ds}{\|u_0\|_{L^2}} = I_\alpha - II_\alpha \tag{43}$$

Since $I_\alpha \longrightarrow 1$ as $\alpha \longrightarrow 0$ (by the calculations for the solutions to the Heat equations). Thus it suffices to show that $II_\alpha \longrightarrow 0$. Noting that (let $H_\alpha = H$ and drop the α)

$$\widehat{H}_1 = \sum \xi_i \widehat{u_i u_j} - \xi_j \widehat{p}$$

$$-\widehat{\Delta p} = -|\xi|^2 \widehat{p} = -\sum \xi_i \xi_j \widehat{u_i u_j} \Longrightarrow$$

$$|\xi_j \widehat{p}| \leq \frac{\sum \xi_i \xi_k}{|\xi|^2} \; \xi_j \; |\widehat{u_i u_k}|$$

Thus

$$|\widehat{H}_j| \leq C \; |\xi| \; |\widehat{u^2}| \leq C \; |\xi| \; \|u\|_{L^2}^2 \tag{44}$$

$$|\widehat{H}_j| \leq C \; |u \cdot \nabla u| \leq C \; \|u\|_{L^2}^2 \; \|\nabla u\|_{L^2}^2 \tag{45}$$

If solution is smooth we have $\|\nabla u\|_{L^2}^2 \leq C_0$. Hence

$$II_\alpha \leq C_* \int_0^t \|e^{-|\xi|^2(t-s)}\|_{L^2} \frac{\|u^\alpha\|_{L^2}}{\|u_0\|_{L^2}} \; ds.$$

Since $\|e^{-|\xi|^2(t-s)}\|_{L^2} \leq C(t-s)^{-3/4}$, by Lebesgue dominated convergence theorem since $\|u^\alpha\|_{L^2} \leq \|u_0^\alpha\|_{L^2} = \|u_0\|_{L^2} = C_0$ and $C_0(t-s)^{-3/4}$ is integrable, we can pass to the limit inside the integral sign. Hence

$$\lim_{\alpha \to 0} II_\alpha = 0,$$

which is a contradiction since combining (43) with the fact that $\lim_{\alpha\to 0} I_\alpha - II_\alpha = 1$ it yields that

$$\lim_{\alpha \to 0} \frac{\|u_\alpha\|}{\|u_0\|} \leq 1.$$

Thus $\lim_{\alpha\to 0} \frac{\|u_\alpha\|}{\|u_0\|} > 0$ and we are done.

9 Decay of Solutions. Part 2 : Uniformity in L^2 $(n \geq 3)$.

From section 8 it is clear that in order to obtain uniform decay of solutions in L^2 that the initial data needs to be in a better space than L^2 alone.
Historically it was first shown that if $u_0 \in L^2 \cap L^1$, then the solutions $u(x,t)$ of Navier-Stokes decay at a rate of $(t+1)^{-n/4}$ (See [S1], [S2]). This rate is optimal in the sense that the solutions to Navier-Stokes decay at the same rate as the solutions to its its linear counterpart, i.e. solutions to the heat equation. The method we use is called Fourier Splitting. It was first applied to parabolic conservation laws [S4]. It consists of obtaining a differential inequality (i.e. an Ordinary Differential Inequality) for the L^2 norms of the Fourier transform of the solutions which combined with a time dependent splitting the of the frequency domain will yield the decay. Specifically, the method can be applied under the following circumstances.

9.1 Fourier Splitting.

Let u satisfy the following energy inequality

$$\frac{d}{dt}\int_{\mathbf{R}^n} |u|^2 \, dx \leq -C \int_{\mathbf{R}^n} |\nabla u|^2 \, dx + C(t) \tag{46}$$

where $C(t) \leq C(t+1)^{-(\alpha+1)}$

$$\xi \in S(t) = \{\xi : |\xi| \leq g(t)\} \quad \text{then} \quad |\widehat{u}(\xi,t)| \leq C_0 \tag{47}$$

Here $g^2(t) = \dfrac{n}{C(t+1)}$. If (46) and (47) hold, then

$$\|u(\cdot,t)\|_2^2 \leq C(t+1)^{-\alpha_0}$$

where $\alpha_0 = \min\,(\alpha, n/2)$.

Remark 9.1 *If in (47)* $|\widehat{u}(\xi,t)| \leq C\,|\xi|$ *, then the decay is of order* $\alpha_0 = min\,(\alpha, n/2{+}1)$.

Remark 9.2 *The function* g *can be replaced essentially by any function such that* $\displaystyle\int^t g^2 = ln\ h(t)$ *and* $h(t) \longrightarrow 0$ *at some rate. We still will get a decay rate but not as good.*

Proof of the Fourier Splitting method :

Note that from (46) $u \in L^2$. Hence $\widehat{u}$ is well defined. By PLancherel (46) can be rewritten as

$$\frac{d}{dt}\int_{\mathbf{R}^n} |\widehat{u}|^2 \, d\xi \leq -C \int_{\mathbf{R}^n} |\xi|^2 \, |\widehat{u}|^2 \, d\xi + C(t).$$

Let $m > \max\,(n, \alpha+1)$. Let

$$S(t) = \Big\{\xi : |\xi| \leq \Big(\frac{m}{C(t+1)}\Big)^{1/2}\Big\},$$

then

$$\frac{d}{dt}\int_{\mathbf{R}^n} |\widehat{u}|^2 \, d\xi \leq -C \int_{S(t)} |\xi|^2 \, |\widehat{u}|^2 \, d\xi - C \int_{S(t)^c} |\xi|^2 \, |\widehat{u}|^2 \, d\xi + C(t).$$

We drop the first term on the right hand side and bound the second one by the fact that $\xi \in S(t)^c$ yields $-c\,|\xi|^2 \leq -\dfrac{m}{t+1}$. Thus

$$\frac{d}{dt}\int_{\mathbf{R}^n} |\widehat{u}|^2 \, d\xi \leq -\frac{m}{t+1}\int_{S(t)^c} |\widehat{u}|^2 \, d\xi + C(t).$$

Hence since $\mathbf{R}^n = S(t) + S(t)^c$

$$\frac{d}{dt}\int_{\mathbf{R}^n} |\widehat{u}|^2 \, d\xi + \frac{m}{t+1}\int_{\mathbf{R}^n} |\widehat{u}|^2 \, d\xi \leq \frac{m}{t+1}\int_{S(t)} |\widehat{u}|^2 \, d\xi + C(t).$$

Now multiplying by $(t+1)^n$, using Plancherel and (47) yields

$$\begin{aligned} \frac{d}{dt}\Big((t+1)^n \int_{\mathbf{R}^n} |u|^2 \, dx\Big) &\leq m\ (t+1)^{m-1}\ C_0\ |S(t)| + (t+1)^m\ C(t) \\ &\leq \tilde{C}\ n\ (t+1)^{m-1}\ (t+1)^{-n/2} + C(t)\ (t+1)^m. \end{aligned} \tag{48}$$

Thus integrating in time again yields

$$(t+1)^m \int_{\mathbf{R}^n} |u|^2 \, dx \leq C(t+1)^{-n/2+m} + \int_o^t C(t)(t+1)^m \, dt$$

and by (47) it follows that

$$(t+1)^m \int_{\mathbf{R}^n} |u|^2 \, dx \leq C(t+1)^{-n/2+m} + C(t+1)^{m-\alpha}.$$

Thus

$$\int_{\mathbf{R}^n} |u|^2 \, dx \leq C(t+1)^{-n/2} + C(t+1)^{-\alpha}.$$

This concludes the proof.

To obtain the result in Remark 9.1 we only need to note that the estimate in (48) of $C_0|S(t)|$ can be improved since

$$\int_{S(t)} |\widehat{u}| \, d\xi \leq \int_{S(t)} |\xi|^2 \leq C(t+1)^{-n/2-1}$$

which gives the extra degree of decay.

Remark 9.3 *We note that condition (47) can be replaced by*

$$\int_{S(t)} |\widehat{u}(\xi,t)|^2 \, d\xi \leq C(t+1)^{-\alpha_0}. \tag{49}$$

We are now in position to apply Fourier splitting to solutions to Navier-Stokes equations. We define the following space for the data (Note Wiegner [W] introduced this space)

$$D^n_\alpha = \{(u_0, f) : \|u_0(t)\|_2^2 + \|f\|_2^2 (t+1)^2 \leq C(t+1)^{-\alpha}\}$$

where $u_0(t)$ is the solution to the heat equation with data u_0 . The proof of the decay for solutions to the Navier-Stokes equations uses Fourier splitting with the modification of the Remark 9.3.

Theorem 9.1 *Let* $n \geq 3$. *Suppose* $(u_0, f) \in D^n_\alpha$ *and* $f \in L'$. *Let* u *be a Leray-Hopf solution (i.e. satisfying the energy inequality) of the Navier-Stokes equations. Then*

$$\|u(\cdot,t)\|^2_{L^2} \leq C_0(t+1)^{-\alpha_0},$$

where $\alpha_0 = min\ (\alpha, n/2)$, C_0 *depends only on* n *and* $\|u_0\|_2$.

PROOF :
The proof is formal. To make it rigorous it needs to be applied to approximating solutions and pass to the limit. We recall that a Leray-Hopf solution is such that it satisfies

$$\int_{\mathbf{R}^n} |u|^2 + \int_0^t \int_{\mathbf{R}^n} |\nabla u|^2\ dx\ ds \leq \int_{\mathbf{R}^n} |u_0|^2 + \int_o^t \int_{\mathbf{R}^n} f \cdot u.$$

The one that [CKN] constructed by retarded mollification is one such solution. We remark first that Wiegner has shown that such solutions belong to L^2 . See [W]. Formally we have

$$\frac{d}{dt}\int_{\mathbf{R}^n} |u|^2 dx \leq -2\int_{\mathbf{R}^n} |\nabla u|^2 dx + \Big(\int_{\mathbf{R}^n} |f|^2 dx\Big)^{1/2}\Big(\int_{\mathbf{R}^n} |u|^2 dx\Big)^{1/2}. \tag{50}$$

Let $C(t) = \|f\|_2\ \|u\|_2 \leq C(t+1)^{-\alpha/2-1}$. From (50) we have by Plancherel

$$\frac{d}{dt}\int_{\mathbf{R}^n} |\widehat{u} d\xi \leq -2\int_{\mathbf{R}^n} |\xi|^2\ |\widehat{u}|^2 d\xi + C(t+1)^{-\alpha/2-1} \tag{51}$$

and thus Fourier splitting yields

$$\frac{d}{dt}[(t+1)^m \int_{\mathbf{R}^n} |u|^2] \leq m(t+1)^{m-1}\int_{S(t)} |\widehat{u}|^2 + (t+1)^{m-\alpha/2-1}. \tag{52}$$

So we will first get an auxiliary decay rate of $(t+1)^{-\beta}$, with $\beta = \min\ (m/2-1, \alpha/2)$. For this we need to show that $|\widehat{u}(\xi,t)| \leq C$ for $\xi \in S(t)$. We first get the auxiliary estimate. Take the Fourier transform of Navier-Stokes

$$\widehat{u}_t + |\xi|^2 = -\widehat{H} = -\widehat{u\ \nabla u} \widehat{\nabla p} \tag{53}$$

Recall that in section 8 we showed

$$|\widehat{H}| \leq |\xi|\ \|u\|^2_2.$$

Thus solving (53)

$$\widehat{u}(\xi,t) = e^{-|\xi|^2 t}\ \widehat{u}_0 - \int_0^t H\ e^{-|\xi|^2(t-s)}\ ds.$$

Hence

$$|\widehat{u}(\xi,t)| \leq |\widehat{u}_0(t)| + \int_0^t |\xi|\ e^{-|\xi|^2(t-s)}\ ds \leq |\widehat{u_0(t)}| + \frac{C}{|\xi|^2}$$

Note that the reason to bound $\widehat{u}(\xi,t)$ for $\xi \in S(t)$ was to obtain a bound of $\int_{S(t)} |\widehat{u}(\xi,t)|^2 \, d\xi$. Here we have since u_0 is solution to heat equation

$$I = \int_{S(t)} |\widehat{u(t)}|^2 \leq 2 \int_{\mathbf{R}n2} |u_0(t)|^2 + 2C \int_{S(t)} \frac{d\xi}{|\xi|^2}.$$

Since $n \geq 3$

$$I \leq C(t+1)^{-\alpha} + C(t+1)^{-n/2+1}.$$

Thus from (52)

$$\frac{d}{dt}[(t+1)^m \int_{\mathbf{R}^n} |u|^2] \leq C(t+1)^{m-1}(t+1)^{-\alpha} + C(t+1)^{m-n/2+1}.C(t+1)^{m-\alpha/2+1},$$

from where we have after integration

$$\int_{\mathbf{R}^n} |u|^2 C(t+1)^{-\beta_0} \qquad -\beta_0 = \min\,(\alpha/2, n/2-1).$$

Now we use this decay to improve the decay of

$$C(t) = |f(t)|_2 \; |u(t)|_2 \leq C(t+1)^{-\alpha/2-\beta} \leq C(t+1)^{-\alpha}.$$

Repeating the above argument we will obtain a decay rate of order $(t+1)^{-\beta_0}$, $\beta_0 = \min\,(n/2-1, \alpha)$. If $\alpha \leq n/2-1$, then we are done. If $n/2-1 < \alpha$, we proceed as follows. Note that $n/2-1 \geq 1/2$. We use the auxiliary decay of $\|u\|_2^2 \leq C(t+1)^{-n/2+1}$. to bound $\widehat{u}(\xi,t)$ when $\xi \in S(t)$ .Recall

$$\begin{aligned}
|\widehat{u}(\xi,t)| &\leq |\widehat{u}_0 \; e^{-|\xi|^2 t}| + \int_0^t e^{-|\xi|^2(t-s)} \; |\xi| \; \|u(s)\|_2^2 \; ds \\
&\leq |\widehat{u}_0 \; e^{-|\xi|^2 t}| + \int_0^t \frac{1}{(t+1)^{1/2}} \; \frac{1}{(s+1)^{n/2-1}} \; ds \\
&\leq |\widehat{u}_0(t)| + \frac{C}{(t+1)^{1/2}} \; (t+1)^{1/2} \\
&\leq |\widehat{u}_0(t)| + C_0.
\end{aligned}$$

Here we used $(s+1)^{-n/2+1} \leq (s+1)^{-1/2}$. Hence

$$\int_{S(t)} |\widehat{u}| \leq C \int_{\mathbf{R}^n} |u_0(t)|^2 + C_0 \int_{S(t)} \leq C(t+1)^{-\alpha} + C(t+1)^{-n}$$

and the decay rate follows by Remark 9.2.

9.2 Special Cases.

Theorem 9.2 *Let* $u_0 \in L^2 \cap L^p$, $1 \leq p < 2$ *. Let* u *be a Leray-Hopff solution to the Navier-Stokes equations (for simplicity let* $f \equiv 0$ *). Then*

$$\|u(\cdot,t)\|_2^2 \leq C(t+1)^{-n/2(2/p-1)}.$$

PROOF :
We only must show

$$(u,0) \in D_\alpha^n \qquad \text{with } \alpha = n/2(2/p-1).$$

That is we need

$$\|u_0(t)\|_{L^2}^2 \leq C(t+1)^{-n/2(2/p-1)}.$$

We used this estimate to show that

$$\int_S \|\widehat{u}_0(t)\|^2 \, d\xi \leq C(t+1)^{-n/2(2/p-1)}.$$

Hence we will show this last inequality . For this we use Hölder's inequality and the Riesz convexity theorem [S]. We recall the Riesz convexity theorem:

Definition 9.1 T *is said to be of type* (p,q) *is for all* $f \in L^p$

$$\|Tf\|_q \leq K \, \|f\|_p.$$

The minimum of such K *is called the* (p,q) *norm of* T *.*

Theorem 9.3 *(Riesz convexity (R.C.))*
Let T *be of the type* (p_i,q_i) *with norm* k_i *,* $i=1,2$ *. Then* T *is of the type* (p_t,q_t) *with norm* k_t *and* $k_t \leq k_0^{1-t} \, k_1^t$ *where* $p_t = \dfrac{1-t}{p_0} + \dfrac{t}{q_0}$ *,* $q_t = \dfrac{1-t}{p_1} + \dfrac{t}{q_1}$ *.*

We apply this theorem with $T=F$ the Fourier transform. It is well known that F is of types $(1,\infty)$ and $(2,2)$. Let $\dfrac{1}{p} = \dfrac{1-\theta}{2} + \dfrac{\theta}{2}$, $\dfrac{1}{q} = \dfrac{1-\theta}{2} + \dfrac{\theta}{2}$ some $0 \leq \theta \leq 1$. Thus $\dfrac{1}{p} + \dfrac{1}{q} = 1$. Then by R.C.

$$\|F(w)\|_{L^q} \leq \|F\|_{(1,\infty)}^{1-t} \, \|F\|_{(2,2)}^{t} \, \|w\|_{L^p} \tag{54}$$

Let $w = u_0(t)$ solution to heat equation with data u_0 . Then

$$\int_{S(t)} |\widehat{u}_0(t)|^2 \, d\xi \leq \Big(\int_{S(t)} |\widehat{u}_0(t)|^q\Big)^{1/r} \Big(\int_{S(t)} |\widehat{u}_0(t)|^p\Big)^{1/s} \tag{55}$$

where $q = \frac{r}{2}$. Choose s so that $1 = \frac{1}{r} + \frac{1}{s} = \frac{2}{q} + \frac{1}{s} = 2 - \frac{2}{p} + \frac{1}{s}$. Hence $\frac{1}{s} = \frac{2}{p} - 1$. Thus by (54)

$$\|Fu_0(t)\| \leq C \, \|u_0(t)\|_{L^p} \leq C_0.$$

Since we know that if $u_0 \in L^p$, then $u_0(\cdot, t) \in L^p$. We also have that

$$\Big(\int_{S(t)} d\xi\Big)^{1/2} \leq C(t+1)^{-n/2(2/p-1)}.$$

Thus by (55)

$$\Big(\int_{S(t)} |\widehat{u}_0(t)|^2 \; d\xi\Big)^{1/2} \leq C(t+1)^{-n/2(2/p-1)}.$$

Hence by Fourier splitting and Theorem 9.1 it follows that

$$\|u(\cdot, t)\|_{L^2}^2 \leq C(t+1)^{-n/2(2/p-1)}.$$

In particular if $u_0 \in L^2 \cap L^1$

$$\|u(\cdot, t)\|_2^2 \leq C(t+1)^{-n/2}.$$

10 Decay of Solutions. Part 3 : Uniformity in L^2 $(n = 2)$.

We first indicate how to obtain an auxiliary rate of decay that will be used in order to obtain an optimal rate of decay.

Theorem 10.1 *Let* $(u_0, f) \in D^2_\alpha, \alpha > 0$. *Let* u *be a solution to the Navier-Stokes equations with data* u_0 . *Then*

$$\|u(\cdot, t)\|_{L^2}^2 \leq C_0 \; [ln \; (t+e)]^{-2} \tag{56}$$

C_0 *depending only on the data.*

PROOF :
The proof follows the same steps as Theorem 9.1 using $g^2(t) = \dfrac{3}{2 \; ln \; (t+e)(t+e)}$. That is using the Fourier splitting. For details of the proof see [S2].

To improve the decay of order $(t+1)^{-1}$ we use Wiegner's extension to the Fourier Splitting method. Specifically we restate Wiegner's theorem [W] and refer the reader to his proof [W].

Theorem 10.2 *(Wiegner)*
Let $n \geq 2$. *Let* u *be a weak solution to Navier-Stokes in the sense of Leray-Hopf and thus suppose that the Leray energy inequality holds. Let*

$$(u_0, f) \in L_2(\mathbf{R}^n)^n \times L_1(R^-, L_2(\mathbf{R^n})^n).$$

Then we have the following

(a) $\|u(t)\|_2 \longrightarrow 0$ *for* $t \longrightarrow \infty$.

(b) $(u_0, f) \in D_\alpha^{(n)}$, *then* $\|u(t)\|_2^2 \leq C(1+t)^{-\overline{\alpha}_0}$ *with* $\overline{\alpha}_0 = min\,(\alpha_0, \frac{1}{2}n+1)$.

(c) Furthermore, the solution u *is asymptotically equivalent with the solution of the heat system with the same data in the sense that*

$$\|u(t) - u_0(t)\|_2^2 \leq h_{\alpha_0}(t)\ (1+t)^{-d}$$

with $d = \frac{1}{2}n+1-2\,max\,(1-\alpha_0, 0)$ *(note that* $d > \overline{\alpha}_0 = \alpha_0$ *as long as* $\alpha_0 < \frac{1}{2}n+1$ *), and*

$$h_{\alpha_0}(t) = \begin{cases} \varepsilon(t) & for\ \alpha_0 = 0,\ \ with\ \varepsilon(t) \searrow 0\ for\ t \longrightarrow \infty \\ C\ ln^2(t+e) & for\ \alpha_0 = 1 \\ C & for\ \alpha_0 \neq 0, 1. \end{cases}$$

For part (c) we need some additional assumption on f *as stated, for example, in* (**??**) *(d):*

$$f \in L_n(\mathbf{R}^n)^n \qquad with \qquad \|f(s)\|_n \leq C\ s^{-\beta},\ \beta = \frac{1}{2}(\alpha_0+1) + \frac{1}{4}n.$$

The constants depend explicitly on the data.

11 Lower Bounds.

We note that solutions to the heat equation can decay at very fast speeds depending on the flatness of the Fourier transform of the solution at the origin. Specifically let

$$R_\alpha^\delta = \{u : |\widehat{u}(\xi)| \geq \delta\ |\xi| \leq \alpha\}$$

$$R_k = \{u : |\widehat{u}(\xi)| = o(|\xi|^k)\ \xi \longrightarrow 0\}.$$

Theorem 11.1 *Let* u *be a solution to heat equation with data* $u_0 \in R_\alpha^\delta \cap L^2(\mathbf{R}^n)$ *. Then*

$$\|u(\cdot, t)\|_{L^2}^2 \geq C(t+1)^{-n/2}$$

with $C = C(\delta, \alpha)$.

PROOF :

$$\int |u|_0^2\ dx = \int |\widehat{u}|^2\ d\xi \geq \int |\widehat{u}_0\ e^{-|\xi|^2 t}|^2\ d\xi \geq$$

$$\int_{|\xi|\leq\alpha} |\widehat{u}_0|^2\ e^{-2|\xi|^2 t}\ d\xi \geq \frac{\delta^2}{t^{n/2}} \int_{y \leq \alpha\sqrt{t}} e^{-y^2}\ dy \geq \frac{C_0}{(t+1)^{n/2}}.$$

Theorem 11.2 *Let* $u_0 \in R_k \cap L^2(\mathbf{R}^n)$ *.Then*

$$\|u(\cdot,t)\|_{L^2}^2 \geq C(t+1)^{-k-n/2}.$$

PROOF :
Same as before by replacing $\widehat{u}_0(\xi)$ by $|\xi|^k$.

Theorem 11.3 *Let* $u_0 \in L^2(\mathbf{R}^n)$ *and* $\widehat{u} = 0$ *in* $N_\epsilon = \{\xi : |\xi| \leq \epsilon\}$ *.Then*

$$\|u(t)\|_{L^2}^2 \geq exp\ (-\epsilon^2\ t).$$

PROOF :

$$\int_{\mathbf{R}^n} |u|^2\ dx = \int_{\mathbf{R}^n} |\widehat{u}|^2\ d\xi \geq \int_{|\xi|\leq\epsilon} |\widehat{u}_0|^2\ e^{-2|\xi|^2 t}\ d\xi \geq e^{-\epsilon^2 t} \int_{|\xi|\leq\epsilon} |\widehat{u}_0|^2\ d\xi \geq C\ e^{-\epsilon^2 t}.$$

The expectation is that for solutions to Navier-Stokes equations there is a lower bound due to the mixing of the terms which is produced by the corrective term. Here we only show that

Theorem 11.4 *If* $u_0 \in R_\alpha^\delta \cap L^2(\mathbf{R}^n)$ *, then*

$$\|u(\cdot,t)\|_{L^2}^2 \geq C(t+1)^{-n/2}, \qquad n \geq 3.$$

PROOF :
Proof is formal. Let $w = u(t,x) - u_0(t)$, where u_0 is solution to the heat equation with same data. Then

$$w_t = \triangle w - (u\ \nabla u + \nabla p).$$

We note that

$$\frac{d}{dt}\int_{\mathbf{R}^n} |w|^2 = -\int_{\mathbf{R}^n} |\nabla w|^2 - \int_{\mathbf{R}^n} (u-u_0)u\ \nabla u - \int_{\mathbf{R}^n} w\ \nabla p = -\int_{\mathbf{R}^n} |\nabla w|^2 + \int_{\mathbf{R}^n} u\ u_0\ \nabla u.$$

The other terms vanish since $\operatorname{div} u = \operatorname{div} w = 0$ (Note $\operatorname{div} u_0 = 0 \Longrightarrow \operatorname{div} u_0(t) = 0$). We also notice that

$$\|u\|_{L^2}^2 \geq \|u_0(t)\|_{L^2}^2 - \|w\|_{L^2}^2.$$

Hence it suffices to show that

$$\|w\|_{L^2}^2 \geq C(t+1)^{-n/2+\epsilon}, \qquad \epsilon > 0.$$

The result will follows by Fourier Splitting.

1) Note first

$$|\int_{\mathbf{R}^n} u_0\ u(t)\ \nabla u|dx = |\int_{\mathbf{R}^n} \sum_i \partial_i u_0(t)\ u_i\ u|dx \leq |\nabla u_0|_{L^\infty}\ \|u\|_{L^2}^2.$$

It is well known that

$$|\nabla u_0|_{L^\infty} \leq C(t+1)^{-n/4-1/2}.$$

Hence

$$|\int_{\mathbf{R}^n} u_0\ u(t)\ \nabla u|dx \leq C(t+1)^{-n/4-1/2-n/2} \leq C(t+1)^{-3n/4-1/2}.$$

2) We need an estimate for

$$\int_{S(t)} |\widehat{w}|\ d\xi \leq \int_{S(t)} \Big(\int_0^t |\xi|\ \|u\|_2^2\ e^{-|\xi|^2(t-s)} \Big)^2\ d\xi.$$

Hence

$$\int_{\mathbf{R}^n} |\widehat{w}|\ d\xi \leq \frac{1}{t+1} \int_{S(t)} \int_0^t \frac{1}{(s+1)^{n/2}}\ ds\ d\xi \leq \frac{1}{t+1}\ \frac{1}{(t+1)^{n/2}}\ C_0.$$

Thus from the usual Fourier splitting we have

$$\frac{d}{dt}[(t+1)^m \int_{\mathbf{R}^n} |w|^2 dx] \leq (t+1)^{m-1}[\int_{S(t)} |\widehat{w}|^2\ d\xi + C(t+1)^{-3n/4-1/2}] \leq$$

$$\leq C\ (t+1)^{m-1}\ C_0\ \Big((t+1)^{-n/2-1} + C\ (t+1)^{-3n/4-1/2} \Big)$$

Let $m > \max\,(n/2+1, 3n/4+1/2)+1$. Hence integrating yields after dividing by m

$$\int_{\mathbf{R}^n} |w|^2 dx \leq C_1(t+1)^{-m} + C_1(t+1)^{-n/2-1} + C_2(t+1)^{-3n/4+1/2}$$

since $n \geq 3,\ 3n/4 - 1/2 \geq n/2 + 1$. Hence

$$\int_{\mathbf{R}^n} |w|^2 dx \leq C(t+1)^{-n/2-1}$$

and we are done.

12 Decay of Higher H^m Norms.

In this section we will study the decay of solutions in the spaces H^m and in H^m of time derivatives of the solution.

We first recall that in section 3 we have shown

1. If $n = 3$, $\widehat{u}_0 \in L^1$, and the solution stays in H^1 , then all the higher derivatives are bounded.
2. If $3 \leq n \leq 5$, $\widehat{u}_0 \in L^1$, and the solution is in $C[[T_0, \infty), W^{2,2}]$, then all higher derivatives are bounded.

Hence here we are going to suppose that either data is sufficiently small so that our solution is smooth or that we are starting at a sufficiently large time so that the solution is small. The decay of the solution in H^m spaces is based on the Fourier splitting method. We remark that for decay of exterior domain there is work of the Kozono, Ogawa, and Sohr [KO],[KOS]. The general idea is to obtain inductively energy inequality of type (46) for the $D^\alpha u$. More precisely we take D^α derivatives of the equation $(|\alpha| = m)$ multiply by $D^\alpha u$ and integrate. After integration by parts and several estimates (we sum all the α derivatives with $|\alpha| = m$) we get an expression of the form

$$\frac{d}{dt} \int_{\mathbf{R}^n} |D^m u|^2 dx \leq -2 \int_{\mathbf{R}^n} |D^{m+1} u|^2 dx + C_m(t),$$

where $C_m(t)$ will decay. The problem is that when we do our induction the first term around $C_m(t)$ will not decay fast enough to give the optimal decay. The procedure then is to obtain auxiliary decays that well step by step allow us to improve our decay to the optimal one (i.e. the same decay rate as for the heat equation).

The results on decays for H^m norms for 2D where first presented in [S3]. Here we present a perhaps less obvious way than in [S3] but more compact way of establishing the decay [SW]. The main result is that each derivative will add one order of decay to the rate in $\| \cdot \|_2^2$. For this the following two Lemmas are needed for smooth solutions to the Navier-Stokes equations.

Lemma 12.1 *For* $m \in \mathbf{N}$ *, we have the inequality*

$$\frac{d}{dt} \|D^m u\|_2^2 + \frac{3}{2} \|D^{m+1} u\|_2^2 \leq C_m(\|u\|_\infty^2 \|D^m u\|_2^2 + R_m)$$

with

$$R_m = \begin{cases} 0 & \mathit{for} \quad m = 1, 2, \\ \sum_{1 \leq j \leq m/2} \|D^j u\|_\infty^2 \|D^{m-j} u\|_2^2 & \mathit{for} \quad m \geq 3. \end{cases}$$

Proof :
Energy methods. See [SW].

Lemma 12.2 *Let* $m \in N$, $T_m = T_0 + 1 - 2^{-m}$ *and assume*

$$\|D^{m-1}u\|_2^2 \leq C_{m-1}\ (t - T_{m-1})^{-\rho_{m-1}} \qquad \mathit{for} \qquad t > T_{m-1}.$$

with $s_i \geq \rho_{m-1} + 2$. *Then*

$$\frac{d}{dt}\ \|D^m u\|_2^2 + \|D^{m+1}u\|_2^2 \leq c_0\ (t - T_{m-1})^{-1}\ \|D^m u\|_2^2 + \sum_{i=1}^{m} c_i\ (t - T_{m-1})^{-s_i},$$

with $\rho_m = 1 + \rho_{m-1}$ *and some* C_m *depending on* C_{m-1} , c_i , s_i , ρ_m , m , *but not on* T_0 .

Proof :
Fourier splitting. See [SW].

From these two lemmas follows the main result. See [SW].

Theorem 12.1 *Suppose* $\|u\|_2^2 \leq C_0\ (t+1)^{-2\mu}$ *, for* $t \leq 0$ *, with some* $\mu \leq 0$ *. Then for* $m \in \boldsymbol{N}$ *, there is some* $C_m = C_m(\mu, C_0)$ *, independent of* T_0 *, with* T_0 *given by (1), such that*

$$\|D^m u\|_2^2 \leq C_m\ (t+1)^{-m-2\mu} \qquad \mathit{for} \qquad t \geq T_0 + 1.$$

Remark 12.1 *If* $n = 2$ *or if* $\|a_n\|$ *is small enough, we have* $T_0 = 0$ *, while* $T_0 = c(n)\ \|a\|_n^{4/(n-2)}$ *is admissible for weak solutions,* $n = 3$ *or* 4 *.*

Proof of Theorem 13.4 :
We want to show by induction the estimate

$$\|D^m u\|_2^2 \leq C_m\ (t - T_m)^{-m-2\mu} \qquad \text{for} \qquad t \geq T_m = T_0 + 1 - 2^{-m}.$$

If $m = 1$ or 2 , we know by Lemma 12.2 and ([**?**]), that

$$\frac{d}{dt}\ \|D^m u\|_2^2 + \|D^{m+1}u\|_2^2 \leq c\ K\ (t - T_0)^{-1}\ \|D^m u\|_2^2.$$

We may apply Lemma 12.2 to get the claim for $m = 1$ or 2 .

For $m \geq 3$, we have the additional term R_m to estimate. We use interpolation inequality

$$\|D^j u\|_\infty \leq c\ \|D^{m+1}u\|_2^{a_j}\ \|u\|_2^{1-a_j} \qquad \text{with} \qquad a_j = (j + \frac{n}{2})/(m+1).$$

Note that

$$\frac{j}{m+1} < a_j < 1 \qquad \text{for} j \leq \frac{m}{2} \qquad \text{and} \qquad m \geq n-1$$

and in the case $n = 5$, $m = 3$, $j = 1$. Then

$$R_m \leq \frac{1}{2} \|D^{m+1}u\|_2^2 + c \|u\|_2^2 \sum_{1 \leq j \leq m/2} \|D^{m-j}u\|_2^{2/(1-a_j)}$$

$$\leq \frac{1}{2} \|D^{m+1}u\|_2^2 + c \sum_j (t - T_{m-1})^{s_j},$$

with

$$s_j = 2\mu + (m+1) \frac{m-j}{m-j+1-\frac{n}{2}} \geq 2\mu + m + 1,$$

where we used the induction hypothesis (weakened to) $\|D^k u\|_2^2 \leq C_k (t - T_k)^{-k}$ for $k \leq m-1$. Hence Lemma 12.2 may be applied again and proves the claim.

Form here the L^p norm decay follows by Gagliardo-Nirenberg's inequality.

Theorem 12.2 *Under the same assumptions, there holds, for* $2 \leq p \leq \infty$ *and* $j \in \mathbf{N}_0$,

$$\|D^j u\|_p \leq c (t - T_0 - 1)^{-j/2 + n/2(1/2 - 1/p) + \mu)}$$

(especially $\|u\|_\infty \leq c (t - T_0 - 1)^{-(\mu + n/4)}$ *).*

Proof :
By interpolation (Gagliardo-Nirenberg)

$$\|D^j u\|_p \leq c \|D^m u\|_2^a \|u\|_2^{1-a} \qquad \frac{1}{p} = \frac{j}{n} + a \left(\frac{1}{2} - \frac{m}{n}\right) + (1-a) \frac{1}{2}.$$

In order to derive estimates also for time derivatives of u , we have to show first a generalization of Lemma 12.1.

Lemma 12.3 *For* $t > T_0$ *, there holds*

$$D^\alpha \frac{d^k}{dt^k} u \in L_2(\mathbf{R}^n)^n$$

for all multi-indices α *and all* $k \in \mathbf{N}$ *.*

Proof :
Follows the same steps as for bounds of space derivatives. See [SW].

Theorem 12.3 *Under the same assumptions as in Theorem* 12.1*, we get, for* $t > T_0 + 2$ *,*

$$\|D^m \frac{d^k}{dt^k} u\|_2 \leq c(t - T_0 - 2)^{-(m/2) - k - \mu}.$$

PROOF :

By Lemma 12.3, we know that we may apply

$$D^\alpha \ \frac{d^{k-1}}{dt^{k-1}}$$

to the equation. After scalar multiplication by

$$D^\alpha \ \frac{d^k}{dt^k} \ u$$

and integration by parts, the pressure drops out and we get with Hölder's inequality

$$\|D^\alpha \ \frac{d^k}{dt^k} \ u\|_2 \leq \|D^\alpha \ \frac{d^{k-1}}{dt^{k-1}} \ (\Delta u - u \cdot \nabla u)\|.$$

Thus, letting $k = 1$, we get , for $t > T_1 + 1$,

$$\|D^\alpha \ \frac{d}{dt} \ u\|_2 \leq c \ \|D^{|\alpha|+2} \ u\|_2 + c \sum_{j \leq |\alpha|/2} \|D^j \ u\|_\infty \ \|D^{|\alpha|-j+1} \ u\|_2$$

$$\leq c(t - T_0 - 1)^{-|\alpha|/2-1-\mu} + c \sum_j (t - T_0 - 1)^{-(j/2+n/4+\mu)-((|\alpha|-j+1)/2+\mu)}$$

$$\leq c(t - T_1 - 1)^{-(|\alpha|/2)-1-\mu},$$

where we used Theorem 12.2. Estimates for

$$\|D^\alpha \ \frac{d}{dt} \ u\|_p, \qquad 2 \leq p \leq \infty,$$

now follow by interpolation. The case of general k is then a consequence of straightforward induction.

13 Decay of Moments of Solutions: Connection with Pointwise Decay.

In this section we discuss briefly the decay of moments of the solutions. We recall that the moments are defined as

$$M_k(u) = \int_{\mathbf{R}^n} |x|^k \ |u|^2 \ dx.$$

Studying this type of decay will lead to obtaining time and spatial point-wise decay. Specifically in [AGSS] rom the decay of moments of higher derivatives, via application of a Gagliardo-Nirenberg inequality yield point-wise decay of the solution, both in space and time . Here we are going to concentrate on the decay of the moments. We remark first that the main part of work is to obtain a construction of a solution for which the moments are bounded. The decay of the moments will follow by a Hölder inequality. Here we will only give a sketch to construct solutions for which the moments are bounded for $k \leq n$. For details we refer the reader to [S3]. At the end of the section we show how to use the bounds of the moments with an appropriate Hölder inequality to obtain the decay of the moments.

To obtain decay of the moments we proceed in several steps

1. We design a construction of weak solutions such that the L^r_ν norms are bounded, where
$$L^r_\nu(\mathbf{R}^n) = \{f : |x|^\nu \ f \in L^r\}.$$
2. We use the L^r_ν bounds to obtain the bounds of the moments.
3. The first bounds are time dependent.
4. We obtain time independent bounds.

To obtain time dependent bounds in L^r_ν we use the following construction.
We show that the approximating solutions which satisfy the linearized equations

$$v_t + (u \cdot \nabla)v + \nabla P(u,v) - \Delta v = 0 \tag{57}$$

where $v = u_{k+1}$, $u = u_k$, are bounded in L^r_ν (More precisely they will have $D^\alpha v \in L^r_\nu$). The bounds will depend on time and u . That is on the approximating solution on level k . We will show that for small data or for solutions bounded in L^r , $r > n$ the bounds will behave independent of k . To understand the linearization (57) we have to define the operator $P(u,v)$.
For this we need to recall the definition of the Riesz transform. Riesz transform is defined such that $R = (R_1, \ldots, R_n)$
$$R_j \ f(\xi) = i \ \frac{\xi_j}{|\xi|} \ \widehat{f}(\xi).$$
Note where it is used. For example, if we need to solve

$$\Delta g = \partial_i \partial_j f \Longrightarrow |\xi|^2 \widehat{g} = -\xi_i \ \xi_j \ \widehat{f} \Longrightarrow$$

$$\widehat{g}(\xi) = -\frac{\xi_i}{|\xi|} \ \frac{\xi_j}{|\xi|} \ \widehat{f} \Longrightarrow g(\xi) = -R_i R_j(f).$$

We use it in
$$-\Delta p = -\sum \partial_i \partial_j (u_i \ u_j) \Longrightarrow p = R_i R_j (u_i \ u_j).$$
We recall that if p , ν are real numbers such that $0 \leq \nu < \dfrac{n}{n-\nu} < q < \infty$, then R_i are bounded operators in L^q_ν , where

$$L^q_\nu(\mathbf{R}^n) = \{f : |x|^\nu \ f \in L^q\}.$$

For a proof see Stein's book.

The bound on the Riesz Transform can be used to obtain appropriate bounds for the solutions of the linearization (57). That is let v be a solution to

$$\begin{cases} v_t + (u \cdot \nabla)v + \nabla P(u,v) - \Delta v &= 0 \\ \text{div } v &= 0 \\ v(x,0) &= u_0, \end{cases}$$

where

$$P(u,v) = \sum_{j,k} R_j R_k (u_j, v_k).$$

The idea is to solve first the integral equation

$$v(x,t) = s(t)\ u_0 - \int_0^t s\ (s-t)\ [u \cdot \nabla v + \nabla P(u,v)]\ ds.$$

Local existence follows by fixed point argument and Gronwall.Specifically we invert the linear part of the equation and use knonw bounds for teh Heat Operator and the Riesz Operator in Sobolev Spaces. From here one can show

Theorem 13.1 *For appropriate data and velocity u the following bound holds*

$$\|D^\alpha v(t)\|_{L^r_\nu} \leq C(T)$$

where C(T) depends on T, norms of the data and u.

Proof : The proof is quite technical. The idea again is to invert the linear part of the corresponding weighted integral equation. The main problem is to bound the nonlinear term. For this we use known we use weighted bounds for the Heat and Riesz Operator. For detail of the proof see [SS]for details.

We remark that if data small enough, the existence can be extended for all time. To obtain the same type of bounds for the solutions to the Navier-Stokes equations we let

$$v = u^{l+1} \qquad\qquad u = u^l,$$

By the theorem above provided we are in a good data space we have the estimate for $\{u^{l+1}\}$, for all l. A technical argument shows that passing to the limit the estimate still holds. It turns out that $\lim u^l$ is a weak solution to Navier-Stokes equations. Thus the result holds for solutions to the Navier-Stokes eqautions. Moroever for small data the bound holds for all time . Since the u^l depend on the data we obtain a result for all time depending on the data. Moroeverone can show that the bound will be independent of time [SS].

Decay of Moments :

Once we have

$$\int |x|^k \; |u|^2 \; dx \leq k < \infty.$$

The decay follows by Hölder inequality.

Theorem 13.2 *Supposing that for some $r > n$ with $2 \leq n \leq 5$ if*

$$\mu(u) = \sup_{k=1.2..,t>0} |u^k(t)|_r < \infty$$

$$u_0 \in L^r_\beta \cap L^r_{\beta/2} \cap L^{r'} \cap H^1 \qquad div \; u_0 = 0 \qquad \beta = n(1 - 2/r)$$

$$\|u\|_2 \leq C(t+1)^{-\mu} \qquad \|u\|_\infty \leq C(t+1)^{-(\mu+n/4}.$$

Then

$$M_k(u) \leq C(t+1)^{-2\mu(1-k/n)} \qquad k \leq n.$$

medskip
Proof:

$$M_k(u) = \int_{\mathbf{R}^n} |x|^k \; |u|^2 \; dx \leq C \; \left(\int_{\mathbf{R}^n} |u|^2 \; dx \right)^{1/p} \; \left(\int_{\mathbf{R}^n} |x|^k \; |u|^2 \; dx \right)^{1/p'},$$

where

$$\frac{1}{p} = \frac{n-k}{n}; \qquad \frac{1}{p'} = \frac{k}{n}.$$

We get

$$M_k(u) \leq C(t+1)^{-2\mu(1-k/n)}.$$

14 Self-similar Solutions : General Description.

In this section interest will be focused on self-similar solutions to the Navier-Stokes equations. We note first that solutions to the Navier-Stokes equations are such that if $u(x,t)$, $p(x,t)$ is a solution so is

$$u_\lambda(x,t) = \lambda \; u(\lambda x, \lambda^2 t)$$
$$p_\lambda(x,t) = \lambda^2 \; p(\lambda x, \lambda^2 t).$$

A solution is said to be auto-similar if it is invariant with respect to the above transformation: specifically for all $\lambda > 0$,

$$u(x,t) = u_\lambda(x,t) = \lambda \; u(\lambda x, \lambda^2 t) \tag{58}$$

$$p(x,t) = p_\lambda(x,t) = \lambda^2 \; p(\lambda x, \lambda^2 t) \tag{59}$$

We note that there are two obvious candidates. We will call these forward self-similar solutions (f.s.s.) and backward self-similar solutions (b.s.s.).

14.1 Forward Self-similar Solutions.

Let $\lambda = \lambda(t) = 1/\sqrt{t}$. Then

$$\begin{cases} u(x,t) = \dfrac{1}{\sqrt{t}}\; U(x/\sqrt{t}) \\ p(x,t) = \dfrac{1}{\sqrt{t}}\; P(x/\sqrt{t}). \end{cases} \tag{60}$$

It is easy to see that (u, p) defined in (60) satisfy (58)and (59). In this part we will briefly recall the construction of a strong self-similar solution with data in a Besov space as was done by Cannone [C] This type of solutions were in the past five years studied by several people V. Meyer school as for example Cannone, Planchon, Barraza.

14.2 Backwards Self-similar Solutions.

Let

$$\begin{cases} u(x,t) = \dfrac{1}{\sqrt{2a(\tau - t)}}\; U(x/\sqrt{2a(\tau - t)}) \\ p(x,t) = \dfrac{1}{\sqrt{2a(\tau - t)}}\; P(x/\sqrt{2a(\tau - t)}), \end{cases} \tag{61}$$

for some $a > 0$, $-\infty < t < T$. We note that for (61) to be a self-similar solution in the sense of (58) and (59) we need $t \longrightarrow t - T$ to be a translation. This is why if the original t varies from zero to infinity the new t will vary from minus infinity to T (Note we are going backwards).

The problem of studying (23) was mentioned first in Leray [L], where he proposes to study such a solution, since if $U \in W^{1,2}(\mathbf{R}^3)$ exists and is nonzero, then the corresponding u (given by (61)) will satisfy :

$$\int_{\mathbf{R}^3} |u(x,t)|^2 \; dx = \sqrt{2a(T-t)} \; \int_{\mathbf{R}^3} |U(y)|^2 \; dy \tag{62}$$

and

$$\int_{\mathbf{R}^3} |\nabla u(x,t)|^2 \; dx = \sqrt{2a(T-t)} \; \int_{\mathbf{R}^3} |\nabla U(y)|^2 \; dy. \tag{63}$$

Thus if there exists a $U \in W^{1,2}$, then u will satisfy that it is a solution to Navier-Stokes such that

$$\int_{\mathbf{R}^3} |u(x,t)|^2 \; dx < \infty \tag{64}$$

and

$$\lim_{t \to T^-} \int_{\mathbf{R}^3} |\nabla u(x,t)|^2 \; dx = \infty. \tag{65}$$

That is we would have constructed an L^2 solution to Navier-Stokes equations where the L^2 norm of the gradient blows up in finite time. A simple calculation shows that u satisfies the elliptic equation

$$\begin{cases} a\ U + a\ y_k \dfrac{\partial}{\partial_k} U + U_k \dfrac{\partial}{\partial_k} U + \nabla P - \nu\ \Delta U & = 0 \\ \operatorname{div} U & = 0 \end{cases} \tag{66}$$

Here and in the future, double indices will be understood as summation, i.e. for example

$$y_k \frac{\partial}{\partial_k} U = \sum_{k=1}^{3} y_k \frac{\partial}{\partial_k} U \tag{67}$$

In particular Leray formulated the problem as follows (in [L], 1934):
If we can find a solution to the elliptic system (66) the $(u(x,t), p(x,t))$ defined by (61) is a solution to Navier-Stokes which becomes irregular at $t = T$.

In 1995, [NRS] showed that (27) has no nonzero solution $U \in W^{1,2}_{loc}(\mathbf{R}^3) \cap L^3(\mathbf{R}^3)$.

In 1996, we showed [MNPS] that under slightly more restrictive conditions, i.e. $U \in W^{1,2}(\mathbf{R}^3)$, all solutions to (66) are zero. The condition is the natural condition and the proof is much simpler, i.e. it is purely geometric. This is the proof we present below in these notes.

14.3 Differences between (f.s.s.) and (b.s.s.).

One of the first questions one has regarding differences between (f.s.s.) and (b.s.s.) is why do we have smooth (f.s.s.) and only zero (b.s.s.). The answer is simple. For (b.s.s.) we look for solutions in L^2 . For (f.s.s.) we look for solutions in a Besov space. It is trivial to show that for (f.s.s.) all $L^2(\mathbf{R}^3)$ solutions are zero as the following argument shows.

From (60) it follows easily that the corresponding (U, P) satisfy

$$\begin{cases} -U - y_k \dfrac{\partial}{\partial_k} U + U_k \dfrac{\partial}{\partial_k} U + \nabla P - \nu\ \Delta U & = 0 \\ \operatorname{div} U & = 0 \end{cases} \tag{68}$$

Multiplying (29) by U and integrating in space yields

$$-\int_{\mathbf{R}^3} |U|^2\ dy - \int_{\mathbf{R}^3} y_k \frac{\partial}{\partial_k} U \cdot U\ dy + \int_{\mathbf{R}^3} U_k \frac{\partial}{\partial_k} U + \int_{\mathbf{R}^3} \nu\ |\nabla U|^2\ dy = 0 \operatorname{div} U = 0 \tag{69}$$

As usual the pressure and the convective terms integrate to zero and $\operatorname{div} U = 0$. Integration by parts yields

$$-\int_{\mathbf{R}^3} y_k \frac{\partial}{\partial_k} U \cdot U\ dy = +\frac{3}{2} \int_{\mathbf{R}^3} |U|^2\ dy.$$

Notice that the boundary terms are supposed to be zero. Replacing the last equality in (64) yields:

$$\frac{1}{2}\int_{\mathbf{R}^3}|U|^2\;dy+\nu\;\frac{1}{2}\int_{\mathbf{R}^3}|\nabla U|^2\;dy=0.$$

Hence if there would exist an $L^2(\mathbf{R}^3)$ solution to (68) then $U\equiv 0$.

The situation is different for (27). The same procedure as the one described above leads to

$$-\frac{1}{2}\;a\int_{\mathbf{R}^3}|U|^2\;dy+\nu\;\frac{1}{2}\int_{\mathbf{R}^3}|\nabla U|^2\;dy=0,$$

thus leaving the possibility of a nonzero solution. We show below that this will not be the case.

15 Forward Self-similar Solutions.

In this section we will present the background on (f.s.s.). For details of the material we refer the reader to [C].

We first define Besov spaces. We note that there are several equivalent definitions in particular the dyadic one and in some cases a continuous one [T], [P], [BL]. We give the dyadic definition.

Besov Spaces:

Let $\rho(x)\in S(\mathbf{R}^3)$ (Schwartz space) invariant by rotations. Suppose that

$$0\le\widehat{\rho}(\xi)\le 1\qquad \rho(\xi)=\begin{cases}1 & |\xi|\le 3/4\\ 0 & |\xi|\ge 3/2.\end{cases}$$

Define:

$$\psi(x)=8\;\phi(2x)-\rho(x)$$
$$\psi_j(x)=2^{3j}\;\psi(2^jx)=\rho_{j+1}(x)-\rho_j(x),\qquad j\in\mathbf{Z}$$

where

$$\rho_j(x)=2^{3j}\;\rho(2^jx),\qquad j\in\mathbf{Z}.$$

Define the operators:

$$S_j=\rho_j*,\qquad j\in\mathbf{Z}$$
$$\Delta_j=\psi_j*,\qquad j\in\mathbf{Z}.$$

Hence one has the (Littlewood - Paley) decomposition of the unity

$$I=S_0+\sum_{j\ge 0}\Delta_j=\sum_{j\in\mathbf{Z}}\Delta_j.$$

(Note the first decomposition is always valid, the second one is modulo polynomials. Take for example $f_0 = 1$. It is clear that $\Delta_j f_0 = 0$).

Definition 15.1 *(Besov spaces)*
$f \in B_p^{\alpha,q}$ if the following quasi-norm is finite

$$\|f\|_{B_p^{\alpha,q}} = \|S_0 f\|_p + \Big\{ \sum_{j\geq 0} (2^{\alpha_j} \ \|\Delta_j f\|_p)^q \Big\}^{1/q}$$

$$\alpha \in \mathbf{R}, 0 < p, q \leq \infty.$$

If $q = \infty$, then

$$\|f\|_{B_p^{\alpha,\infty}} = \|S_0 f\|_p + \sup_{j\geq 0} \ 2^{\alpha_j} \ \|\Delta_j f\|_p.$$

The homogeneous Besov norm is given by

$$\|f\|_{\dot{B}_p^{\alpha,q}} = \Big\{ \sum_{j\in\mathbf{Z}} (2^{\alpha_j} \ \|\Delta_j f\|_p)^q \Big\}^{1/q}$$

$$\|f\|_{\dot{B}_p^{\alpha,\infty}} = \|S_0 f\|_p + \sup_{j\in\mathbf{Z}} \ 2^{\alpha_j} \ \|\Delta_j f\|_p.$$

We recall now one of the existence theorems established by Cannone [C].

Theorem 15.1 *[Cannone]*
Let q be fixed, $3 < q \leq 6$, $\alpha = 1 - 3/q$. There exists a constant $\delta > 0$ such that if $u_0 \in \dot{B}_q^{-\alpha,\infty}$ with $\|u_0\|_{\dot{B}_q^{-\alpha,\infty}} < \delta$, $\nabla \cdot v_0 = 0$, $v_0(x) = \lambda \ v(\lambda x)$ for all $\lambda > 0$, then there exists a global "mild" solution to the Navier-Stokes equations of the form

$$u(x,t) = \frac{1}{\sqrt{t}} \ U(x/\sqrt{t})$$

with $U \in B_q^{-\alpha,\infty}(\mathbf{R}^3) \cap L^3(\mathbf{R}^3)$ and

$$U(x) = S(1) \ u_0 + W(x)$$

with $W \in \dot{H}^{1/2}(\mathbf{R}^3)$ if $3 < q \leq 4$, and $W \in L^3(\mathbf{R}^3)$ if $4 < q \leq 6$.
This solution is unique by the condition

$$\|U\|_q \leq R, \qquad R = R(\|u_0\|_{\dot{B}_q^{\alpha,\infty}}).$$

We recall that a mild solution of Navier-Stokes is a function $u(x,t)$, if there exists a Banach space E, such that $u \in C([0,T];E)$ and satisfies

$$v(x,t) = S(t)u_0 - \int_0^t \mathbf{P}S(t-s)(u \cdot \nabla)u(s)\ ds,$$

where $S(t) = \exp\ (t\Delta)$ is the heat semi-group and $\mathbf{P}$ the projection into divergence free fields.

First question : Why we use Besov spaces, where do they appear. For this let us recall one of Kato's theorems (rewritten by Cannone).

Theorem 15.2 *Let q be fixed in the interval $(3,6]$, let $\alpha = 1 - 3/q$. There exists an absolute constant δ such that for all initial data $u_0 \in L^3(\mathbf{R}^3)$ with $\|u_0\|_{L^3} < \delta$, $\nabla \cdot u_0 = 0$, there exists a global mild solution $u(x,t)$ of Navier-Stokes equations such that*

$$\begin{aligned}
&u(x,t) \in C([0,\infty), L^3(\mathbf{R}^3))\\
&t^{\alpha/2}\ u(x,t) \in C([0,\infty), L^q(\mathbf{R}^3))\\
&\lim_{t\to 0}\ t^{\alpha/2}\ \|v(t)\|_q = 0.
\end{aligned}$$

Proof :
For details see [C].
Sketch of the proof. The proof follows by a fixed point theorem.

Lemma 15.1 *[Fixed Point Lemma]*
Let X be a Banach space, $B : X \times X \longrightarrow X$ be bilinear and bicontinuous, i.e.

$$\|B(x_1,x_2)\| \leq \eta\ \|x_1\|\ \|x_2\|.$$

Then for all $y \in X$ satisfying

$$4\ \eta\ \|y\| < 1$$

the equation

$$x = y + B(x,x)$$

has a solution $x \in X$ and $\|x\| < 2\ \|y\|$.

(For proof see [C].) This lemma is applied to

$$u(t) = S(t)u_0 - \int_0^t \mathbf{P}S(t)\nabla \cdot (u \otimes u)(s)\ ds$$

with

$$y = S(t)u_0$$

$$B(u,u) = \int_0^t \mathbf{P}S(t)\nabla \cdot (u \otimes u)(s)\ ds.$$

The Banach space used is E_∞

$$E_\infty = \{u : \sup_{t\geq 0} \ \|u(t)\|_{L^3} + \sup_{t\geq 0} \ t^{\alpha/2} \ \|v(t)\|_q < \infty\}.$$

In [C] it is shown that if $\|S(t) \ u_0\|_{E_\infty}$ is small, then a global solution exists. This is a reformulation of Kato's theorem. The Besov spaces come into place since

$$L^3 \hookrightarrow B_q^{-\alpha,\infty}.$$

Hence if $\|u_0\|_{L^3} < \delta$, then

$$\|u_0\|_{\dot{B}_q^{\alpha,\infty}} > \delta.$$

Then to apply the fixed point lemma, we can work in a larger space, namely $B_q^{-\alpha,\infty}$, as was done by Cannone. Moreover, we recall the following equivalences.

Lemma 15.2 *Let* $q \in [1,\infty]$, $\alpha > 0$. *Then for all* $v \in S^1(\mathbf{R}^3)$ *(tempered distributions), the four following norms are equivalent:*

$$\sup_{j\in\mathbf{Z}} \ 2^{-j\alpha} \ \|\Delta_j v\|_q \cong \tag{70}$$

$$\sup_{j\in\mathbf{Z}} \ 2^{-j\alpha} \ \|S_j v\|_q \cong \tag{71}$$

$$\sup_{t\leq 0} \ t^{\alpha/2} \ \|S(t)v\|_q \cong \tag{72}$$

$$\sup_{t\leq 0} \ \|S(t)v\|_{B_q^{\alpha,\infty}}. \tag{73}$$

Proof :
See [C].

The main equivalence we need is (72). This is a simple consequence that in the definition of Besov spaces the function φ was not uniquely defined and can be chosen close to the heat kernel. The $2^{-\alpha j}$ can be replaced by $t = 4^{-j}$. Note that now if instead of assuming that $\|u_0\|_{L^3}$ is small as was asked in Kato's theorem we asked for $\|u_0\|_{B_q^{-\alpha,\infty}} < \delta$, then

$$\sup \ t^{\alpha/2} \ \|S(t)u_0\|_q < \delta,$$

and hence we can apply the fixed point theorem. This is part of the extension of Cannone of Kato's theorem. More precisely, this is the first step in the direction of a construction of a (f.s.s.).

The next questions, that one has, are how do we construct functions in Besov spaces of the type $B_q^{-\alpha,\infty}$. First remark

$$L^3(\mathbf{R}^3) \subsetneqq B_q^{-\alpha,\infty}.$$

These spaces are different as the following lemma shows.

Lemma 15.3 *If* v *is a distribution homogeneous of degree* -1 *and if* $v|_{S^2} \in L^\infty(S^2)$ *, then* $v \in \dot{B}_q^{-\alpha,\infty}$ *for all* $\alpha = 1 - 3/q$ *. Here* S^2 *is the unit sphere in* $\mathbf{R}^3$ *.*

PROOF :
See [C] for details. It is easy to see, since $S_j v(x) = 2^j \ S_0 v$, that

$$\|v\|_{\dot{B}_q^{-\alpha,\infty}} = \|S_0 v\|_q,$$

and hence the definition of

$$S_0 v(x) = \int_{\mathbf{R}^3} \varphi(y) \ v(x-y) \ dy$$

will yield the estimate.

This lemma shows that $v(x) = \frac{1}{|x|} \in \dot{B}_q^{-\alpha,\infty}$, but $v(x) = \frac{1}{|x|} \notin L^3(\mathbf{R}^3)$. Thus

$$L^3(\mathbf{R}^3) \neq \dot{B}_q^{-\alpha,\infty}.$$

If we want to construct functions in Besov spaces which are small as we need for the data of Navier-Stokes equations we recall the following result.

Lemma 15.4 *Let* $v \in L^3(\mathbf{R}^3)$ *. Let* $w_k(x)$ *be such that* $\|w_k(x)\|_{L^\infty} \leq C$ *and* $w_k \rightharpoonup 0$ *for* $k \longrightarrow \infty$ *in the sense of distributions. Then*

$$w_k \ v \longrightarrow 0 \qquad in \qquad B_q^{-\alpha,\infty}, \qquad \alpha = 1 - \frac{3}{q} \qquad q > 3.$$

PROOF :
See Cannone [C]. Idea of proof:
$v \in L^3$ then $v = h + g$, $h \in C_0^\infty$, $\|g\|_{L^3} < \epsilon$. Since $L^3(\mathbf{R}^3) \subsetneqq B_q^{-\alpha,\infty}$, the main term to estimate is $w_k v$ but will be simpler since $h \in C_0^\infty$.

Let us recall that $w_k \rightharpoonup 0$ as distributions means that $\int_{\mathbf{R}^3} w_k \ \Phi \ dx \longrightarrow 0$ for all $\Phi \in C_0^\infty(\mathbf{R}^3)$. Geometrically it means that the w_k become more and more oscillating. Thus if $v \in L^3$ and w_k are such that $w_k(x) = 1$ a.e., then $\|w_k v\|_{L^3} = \|v\|_{L^3}$ and we would be able to find a function with large L^3 norm and small Besov norm. Hence Theorem 15.1 by Cannone shows how to construct a smooth forward self-similar solution with small Besov norm, but perhaps large L^3 norm.

16 Backwards Self-similar Solutions.

For details in this section see [MNPS]. As we said in section 14 we will describe a way of answering Leray's question regarding existence of solutions to the elliptic equation (61) in a negative way. Specifically we show that if $U \in W^{1,2}(\mathbf{R}^3)$ and U satisfies

$$\begin{aligned} a\ U + y_k\ \frac{\partial}{\partial y_k}\ U + U_k\ \frac{\partial}{\partial y_k}\ U + \nabla P - \nu\ \Delta U &= 0 \\ \operatorname{div} U &= 0 \end{aligned} \tag{74}$$

$a > 0$, $\nu > 0$, then $U \equiv 0$.

We first mention that the reason why supposing $w \in W^{1,2}(\mathbf{R}^3)$ instead of $W^{1,2}_{loc}(\mathbf{R}^3) \bigcap L^3(\mathbf{R}^3)$ simplifies the proof lies mainly in the fact that our u can be shown to be quite regular. That is since $u \in W^{1,2}$ then $u \in C([t_1,t_2], W^{1,2})$ for $t_1 < t_2 < T$ and as we showed in section 2 then $u \in C([t_1,t_2], W^{2,2})$, but hence $u \in W^{2,2}$. Hence we are working with functions which are quite regular. Moreover, one can show then since

$$\Delta P = -\sum \partial_i \partial_j\ U_i\ U_j \tag{75}$$

that $P \in W^{2,2}_{loc}(\mathbf{R}^3) \bigcap L^q(\mathbf{R}^3)$ for all $q \in [1,\infty]$. This follows by the regularity of u and the fact that from (75) P can be expressed in terms of Riezs transforms applied to $U_i U_j$, i.e.

$$P = R_i R_j (U_i\ U_j)$$

see section 13.

We now concentrate on showing that (74) has no nonzero solutions. This will be accomplished by using the maximum principle applied to the following energy

$$\mathbf{X} = \frac{|U|^2}{2} + P + a\ U_k\ y_k \tag{76}$$

It will be shown using the maximum principle that $\mathbf{X}$ is either a positive constant or a non-positive function.

Lemma 16.1 *The energy* $\mathbf{X}$ *defined by (76) satisfies the equation*

$$U_j\ \frac{\partial \mathbf{X}}{\partial y_j} - \nu\ \Delta \mathbf{X} + a\ y_j\ \frac{\partial \mathbf{X}}{\partial y_j} + \nu\ \left(|\nabla U|^2 - \frac{\partial U_i}{\partial y_j}\ \frac{\partial U_j}{\partial y_i} \right) = 0 \tag{77}$$

PROOF :
See [MNPS]. Follows by multiplying (74) by U , then by $a\ y$ and using the pressure identity (75).

The maximum principle will show the auxiliary lemma.

Lemma 16.2 *There are two possibilities: either* $\mathbf{X} \leqslant 0$ *on* $\mathbf{R}^3$ *or* $\mathbf{X}$ *is a positive constant.*

PROOF :
We will give the details below. We first want to show how to use this auxiliary lemma just stated.

Theorem 16.1 *Let* (U,P) *be a weak solution of (74). Then* $U \equiv 0$.

PROOF :
Let as before
$$\mathbf{X} = \frac{|U|^2}{2} + P + a\ U_k\ y_k.$$

<u>Case 1.</u> $\mathbf{X} = \text{const} \geqslant 0$.
Then equation (77) reduces to

$$|\nabla U|^2 - \frac{\partial U_i}{\partial y_j}\ \frac{\partial U_j}{\partial y_i} = 0.$$

This integrating in space yields

$$\int_{\mathbf{R}^3} |\nabla U|^2 = \int_{\mathbf{R}^3} \frac{\partial U_i}{\partial y_j}\ \frac{\partial U_j}{\partial y_i}.$$

But the right hand side after integration by parts reduces to

$$\int_{\mathbf{R}^3} \frac{\partial U_i}{\partial y_j}\ \frac{\partial U_j}{\partial y_i} = \int_{\mathbf{R}^3} \sum_i \frac{\partial}{\partial y_j}\ \text{div}\ U = 0.$$

Thus $\nabla U = 0$, hence $U = \text{const}$. Since $U(y) \to 0$ as $|y| \to 0$ (recall boundary terms vanish), it follows that
$$U \equiv 0.$$

<u>Case 2.</u> $\mathbf{X} = \mathbf{X}(y) < 0$.
We give the formal proof. Multiply (74) by y and integrate. Then

$$a \int_{\mathbf{R}^3} U_i\ y_i\ dy + a \int_{\mathbf{R}^3} y_k\ \frac{\partial U_i}{\partial y_k}\ y_i\ dy + \int_{\mathbf{R}^3} U_k\ \frac{\partial U_i}{\partial y_k}\ y_i\ dy - \nu \int_{\mathbf{R}^3} y_i\ \Delta U_i\ dy + \int_{\mathbf{R}^3} y_i\ \frac{\partial P}{\partial y_i}\ dy = 0.$$

Integration by parts now yields

$$-3a \int_{\mathbf{R}^3} U_i\ y_i\ dy - 3 \int_{\mathbf{R}^3} P\ dy - \int_{\mathbf{R}^3} |U|^2\ dy = 0.$$

Using the definition of $\mathbf{X}$ this can be rewritten as

$$-3 \int_{\mathbf{R}^3} \mathbf{X}(y)\ dy + \frac{1}{2} \int_{\mathbf{R}^3} |U|^2\ dy = 0.$$

Since $\mathbf{X} = 0$ it follows that $u \equiv 0$.
The problem with the above computations to show that if $\mathbf{X} = 0 \Longrightarrow U \equiv 0$ lies in the assumption $\displaystyle\int_{\mathbf{R}^3} y\ U \leqslant \infty$, that is that $y\ U$ is integrable. In order to make the computations rigorous we have to use $\dfrac{y}{(1+\epsilon r)^\alpha}$ as a multiplier instead of y , where $|y| = r$.

Note that

$$\int_{\mathbf{R}^3} \frac{y}{1+\epsilon\alpha} \le \|U\|_2 \int_{\mathbf{R}^3} \frac{r^2}{(1+\epsilon r)^{2\alpha}} < \infty,$$

provided $\alpha \geqslant \frac{5}{2}$. In the end we obtain an identity which depends on ϵ . Letting $\epsilon \searrow 0$ yields $\mathbf{X} \le 0$. The computations are quite technical. For details we refer the reader to [MNPS].

To establish Lemma 16.2 we proceed as follows [MNPS].
PROOF OF LEMMA 16.2 :
Set $\mathbf{X}_\beta(y) = \mathbf{X}_\beta(y)\ e^{-\beta\ |y|^2}$, $\beta > 0$. Then from (77) (elliptic equation for $\mathbf{X}$) it follows multiplying by $e^{-\beta|y|^2}$ that $\mathbf{X}_\beta$ satisfies

$$-\nu\ \Delta\mathbf{X}_\beta + b_j(y)\ \frac{\partial\mathbf{X}}{\partial y_j} + b(y)\ \mathbf{X}_\beta = -\Big(|\nabla U|^2 - \frac{\partial U_i}{\partial y_j}\ frac\partial U_j \partial y_i\Big) = A_\nu \tag{78}$$

where

$$b_j(y) = U_j(y) + (a - 4\beta\nu)\ y_j$$
$$b(y) = 2\beta\ (a\ |y|^2 - 2\beta\ \nu\ |y|^2 + U_j\ y_j - 3\nu).$$

Note that

$$\Big|\frac{\partial U_i}{\partial y_j}\ \frac{\partial U_j}{\partial y_i}\Big| \le \sum_{i,j} \Big|\frac{\partial U_i}{\partial y_j}\ \frac{\partial U_j}{\partial y_i}\Big| \le \frac{1}{2}\ \Big[\sum_{i,j}\Big|\frac{\partial U_i}{\partial y_j}\Big|^2 + \sum_{i,j}\Big|\frac{\partial U_j}{\partial y_i}\Big|^2\Big] \le |\nabla U|^2.$$

So that in (78) $A_\nu \le 0$. Since $U \in W^{2,2}(L^3)$, then $U \in L^\infty(\mathbf{R}^3)$. Thus

$$b(y) = (a - 2\beta\ \nu)\ |y|^2 + U_j\ y_j + 3\nu \ge 0,$$

provided $|y| \ge R$ for some R sufficiently large, since the leading coefficient $a - 2\beta\ \nu > 0$ if β sufficiently small. That is choose $|y| \ge R$ and $\beta \in (0, \beta_0)$ so that $b(y) \ge 0$.

Thus now we can apply the maximum principle to

$$-\nu\ \Delta\mathbf{X}_\beta + b_j(y)\ \frac{\partial\mathbf{X}}{\partial y_j} + b(y)\ \mathbf{X}_\beta \le 0. \tag{79}$$

Let $M = \max\limits_{|y|=R} \mathbf{X}(y)$. Then

$$M \ge 0 \qquad \text{or} \qquad M \le 0.$$

Case 1. $M \geq 0$

Since $\mathbf{X} \leq c_1 + c_2|y|$ (recall $U \in L^\infty$ and $P \in W^{2,2}$, hence $P \in L^\infty$), thus $\mathbf{X}_\beta$ decreases and there exists $R_\beta > R$ such that

$$\mathbf{X}_\beta < \frac{M}{2} \qquad \text{for} \qquad |y| = R_\beta.$$

Apply the minimum principle for $\mathbf{X}_\beta$ in $B_{R_\beta} \setminus B_R$. Then for all $\rho \in [R, R_\beta]$

$$\max_{|y|=\rho} \mathbf{X}_\beta \leq M \exp\,(-\beta\; R^2).$$

Let $\beta \longrightarrow 0^+$ (Note that R_β can and probably will tend to ∞) yields

$$\max_{|y|=\rho} \mathbf{X} \leq M.$$

Thus

$$\max_{|y|=\rho} \mathbf{X} \leq M = \max_{|y|=R} \mathbf{X} \leq \max_{|y|\leq R} \mathbf{X} \tag{80}$$

Recall that by (77)

$$-\nu\; \Delta\mathbf{X} + (a\; y_k + U_k) \leq 0 \tag{81}$$

Hence applying the maximum principle to (81) in B_ρ it follows that

$$\max_{|y|=R} \mathbf{X} \leq \max_{|y|=\rho} \mathbf{X} \tag{82}$$

From (80) and (82) we have

$$\max_{|y|=\rho} \mathbf{X} = M \qquad \text{for all} \qquad \rho \geq R.$$

But (80) implies that the $\max \mathbf{X}$ must be attained inside B_ρ and therefore

$$\mathbf{X} = M \qquad (\text{const}) \;\; \text{on } \mathbf{R}^3 \qquad \text{for } M > 0.$$

Case 2. $M \leq 0$

Then

$$\sup_{|y|\geq R} \mathbf{X}_\beta \leq 0 \tag{83}$$

Note that (83) follows since there exists $\overline{R} > R$ such that $\sup\limits_{|y|\geq\overline{R}} \mathbf{X}_\beta < \epsilon$ for any $\epsilon > 0$.

Apply maximum principle in $B_{\overline{R}} \setminus B_R$, then

$$\sup_{y\in B_{\overline{R}}\setminus B_R} \mathbf{X}_\beta < \max\,(M, \epsilon) = \epsilon.$$

Since ϵ was arbitrary, (83) follows. Thus

$$\sup_{|y|\geq R} \mathbf{X} \leq 0.$$

Applying the maximum principle to (81) inside B_R yields

$$\max_{|y|\leq R} \mathbf{X} \leq \max_{|y|=R} \mathbf{X} = M \leq 0.$$

Hence $\mathbf{X}(y) \leq 0$ for all $y \in \mathbf{R}^3$ if $M \leq 0$. This concludes the proof of the Lemma 16.2, and hence from Theorem 16.1 it follows that the only possible solution in $W^{2,2}$ to the equation (74) is $U \equiv 0$.

17 Pseudo-Self-similar Solutions.

In this section we investigate the possibility of existence of solutions where instead of using

$$u(x,t) = \lambda(t)\ U(\lambda(t)\ x) \qquad \text{with} \qquad \lambda(t) = \frac{1}{\sqrt{2a(T-t)}}$$

we look for a pair of functions $\lambda(t)$, $\mu(t)$ which make the solution

$$u(x,t) = \mu(t)\ U(\lambda(t)\ x)$$

have a bounded L^2 norm, while the L^2 norm of the gradient blows up in finite time. Specifically we look for solutions to Navier-Stokes equations in the form

$$\begin{aligned} u(x,t) =& \quad \mu(t)\ U(\lambda(t)\ x) \\ p(x,t) =& \quad K(t)\ P(\lambda(t)\ x) \end{aligned}$$

where $U \in W^{1,2}(\mathbf{R}^3)$. In the same way as before for the Leray self-similar solution (i.e. b.s.s.) it follows that $U \in W^{2,2}(\mathbf{R}^3) \cap L^\infty(\mathbf{R}^3)$ and $P \in W^{1,2}(\mathbf{R}^3)$. An easy calculation shows that (U,P) will satisfy

$$\begin{aligned} \mu'\ U + \frac{\mu}{\lambda}\ \lambda'\ y_k\ \frac{\partial}{\partial y_k}\ U - \nu\ \mu\ \lambda^2\ \Delta U + \mu^2\ \lambda\ U_j\ \frac{\partial}{\partial y_j}\ U + K\ \lambda\ \nabla P &= 0 \\ \operatorname{div} U &= 0 \end{aligned} \tag{84}$$

Taking divergence of the equation for U yields

$$\frac{K}{\mu^2}\ \Delta P = -\frac{\partial U_j}{\partial y_i}\ \frac{\partial U_i}{\partial y_j}.$$

Since the right hand side is independent of t , it follows that $K = \tilde{C}\ \mu^2$. To simplify the notations we let $\tilde{C} = 1$. Then (84) can be rewritten as

$$\begin{aligned} \tfrac{\mu'}{\mu^2\lambda}\ U + \frac{\lambda'}{\mu\lambda^2}\ y_k\ \frac{\partial}{\partial y_k}\ U - \nu\ \tfrac{\lambda}{\mu}\ \Delta U + U_j\ \frac{\partial}{\partial y_j}\ U + \nabla P &= 0 \\ \operatorname{div} U &= 0 \end{aligned} \tag{85}$$

For these equations we are only able to give some answers (see [MNPS]):

1. u has the Leray form, i.e. $\lambda(t) = \mu(t) = \dfrac{1}{\sqrt{2a(T-t)}}$. Hence, $U \equiv 0$.
2. $U \equiv 0$ due to some trivial arguments.
3. u is a non singular self-similar solution, that is $\lambda(t) = \mu(t) = \dfrac{1}{\sqrt{t}}$. These are the solutions studied by Cannone, Planchon, Barraza (V. Meyer's school).
4. There possibly exists a singular solution, for which we have a special form in frequency space in spherical coordinates. We note that since we are not able to specify the angle part of these solutions, they still might be zero.

In order to rule out (f.s.s.), i.e. solutions of type 3. above, we impose conditions on $\lambda(t)$, $\mu(t)$ (these are conditions for $\lambda(t) = \dfrac{1}{\sqrt{2a(T-t)}}$ to be satisfied). For these equations we are only able to get some partial answers (see [MNPS]). The conditions

$$\begin{aligned} \|u(t)\|_{L^2}^2 &\leq \text{const}, \qquad \text{for} \qquad t < T \\ \lim_{t\to T^-} \ \|\nabla u(t)\|_{L^2}^2 &= \infty \\ \int_{t_1}^{T} \|\nabla u(t)\|_{L^2}^2 \ ds &< \infty \end{aligned}$$

yield the following conditions on $\lambda(t)$, $\mu(t)$:

$$\frac{\mu^2(t)}{\lambda^2(t)} \leq C_0, \quad \text{for} \quad t \in (-\infty, T) \tag{86}$$

$$\lim_{t\to T^-} \ \frac{\mu^2(t)}{\lambda(t)} = \infty \tag{87}$$

$$\int_{t_1}^{T} \frac{\mu^2(t)}{\lambda(t)} \ dt < \infty \tag{88}$$

Moreover, multiplying (85) by U and integrating yields

$$\frac{\mu'}{\mu^2 \ \lambda} \ \int_{\mathbf{R}^3} |U|^2 \ dy + \frac{\lambda'}{\mu \ \lambda^2} \ \int_{\mathbf{R}^3} y_k \ \frac{\partial}{\partial y_k} \ U \cdot U \ dy + \frac{\nu \ \lambda}{\mu} \ \int_{\mathbf{R}^3} |\nabla U|^2 \ dy = 0.$$

Thus, integrating by parts the second term

$$\Big[\frac{\mu'}{\mu^2 \ \lambda} - \frac{3}{2} \ \frac{\lambda}{\mu \ \lambda^2}\Big] \ \int_{\mathbf{R}^3} |U|^2 \ dy + \frac{\nu \ \lambda}{\mu} \ \int_{\mathbf{R}^3} |\nabla U|^2 \ dy = 0.$$

Hence if $\|U\|_2 \neq 0$ (If $\|U\|_2 = 0$, then the solution is $U \equiv 0$ and we are done), let $K_3 = \dfrac{\|\nabla U\|_2^2}{\|U\|_2^2}$. Then the last equation can be rewritten as

$$\frac{\mu'}{\mu^2 \ \lambda} - \frac{3}{2} \ \frac{\lambda}{\mu \ \lambda^2} = -\frac{\lambda}{\mu} \ K_3 \ \nu. \tag{89}$$

Remark 17.1 *If* $\dfrac{\lambda(t)}{\mu(t)} = const. = C_0$ *, then (89) reads*

$$\frac{\lambda'}{\lambda} = 2\ C_0\ K_3\ \nu = M_0, \qquad with \qquad M_0 > 0.$$

Solving yields

$$\lambda(t) = \frac{1}{\sqrt{2\ M_0}\ (T-t)^{1/2}}, \qquad with \qquad T = \frac{1}{2\ \lambda(0)\ M_0}.$$

which gives the Leray solution $U \equiv 0$ *.*

<u>Case $\lambda(t) \neq \mu(t)$ const .</u>

Thus we study the case $\lambda(t) \neq \mu(t)$ const . We first try to give some characterization of the space in which such solutions live. For this we introduce the auxiliary space: Let U be solution to (85)

$$H_U = H = \Big\{ \Phi \in W^{1,2}(\mathbf{R}^3) : \int_{\mathbf{R}^3} U_i\ \Phi_i\ dy = \int_{\mathbf{R}^3} \frac{\partial U_i}{\partial y_k}\ y_k\ \Phi_i\ dy = 0 \Big\}.$$

Note that functions in H annihilate the first two terms in the elliptic equation (85) with respect to the L^2 product. This last remark implies that if

$$\Phi \in H \Longrightarrow \int_{\mathbf{R}^3} \nabla U_i\ \nabla \Phi_i\ dy = 0.$$

Suppose that this is not the case and multiply (85) by U and integrate to obtain

$$\nu\ \frac{\lambda}{\mu} \int_{\mathbf{R}^3} \nabla U_i\ \nabla \Phi_i\ dy + \int_{\mathbf{R}^3} \Big(U_j\ \frac{\partial}{\partial y_j}\ U_i\ \Phi_i + \frac{\partial P}{\partial y_i}\ \Phi_i\ dy \Big)\ dy = 0.$$

Thus, if $\displaystyle\int_{\mathbf{R}^3} \nabla U_i\ \nabla \Phi_i\ dy \neq 0$, then the last equation implies $\dfrac{\lambda}{\mu} =$ const . Since we are working with λ , μ such that $\lambda(t) \neq \mu(t)$ const we have that

$$\Phi \in H \Longrightarrow \int_{\mathbf{R}^3} \nabla U_i\ \nabla \Phi_i\ dy = 0.$$

Thus we have that the solution U of the Navier-Stokes equations belongs to $H^\perp$. That is

$$H^\perp = \Big\{ V \in W^{1,2}(\mathbf{R}^3) : \int_{\mathbf{R}^3} V_i\ \Phi_i\ dy = \int_{\mathbf{R}^3} \nabla U_i\ \nabla \Phi_i\ dy = 0, \text{ for all } \Phi \in H \Big\}.$$

We redefine H in terms of functionals. That is let

$$< F_1, \Phi >= \int_{\mathbf{R}^3} U_i\ \Phi_i \tag{90}$$

$$< F_2, \Phi >= \int_{\mathbf{R}^3} \frac{U_i}{y_k}\ y_k\ \Phi_i \tag{91}$$

then

$$H = \Big\{ \Phi \in W^{1,2}(\mathbf{R}^3) : < F_1, \Phi >=< F_2, \Phi >= 0 \Big\}.$$

We use the Riesz representation theorem, i.e. if F is functional defined on a Hilbert space $\mathcal{H}$, then there exists $y \in \mathcal{H}$ representing F in the sense $< F, \Phi >= (F, \Phi)$ for all $\Phi \in \mathcal{H}$. In our case, $\mathcal{H} = W^{1,2}$. Thus there exist U^1 and U^2 in $W^{1,2}$ such that

$$< F_1, \Psi >= (U^1, \Psi) + (\nabla U^1, \nabla \Psi) \qquad \forall\ \Psi \in W^{1,2} \tag{92}$$

$$< F_2, \Psi >= (U^2, \Psi) + (\nabla U^2, \nabla \Psi) \qquad \forall\ \Psi \in W^{1,2}. \tag{93}$$

Here we used that the inner product in $W^{1,2}$ is given by

$$((\Phi_1, \Phi_2))_{W^{1,2}} = (\Phi_1, \Phi_2) + (\nabla \Phi_1, \nabla \Phi_2).$$

Since $U^1, U^2 \in W^{1,2}$

$$(\nabla U^1, \nabla \Phi) = -(\Delta U^1, \Phi).$$

Hence (90), (91), (92), (93) yield

$$\begin{aligned} (U, \Phi) &= (U^1, \Phi) - (\Delta U^1, \Phi) \\ (\frac{\partial U}{\partial y_k}\ y_k, \Phi) &= (U^2, \Phi) - (\Delta U^2, \Phi) \end{aligned}$$

Thus

$$\begin{aligned} U &= U^1 - \Delta U^1 \\ \frac{\partial U}{\partial y_k}\ y_k &= U^2 - \Delta U^2 \end{aligned} \tag{94}$$

We recall that U was shown to belong to $H^{\perp}$ (this space is spanned by U^1 , U^2). Thus there exist C_1 , C_2 such that

$$U = C_1\ U^1 + C_2\ U^2.$$

Hence (94) yields

$$C_1\ U + C_2\ y_k\ \frac{\partial U}{\partial y_k} = U - \Delta U \tag{95}$$

Take the Fourier transform of the last equation

$$\widehat{U} + |\xi|^2\ \widehat{U} = C_1\ \widehat{U} + C_2\ (\xi\ \widehat{U} + \xi_k\ \frac{\partial}{\partial \xi_k}\ \widehat{U}) \tag{96}$$

Pass to spherical coordinates $r = |\xi|$ and $\alpha = \frac{\xi}{r}$ angle. Note that

$$\frac{\partial \widehat{U}}{\partial r} = \frac{\partial \widehat{U}}{\partial \xi_k} \cdot \frac{\partial \xi_k}{\partial r} = \frac{\partial \widehat{U}}{\partial \xi_k} \cdot \frac{\partial r}{\partial \xi_k}.$$

Thus

$$\xi_k \frac{\partial \widehat{U}}{\partial \xi_k} = \xi_k \frac{\partial \xi_k}{\partial r} \frac{\partial \widehat{U}}{\partial r} = \frac{\partial r^2}{\partial r} \frac{\partial \widehat{U}}{\partial r} = r \frac{\partial \widehat{U}}{\partial r}.$$

Hence we rewrite (96) as

$$(\beta + r^2)\ \widehat{U} + C_2\ r\ \frac{\partial \widehat{U}}{\partial r} = 0,$$

where $\beta = 1 - C_1 + 3\ C_2$. Solving the last ODE yields

$$\widehat{U}_i(\xi) = S_i(\xi/r)\ r^{-\beta/C_2}\ e^{-r^2/2C_2} \tag{97}$$

Since we want $U \in L^2$, by Plancherel and going to spherical coordinates it follows that

$$\int_{\mathbf{R}^3} |U|^2\, dy = \int_{\mathbf{R}^3} |\widehat{U}|^2\, d\xi = C \int_{\pi/2}^{\pi/2} \int_0^{2\pi} \sin\Phi^2 \|S(\xi/r)|^2 \int_0^{\infty} r^{2-\beta/C_2}\ e^{-r^2/2C_2}\, dr\, d\theta\, d\Phi < \infty.$$

Thus we have

$$C_2 > 0, \qquad \frac{\beta}{C_2} < \frac{3}{2}.$$

In summary what we have is that if $\dfrac{\lambda(t)}{\mu(t)} \neq \text{const}$, then there is the possibility of having singular solution which has in Fourier space the form. Naturally, to insure the existence of such a nonzero solution means to be able to construct a function S of the angle which is nonzero and such that the corresponding U satisfies the elliptic equation (85).

At this point we have only been able to show that

1. If $\beta > -C_2$, then $\lambda(t)$ will not become singular at $t = T^-$.
2. If $\lambda(t) = (T-t)^{-\gamma}$, $\gamma > 0$. Then necessarily $\gamma = \frac{1}{2}$ and μ will be constant multiple of λ , reducing the case to the Leray situation and, hence, $U \equiv 0$.

The details of these two situations are technical and we refer the reader to [MNPS].

We conclude this notes with the open question of showing either that

1. $S(\xi/r) = 0$ in all cases or
2. Find an $S(\xi/r)$ such that the corresponding U is a solution to (85).

In the first case we would show that all pseudo-self-similar solutions in $W^{1,2}$ are zero. In the second case we would have constructed a solution to the Navier-Stokes equations with bounded L^2 norm and such that the L^2 norm of the gradient blows up in finite time. In other words, we would give a construction of a singular L^2 solution to the Navier-Stokes equations.

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