ASYMPTOTIC BEHAVIOR TO DISSIPATIVE QUASI-GEOSTROPHIC FLOWS

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ABSTRACT. We consider the long time behavior of solutions of dissipative Quasi-Geostrophic flow (QG) with sub-critical powers. The flow under consideration is described by the nonlinear scalar equation

$$\frac{\partial\theta}{\partial t} + u\cdot\nabla\theta + \kappa(-\triangle)^{\alpha}\theta = f, \qquad \theta|_{t=0} = \theta_0 \tag{0.1}$$

Rates of decay are obtained, for both the solutions and higher derivatives in different Sobolev spaces.

1. Introduction

In this paper we are concerned with the long time behavior of the solutions to a special case of surface 2D dissipative Quasi-Geostrophic flows (DQG) with sub-critical powers α

$$\frac{\partial\theta}{\partial t} + u\cdot\nabla\theta + \kappa(-\triangle)^{\alpha}\theta = f, \qquad \theta|_{t=0} = \theta_0 \tag{1.2}$$

Here $\alpha \in (0,1]$, $\kappa > 0$, $\theta(t)$ is a real function of two space variables $x \in \mathbb{R}^2$ and a time variable t. The function $\theta(t) = \theta(x,t)$ represents the potential temperature. The fluid velocity u is determined from θ by a stream function ψ

$$(u_1, u_2) = \left(-\frac{\partial\psi}{\partial x_2}, \frac{\partial\psi}{\partial x_1}\right) \tag{1.3}$$

where the function ψ satisfies

$$(-\triangle)^{\frac{1}{2}}\psi = -\theta$$

Equation (1.2) is obtained when dissipative mechanisms are incorporated into the inviscid 2D-Quasi-Geostrophic equation (2DQG). The 2DQG is derived from the General Quasi Geostrophic (GQG) equations by reduction to the special case of solutions with constant potential vorticity in the interior and constant buoyancy frequency [3]. For information on the GQG equations we refer the reader to [8]. The fractional power $\alpha = 1/2$ is perhaps the most interesting one since it corresponds to a fundamental model of quasi-geostrophic equations, see [4] and [8]. As pointed out in [4] "Dimensionally the 2DQG equation with $\alpha = 1/2$ is the analogue of the 3D Navier-Stokes equations."

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Two main problems will be considered. In the first, the power α will range in the interval $(\frac{1}{2}, 1]$. In this case when $\alpha \in (1/2, 1]$ it is known that the solutions are smooth on the torus, see [4]. In [16] Wu establishes regularity of solutions for certain type of data and forcing functions. Here we obtain smooth solutions in $\mathbb{R}^2$ by establishing uniform bounds in the H^m norms for solutions with appropriate data and forcing term. Interest will be then focused on the analysis of the asymptotic behavior of the energy of derivatives of all orders.

To establish decay in $H^m(I\!\!R^n)$ spaces the main tool will be the Fourier splitting method [11], [12]. This technique was used among others to treat solutions to Parabolic Conservation Laws (PCL) and Navier-Stokes Equations (NSE). What makes the approach different here is that unlike the case of PCL and NSE the dissipative mechanism is not given by a straightforward Laplacian but by a fractional power of the Laplacian and new estimates are necessary. Before even addressing questions of decay new estimates are necessary to establish uniform bounds for the derivatives.

Some of the proofs presented in this paper only consider the case when $\alpha \in (1/2, 1]$; these proofs could be extended to the case $\alpha \in (0, 1]$ provided there were an a priori bound (possibly time dependent) of the derivatives of the solutions in the space L^2. In particular the estimate obtained by Wu in [16] could be used once a uniform bound on the $W^{1,\infty}$ norm of the velocity u is established.

The second question we address is the decay of the solutions in L^p. Given the decay in L^2 obtained in [4], the new H^m decay obtained in the first part of the paper will yield immediately, via a Gagliardo-Nirenberg inequality, decay in all L^p spaces with $p \geq 2$. Decay rate in L^p had been already obtained in [16], for $p > 1$. The problem now is to improve this decay by imposing conditions on the initial data which insure the decay of the L^1 norm of the solutions. Two cases are considered. First the weak solution will be analyzed when $\alpha = 1/2$ and decay will be shown in L^1. Second, decay is established for solutions in $W^{q,p}$, $p \geq 2$ and $q \geq 1$, in the case where $\alpha \in (1/2, 1]$.

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1.1. **Notation and Preliminaries.** The Fourier transform of $v \in \mathcal{S}(\mathbb{R}^2)$ is defined by $\widehat{v}(\xi) = (2\pi)^{-1} \int_{\mathbb{R}^2} e^{-ix\cdot\xi} v(x)\, dx$. It is then extended as usual to $\mathcal{S}'$. Given a multi-index $\gamma = (\gamma_1, \gamma_2)$ and $m = |\gamma| = \gamma_1 + \gamma_2$, we denote

$$\partial^\gamma = \frac{\partial^{|\gamma|}}{\partial_{x_1}^{\gamma_1} \partial_{x_2}^{\gamma_2}}$$

and

$$D^m = \sum_{|\alpha|=m} \partial^\gamma.$$

If k is a nonnegative integer, $W^{k,p}(\mathbb{R}^2)$ will be, as is standard, the Sobolev space consisting of functions in $L^p(\mathbb{R}^2)$ whose generalized derivatives up to order k belong to $L^p(\mathbb{R}^2)$. As usual, when $p = 2$, then $W^{k,2}(\mathbb{R}^2) = H^k(\mathbb{R}^2)$ where (also as usual) the space H^s is defined for all $s \in \mathbb{R}$ as the space of all $f \in \mathcal{S}'$ such that $(1 + |\xi|^2)^{s/2} \hat{f}(\xi) \in L^2$.

Following Constantin and Wu [4], we denote by

$$\Lambda = (-\triangle)^{\frac{1}{2}} \tag{1.4}$$

the operator defined by $\widehat{\Lambda f}(\xi) = |\xi|\hat{f}(\xi)$. More generally, if $s \geq 0$, we define Λ^s by

$$\widehat{\Lambda^s f}(\xi) = |\xi|^s \hat{f}(\xi).$$

Clearly $\Lambda^s f$ is well defined (and in L^2) if $f \in H^s$. More generally, one can define the domain of Λ^s as consisting of all elements $f \in \mathcal{S}'$ such that $\hat{f}$ is a function (i.e., locally integrable); it is then clear that the definition given above defines $\Lambda^s f$ as a tempered distribution.

We denote by $\mathcal{R}_1, \mathcal{R}_2$ the Riesz-transforms in $\mathbb{R}^2$; i.e., $\widehat{\mathcal{R}_j f}(\xi) = -i(\xi_j/|\xi|)\hat{f}(\xi)$. The operator $\mathcal{R}^\perp$ taking scalar valued functions to vector valued functions is defined by

$$\mathcal{R}^\perp f = (-\partial_{x_2}\Lambda^{-1} f, \partial_{x_1}\Lambda^{-1} f) = (-\mathcal{R}_2 f, \mathcal{R}_1 f). \tag{1.5}$$

The relation between u and θ in (1.2) can then briefly be stated as $u = \mathcal{R}^\perp \theta$.

If F is a function defined on $\mathbb{R}^2 \times [0, \infty)$, we define for $t \geq 0$ the function $F(t)$ on $\mathbb{R}^2$ by $F(t)(x) = F(x, t)$. For such F, the Fourier transform (and inverse Fourier transform) is always with respect to the space variables; thus

$$\hat{F}(\xi, t) = \widehat{F(t)}(\xi)$$

for all $t \geq 0$. The letters C, C_0, C_1, etc., will denote generic positive constants, which may vary from line to line during computations.

2. Uniform Estimates

In this section we suppose $\alpha \in (1/2, 1]$. We show that $\Lambda^\beta \theta$ decays in L^2-norm for $\beta \geq 0$; in particular we establish the uniform boundedness of the solution θ in H^m if the initial datum $\theta_0 \in H^m$. Our results in Theorem 2.4 can easily be adapted to the torus and as such extend those of Constantin and Wu [4, Theorem 2.1]. The decay we obtain in this section is not optimal, but is needed to obtain the optimal rate of decay in the next section. In the last part of this section we establish uniform estimates on the L^∞ norms of the solutions. These estimates are obtained by bounding the L^1 norm of $\hat{\theta}$. We will need to use Theorem 3.1 from [4], we state it here for ease of reference.

Theorem 2.1. *Let $\alpha \in (0, 1]$ and $\theta_0 \in L^1 \cap L^2$. Assume that $f \in L^1([0, \infty); L^2)$, satisfying*

$$\|f(t)\|_2 \leq C_0(1+t)^{-\frac{1}{\alpha}-1}, \; |\hat{f}(\xi, t)| \leq C_0|\xi|\alpha \tag{2.1}$$

for some constant C_0. Then there exists a weak solution θ of the 2DQG equation

$$\frac{\partial \theta}{\partial t} + u \cdot \nabla \theta + \kappa(-\triangle)^\alpha \theta = f, \quad \theta|_{t=0} = \theta_0 \tag{2.2}$$

such that

$$\|\theta(\cdot, t)\|_{L^2(\mathbb{R}^2)} \leq C(t+1)^{-\frac{1}{2\alpha}} \tag{2.3}$$

where C is a constant depending on L^1 and L^2 norms of θ_0, on the $L^1(L^2)$ norm of f, and on C_0.

We also need the following Sobolev type estimate.

Lemma 2.2. *Let $2 < p < \infty$ and let $\sigma = 1 - \frac{2}{p}$. There exists a constant $C \geq 0$ such that if $f \in \mathcal{S}'$ is such that $\hat{f}$ is a function, then*

$$\|f\|_p \leq C\|\Lambda^\sigma f\|_2.$$

Proof: Since $\hat{f}$ is a function, we have $\hat{f}(\xi) = |\xi|^{-\sigma}|\xi|^\sigma \hat{f}(\xi)$. Taking the inverse Fourier transform, we get $f = I_\sigma(\Lambda^\sigma f)$ where I_σ is the Riesz potential of order σ. It is well known (cf. [13, Chapter V, Theorem 1]) that I_σ is bounded from $L^2(\mathbb{R}^2)$ to $L^p(\mathbb{R}^2)$ if $\frac{1}{p} = \frac{1}{2} - \frac{\sigma}{2}$. The Lemma follows.

Next, a simple observation connecting the L^2 norms of the temperature and the velocity (or transport term) that will be used repeatedly.

Remark 2.3. Let $1 < p < \infty$. There exists a constant C_p depending only on p such that

$$\|\Lambda^\beta u(t)\|_p \leq C_p\|\Lambda^\beta \theta(t)\|_p \tag{2.4}$$

for all $\beta \geq 0$, $t \geq 0$. If $p = 2$, this inequality can be strengthened to

$$\|\Lambda^\beta u(t)\|_2 = \|\Lambda^\beta \theta(t)\|_2. \tag{2.5}$$

In fact, (2.4) is immediate from the fact that $u = \mathcal{R}^\perp \theta$, the fact that the Riesz transforms commute with Λ^β and the boundedness of the Riesz transforms in L^p. Concerning (2.5), it suffices to observe that

$$\widehat{\Lambda^\beta u}(\xi, t) = \frac{i}{|\xi|}(\xi_2, \xi_1)|\xi|^\beta \hat{\theta}(\xi, t)$$

and the norm equality follows.

We are ready to state and prove the main result of this section. This first theorem gives a uniform bound for the derivatives of the solution $\theta(t)$ of the two dimensional DQG and, for a sufficiently fast decaying f, an auxiliary rate of decay that will be improved in the next section.

Theorem 2.4. *Let $\alpha \in (1/2, 1]$, $\beta \geq \alpha$ and assume q satisfies $2/(2\alpha - 1) < q < \infty$. Suppose $\theta_0 \in L^1 \cap L^2$, $\Lambda^\beta \theta_0 \in L^2$, $f \in L^1([0, \infty] : L^q \cap L^2)$, satisfies (2.1) and $\Lambda^{\beta-\alpha} f \in L^2((0, \infty), L^2)$. If θ is a solution to (1.2) with initial datum θ_0 then*

$$\|\Lambda^\beta \theta(t)\|_{L^2} \leq C_0(1 + t)^{-\frac{1}{2\alpha}} + C_1 \left(\int_0^t \|\Lambda^{\beta-\alpha} f(s)\|_2^2 \, ds\right)^{1/2}, \tag{2.6}$$

for $t \geq 0$, where C_0, C_1 are constants depending only on norms of the initial datum and f. In particular, if $f = 0$, then

$$\|\Lambda^\beta \theta(t)\|_{L^2} \leq C_0(1 + t)^{-\frac{1}{2\alpha}} \tag{2.7}$$

for all $t \geq 0$.

Remark 2.5. In [4, Theorem 2.1] the authors assume, in case $\beta < 1$, that $q = 2/(1 - \beta)$. This choice is consistent with our more general one, since it is also assumed in [4] that $\beta + 2\alpha > 2$, which implies $2/(1 - \beta) > 2/(2\alpha - 1)$. The assumption $\theta_0 \in L^1 \cap L^2$ (as well as f satisfying (2.1)) is needed to apply Theorem 2.1.

Proof:

The first part of the proof we present is formal. At the end of the proof we give a sketch on how to make the arguments rigorous. To obtain (2.6) multiply both sides of (1.2) by $\Lambda^{2\beta}\theta(t)$ and integrate in space,

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^2}|\Lambda^\beta\theta(t)|^2\,dx+\kappa\int_{\mathbb{R}^2}|\Lambda^{\alpha+\beta}\theta(t)|^2\,dx= -\int_{\mathbb{R}^2}(u\cdot\nabla\theta)\Lambda^{2\beta}\theta\,dx+\int_{\mathbb{R}^2}f\Lambda^{2\beta}\theta.\,dx \tag{2.8}$$

We estimate the second term on the right hand side of the last equation by

$$\int_{\mathbb{R}^2}f\Lambda^{2\beta}\theta\,dx\le\frac{\kappa}{8}\int_{\mathbb{R}^2}|\Lambda^{\alpha+\beta}\theta(t)|^2\,dx+\frac{2}{\kappa}\int_{\mathbb{R}^2}|\Lambda^{\beta-\alpha}f|^2\,dx. \tag{2.9}$$

Estimating the first term will take a little bit longer. We claim that there exists a constant $C(\kappa,\theta_0,f)$, depending only the initial datum θ_0, the $L^1(0,\infty,L^q)$-norm of the external force f, and κ, such that

$$\left|\int_{\mathbb{R}^2}(u\cdot\nabla\theta)\Lambda^{2\beta}\theta\,dx\right|\le\frac{\kappa}{8}\|\Lambda^{\alpha+\beta}\theta\|_2^2+C(\theta_0,f,\kappa)\|\Lambda^{s+1-(2/p)}\theta\|_2^2, \tag{2.10}$$

where $s=\beta-\alpha+1$ and p is determined by $\frac{1}{p}+\frac{1}{q}=\frac{1}{2}$ and q is as in the statement of the theorem. The meaning of s,p,q will not change for the remainder of this proof. To establish the claim, we begin observing that because div $u=0$ we can write

$$u\cdot\nabla\theta=\mathrm{div}(u\theta)-\theta\mathrm{div}u=\mathrm{div}(u\theta);$$

thus, by Plancherel, Hölder, and again Plancherel

$$\begin{aligned}\left|\int_{\mathbb{R}^2}(u\cdot\nabla\theta)\Lambda^{2\beta}\theta\,dx\right| &= \left|\int_{\mathbb{R}^2}(\xi_1\widehat{\theta u_1}(\xi)+\xi_1\widehat{\theta u_2}(\xi))|\xi|^{2\beta}\hat{\theta}(\xi)\,d\xi\right|\\ &\le\sum_{i=1}^{2}\int_{\mathbb{R}^2}|\xi|^{\beta-\alpha+1}|\widehat{\theta u_i}(\xi)||\xi|^{\alpha+\beta}|\hat{\theta}(\xi)|\,d\xi\\ &\le\sum_{i=1}^{2}\|\Lambda^{\beta-\alpha+1}(\theta u_i)\|_2\|\Lambda^{\alpha+\beta}\theta\|_2,\end{aligned}$$

hence

$$\left|\int_{\mathbb{R}^2}(u\cdot\nabla\theta)\Lambda^{2\beta}\theta\,dx\right|\le\frac{\kappa}{8}\|\Lambda^{\alpha+\beta}\theta\|_2^2+\frac{2}{\kappa}\sum_{i=1}^{2}\|\Lambda^s(\theta u_i)\|_2^2. \tag{2.11}$$

We estimate $\|\Lambda^s(\theta u_i)\|_2$ by the calculus inequality, getting

$$\|\Lambda^s(\theta u_i)\|_2\le C\left(\|u_i\|_q\|\Lambda^s\theta\|_p+\|\theta\|_q\|\Lambda^s u_i\|_p\right)$$

for $i=1,2$. This inequality follows easily by combining Hölder's inequality with the Gagliardo-Nirenberg and Young inequalities, see also inequality (3.1.59) on page 74 of [14]). Since $u_i=\pm\mathcal{R}_j\theta$ ($i,j\in\{1,2\}$, $i\ne j$) and the Riesz transforms commute with Λ and are bounded in L^p,L^q (notice that $2<p,q<\infty$), we have $\|\Lambda^s u_i\|_p\le C\|\Lambda^s\theta\|_p$ and $\|u_i\|_q\le C\|\theta\|_q$ for $i=1,2$. Applying this to the previous estimate, we get

$$\|\Lambda^s(\theta u_i)\|_2\le C\|\theta\|_q\|\Lambda^s\theta\|_p \tag{2.12}$$

for $i = 1, 2$. To continue, we estimate $\|\theta\|_q$ by the following maximum principle,

$$\|\theta\|_{L^q} \leq \|\theta_0\|_{L^q} + \int_0^t \|f(\tau)\|_{L^q}\, d\tau. \tag{2.13}$$

For details on this inequality and its proof we refer the reader to [10], [1], but we briefly describe the main idea, as given by Wu [15]. Specifically, (2.13) follows by multiplying both sides of (1.2) by $q|\theta|^{q-2}\theta$ and integrating with respect to x to get

$$\begin{aligned} \frac{d}{dt}\|\theta\|_{L^q}^q \quad &\leq \quad q\left(\int |\theta|^{q-2}\theta f\, dx\right. \\ &\qquad \left. - \int |\theta|^{q-2}\theta(u \cdot \nabla\theta)\, dx - \int |\theta|^{q-2}\theta\kappa(-\triangle)^\alpha\theta)\, dx\right). \end{aligned}$$

One sees that the second integral on the right is zero. The last integral on the right can be shown to be positive [10], [15]. Thus

$$\frac{d}{dt}\|\theta\|_{L^q}^q \leq q\int |\theta|^{q-2}\theta f\, dx \leq q\|f\|_{L^q}\|\theta\|_{L^q}^{q-1}$$

and (2.13) follows. Because $f \in L^1(0,\infty; L^q)$, we proved

$$\|\Lambda^s(\theta u_i)\|_2 \leq C(\theta_0, f)\|\Lambda^s\theta\|_p$$

for $i = 1, 2$, where $C_0(\theta_0, f)$ is independent of t, depends only on θ_0, f. By Lemma 2.2,

$$\|\Lambda^s(\theta u_i)\|_2 \leq C(\theta_0, f)\|\Lambda^{s+1-(2/p)}\theta\|_2$$

for $i = 1, 2$. Using this in (2.11), (2.10) follows, establishing our claim, with $C(\kappa, \theta_0, f) = \frac{4}{\kappa}C(\theta_0, f)^2$. Combining (2.8), (2.9) and (2.10) yields

$$\frac{1}{2}\frac{d}{dt}\|\Lambda^\beta\theta(t)\|_2^2 + \frac{3\kappa}{4}\|\Lambda^{\alpha+\beta}\theta(t)\|_2^2 \leq C_0\|\Lambda^\gamma\theta\|_2^2 + \frac{2}{\kappa}\|\Lambda^{\beta-\alpha}f\|_2^2 \tag{2.14}$$

where $C_0 = C(\kappa, \theta_0, f)$ and we introduced $\gamma = s + 1 - \frac{2}{p} = \beta - \alpha + 2(1 - \frac{1}{p})$. To continue estimating, let $B_M = \{\xi : |\xi|^2 \leq M\}$, with $M > 0$ to be determined appropriately below. The choice $2/(2\alpha-1) < q < \infty$ implies $\frac{1}{2} > \frac{1}{p} = \frac{1}{2} - \frac{1}{q} > 1-\alpha$, hence $\frac{1}{p} + \alpha - 1 > 0$. Thus $\gamma = \alpha + \beta - 2(\frac{1}{p} + \alpha - 1) < \alpha + \beta$ and

$$\begin{aligned} \|\Lambda^\gamma\theta(t)\|_2^2 dx \quad &= \quad \int_{B_M} |\xi|^{2\gamma}|\hat{\theta}(t)|^2\, d\xi + \int_{B_M^c} |\xi|^{2\gamma}|\hat{\theta}(t)|^2\, d\xi \\ &\leq \quad M^{2\gamma}\|\theta(t)\|_2^2 + M^{-4(\frac{1}{p}+\alpha-1)}\|\Lambda^{\alpha+\beta}\theta(t)\|_2^2 \end{aligned}$$

Selecting M large enough to satisfy $M^{-4(\frac{1}{p}+\alpha-1)} < \kappa/(4C_0)$, it follows that

$$C_0\|\Lambda^\gamma\theta(t)\|_2^2 dx \leq \frac{\kappa}{4}\|\Lambda^{\alpha+\beta}\theta(t)\|_2^2 + C_0M^{2\gamma}\|\theta(t)\|_2^2. \tag{2.15}$$

Next,

$$\begin{aligned} \|\Lambda^{\alpha+\beta}\theta\|_2^2 \quad &\geq \quad \int_{B_M^c} |\xi|^{2(\alpha+\beta)}|\hat{\theta}|^2\, d\xi \geq M^{2\alpha}\int_{B_M^c} |\xi|^{2\beta}|\hat{\theta}|^2\, d\xi \\ &= \quad M^2\alpha\|\Lambda^\beta\theta\|_2^2 - M^{2\alpha}\int_{B_M} |\xi|^{2\beta}|\hat{\theta}|^2\, d\xi \end{aligned}$$

implying

$$\|\Lambda^{\alpha+\beta}\theta\|_2^2 \geq M^2\alpha\|\Lambda^\beta\theta\|_2^2 - M^{2(\alpha+\beta)}\|\theta(t)\|_2^2. \tag{2.16}$$

By Theorem 3.1 in [4] (stated as Theorem 2.1 in this article), $\|\theta(t)\|_2$ decays at the rate of $(1+t)^{-1/\alpha}$. Using this estimate in (2.15) and (2.16), and then returning to (eq:L1), we get

$$\frac{d}{dt}\|\Lambda^\beta\theta(t)\|_2^2 + \kappa M^{2\alpha}\|\Lambda^\beta\theta(t)\|_2^2 \le \tilde{C}_0 M^c(1+t)^{-1/a} + \frac{2}{\kappa}\|\Lambda^{\beta-\alpha}f(t)\|_2^2 \tag{2.17}$$

where $\tilde{C}_0$ is a new constant depending only on f, θ_0, κ and $c = \max(2\gamma, 2\alpha+2\beta)$. For convenience, let $\nu = \kappa M^{2\alpha}$. Multiplying both sides of (2.17) by $e^{\nu t}$ and integrating in time we see that

$$\begin{aligned} \|\Lambda^\beta\theta(t)\|_2^2 \quad &\le \quad e^{-\nu t}\|\Lambda^\beta\theta_0\|_2^2 + \tilde{C}_0 M^c \int_0^t e^{-\nu(t-s)}(s+1)^{-\frac{1}{\alpha}}\,ds \\ &\quad + \frac{2}{\kappa}\int_0^t e^{-\nu(t-s)}\|\Lambda^{\beta-\alpha}f(s)\|_2^2\,ds. \end{aligned}$$

The desired estimate (2.6) now follows, since

$$\int_0^t e^{-\nu(t-s)}(1+s)^{-\frac{1}{\alpha}}\,ds \quad \le \quad C(1+t)^{-\frac{1}{\alpha}}, \tag{2.18}$$

$$\int_0^t e^{-\nu(t-s)}H(s)\,ds \quad \le \quad \int_0^t H(s)\,ds \tag{2.19}$$

for all $t \ge 0$, some C.

This completes the formal part of the proof. To make the above arguments rigorous, apply the same proof to the "retarded mollifications θ_n" which are solutions of the sequence of approximate equations

$$\frac{\partial\theta_n}{\partial t} + u_n\cdot\nabla\theta_n + \kappa(-\triangle)^\alpha\theta_n = f, \tag{2.20}$$

where $u_n = \Psi_{\delta_n}(\theta_n)$ is obtained from θ_n by

$$\Psi_{\delta_n}(\theta_n) = \int_0^t \phi(\tau)\mathcal{R}^\perp\theta_n(t-\delta_n\tau)\,d\tau. \tag{2.21}$$

and $\mathcal{R}^\perp$ is defined by (1.5).

The function ϕ is smooth, has support in $[1,2]$ and $\int_0^\infty \phi(t)\,dt = 1$. This construction is similar to the one used by Caffarelli, Kohn and Nirenberg in [2] for solutions to the Navier-Stokes equations. It is easy to see that for each n the values of u_n depend only on the values of θ_n in $[t-2\delta_n, t-\delta_n]$. As stated in [4] the θ_n converge to a weak solution θ and strongly in L^2 almost everywhere in t. Since the bounds for the $\Lambda^\beta\theta_n$ are independent of n it follows that they hold for the limiting solution θ.

This concludes the proof of the theorem

□

Remark 2.6. In proving Theorem 2.4 we estimated (see (2.19))

$$\int_0^t e^{-\nu(t-s)}\|\Lambda^{\beta-\alpha}f(s)\|_2^2\,ds \le \int_0^t \|\Lambda^{\beta-\alpha}f(s)\|_2^2\,ds$$

to get the second term on the right hand side of (2.6). The assumption $f=0$ then causes the L^2-norm of $\Lambda^\beta\theta(t)$ to decay in time. However, by (2.18), it follows that we have decay of this norm as long as $\|\Lambda^{\beta-\alpha} f\|_2$ decays fast enough. For example, if

$$\|\Lambda^{\beta-\alpha} f(t)\|_2 \leq C(1+t)^{-\delta}$$

for some $\delta > 0$, then (2.7) can be replaced by

$$\|\Lambda^{\beta}\theta(t)\|_{L^2} \leq C_0(1+t)^{-\min(\frac{1}{2\alpha},\delta)} \tag{2.22}$$

where C_0 only depends on f and the initial datum θ_0.

The remainder of this section deals with obtaining L^∞ bounds of the solution; more precisely, L^1-bounds of the Fourier transform of the solution. If the hypotheses of Theorem 2.4 are satisfied with $\beta > 1$, it is clear that $\hat{\theta}(t) \in L^1$ and $\|\hat{\theta}(t)\|_1$ is uniformly bounded in t. In fact, $\theta \in L^2 \cap \tilde{H}^\beta = H^\beta$, hence

$$\int_{\mathbb{R}^2} |\hat{\theta}(\xi)|\, d\xi \leq C \left(\int_{\mathbb{R}^2} (1+|\xi|^2)^\beta |\hat{\theta}(\xi)|^2\, d\xi \right)^{1/2}$$

with

$$C = \left(\int_{\mathbb{R}^2} (1+|\xi|^2)^{-\beta} |\, d\xi \right)^{1/2} < \infty.$$

In the next lemma, we show that we also have $\hat{\theta}(t) \in L^1$, with a uniformly bounded L^1-norm, if $\beta = 1$.

The next lemma gives an a priori bound of the L^1 norm of $\hat{\theta}(t)$. It then suffices to establish a local existence theorem to obtain a global uniform bound.

Lemma 2.7. *(A priori bound) Assume the hypothesis of Theorem (2.4) with $\beta \geq 1$. If $\beta = 1$, assume also that $\hat{\theta}_0 \in L^1$ and that $\hat{f} \in L^1(0,\infty,L^1)$. It follows that there exists $C \geq 0$ such that*

$$\|\hat{\theta}(t)\|_1 \leq C$$

for all $t \geq 0$

Remark 2.8. The hypothesis on f in case $\beta = 1$ can be considerably weakened, but the proof becomes somewhat more involved.

Proof: Since we only want an a priori bound the proof is formal. The case $\beta > 1$ was dealt with in the remarks preceding this lemma; we assume from now on that $\beta = 1$. By Theorem 2.4, there exists $C \geq 0$ such that

$$\|\nabla\theta(t)\|_2 = \|\Lambda\theta(t)\|_2 \leq C$$

for all $t \geq 0$. An easy calculation yields

$$\hat{\theta} = e^{-\kappa|\xi|^{2\alpha}t}\hat{\theta}_0 - \int_0^t e^{-\kappa|\xi|^{2\alpha}(t-s)} \widehat{u \cdot \nabla\theta}\, ds + H(t)$$

where

$$H(t) = \int_0^t e^{-\kappa|\xi|^{2\alpha}(t-s)} \hat{f}(s)\, ds.$$

By the additional hypothesis on f, it is obvious that $H(t)$ is uniformly bounded in the L^1-norm. Hence

$$\|\hat{\theta}(t)\|_1 \leq \|\hat{\theta}_0\|_1 + \int_0^t \|e^{-\kappa|\xi|^{2\alpha}(t-s)} \widehat{u \cdot \nabla\theta}\|_1\, ds + C \tag{2.23}$$

where C is chosen so that $\|H(t)\|_1 \leq C$ for all $t \geq 0$. Since the first term of the right hand side of (2.23) is bounded by hypothesis, we only need to bound the second term. For this purpose, we split it into two parts for an appropriate value of $\epsilon > 0$ as follows.

$$\int_0^t \|e^{-\kappa|\xi|^{2\alpha}(t-s)}\widehat{u \cdot \nabla\theta}\|_1 \, ds = I + II$$

where, if $t \geq \epsilon$,

$$\begin{aligned} I &= \int_0^{t-\epsilon} \|\widehat{u \cdot \nabla\theta}\|_1 \, ds, \\ II &= \int_{t-\epsilon}^t \|e^{-\kappa|\xi|^{2\alpha}(t-s)}\widehat{u \cdot \nabla\theta}\|_1 \, ds; \end{aligned}$$

if $0 \leq t < \epsilon$, then $I = 0$ and $II = \int_0^t \|\widehat{u \cdot \nabla\theta}\|_1 \, ds$. We begin bounding II, assuming $t > \epsilon$.

$$\begin{aligned} II &\leq \int_{t-\epsilon}^t \|e^{-\kappa|\xi|^{2\alpha}(t-s)}\|_2\|\widehat{u \cdot \nabla\theta}\|_2 \, ds \leq C\int_{t-\epsilon}^t \frac{1}{(t-s)^{\frac{1}{2\alpha}}}\|\nabla\theta\|_2\|u\|_\infty \, ds \\ &\leq C\sup_{t\geq 0}\|\nabla\theta(t)\|_2 \sup_{0\leq s\leq t}\|\hat{\theta}(s)\|_1 \epsilon^{1-\frac{1}{2\alpha}}, \end{aligned}$$

where we used that $\|u(t)\|_\infty \leq C\|\hat{u}(t)\|_1 \leq C\|\hat{\theta}(t)\|_1$, since the components of $\hat{u}$ are obtained multiplying $\hat{\theta}$ by functions of absolute value 1. Since $\|\nabla\theta(t)\|_2$ is bounded in t, we can select $\epsilon > 0$ so that

$$II \leq \frac{1}{2}\sup_{0\leq s\leq t}\|\hat{\theta}(s)\|_1 \tag{2.24}$$

for all $t \geq \epsilon$. Assuming now $t < \epsilon$, we estimate essentially the same way to get

$$II \leq C\sup_{t\geq 0}\|\nabla\theta(t)\|_2 \sup_{0\leq s\leq t}\|\hat{\theta}(s)\|_1 \int_0^t (t-s)^{-\frac{1}{2\alpha}} \, ds \leq C\sup_{0\leq s\leq t}\|\hat{\theta}(s)\|_1\epsilon^{1-\frac{1}{2\alpha}}.$$

Decreasing the size of $\epsilon > 0$ if necessary, we can assume that (2.24) also holds for

$0 < t < \epsilon$.

To bound I, we use the fact that $\|u(s)\|_2 = \|\theta(s)\|_2 \leq C(1+s)^{-1/2\alpha} \leq C$, $\|\nabla\theta(s)\|_2 \leq C$ for all $s \geq 0$ (some constant C). We assume $t \geq \epsilon$ (otherwise $I = 0$).

$$\begin{aligned} I &\leq \int_0^{t-\epsilon} \|e^{-\kappa|\xi|^{2\alpha}(t-s)}\|_1\|\widehat{u \cdot \nabla\theta}\|_\infty \, ds \leq C\int_0^{t-\epsilon} \frac{1}{(t-s)^{\frac{1}{\alpha}}}\|\widehat{u \cdot \nabla\theta}\|_\infty \, ds \\ &\leq C\int_0^{t-\epsilon} \frac{1}{(t-s)^{\frac{1}{\alpha}}}\|u\|_2\|\nabla\theta\|_2 \, ds \leq C\int_0^{t-\epsilon} \frac{1}{(t-s)^{\frac{1}{\alpha}}} \, ds. \end{aligned}$$

The last integral is bounded by $C\epsilon^{1-\frac{1}{\alpha}}$ if $\alpha < 1$, by $C\log(1/\epsilon)$ if $\alpha = 1$; in either case by a constant since ϵ has been fixed. In other words I is bounded independently of t; using this and (2.24) in (2.23), we get

$$\|\hat{\theta}(t)\|_1 \leq C + \frac{1}{2}\sup_{0\leq s\leq t}\|\hat{\theta}(s)\|_1$$

for all $t \geq 0$, C independent of t. The lemma follows.

□

Corollary 2.9. Under the hypotheses of Lemma 2.7 one has that $\|\theta(t)\|_\infty$, $\|\hat{u}(t)\|_1$, and $\|u(t)\|_\infty$, are uniformly bounded in t.

Proof: Since the components of u are Riesz transforms of θ (hence, the components of $\hat{u}$ differ from $\hat{\theta}$ by factors of absolute value 1), it is immediate from Lemma 2.7 that $\|\hat{u}(t)\|_1$ is uniformly bounded in time. The uniform bound on the L^∞ norms now follows.

□

The next Lemma gives the local existence for solutions with data θ_0, where $\theta_0 \in H^1$ and $\hat{\theta_0} \in L^1$

Lemma 2.10. *Let $\theta_0 \in H^1$ and $\hat{\theta_0} \in L^1$, and f satisfies the hypothesis of Theorem (2.4). Let $\alpha \in (1/2, 1]$, $\beta = 1$. Then there exists $T > 0$ and a solution θ of(0.1) such that $\theta \in L^\infty([0,T] : H^1)$ and $\hat{\theta} \in L^\infty([0,T] : L^1)$*

Proof: The proof follows by a straightforward application of the Contraction Mapping Theorem to the sequence of solutions of the equations

$$\begin{aligned} \frac{\partial \theta_n}{\partial t} + (-\mathcal{R}_2\theta_{n-1}, \mathcal{R}_1\theta_{n-1}) \cdot \nabla\theta_n + \kappa(-\triangle)^\alpha\theta_n &= f, \\ \theta|_{t=0} = \theta_0 \end{aligned}$$

Theorem 2.11. *Under the conditions of Theorem (2.4) there exists a global solution $\theta \in L^\infty([0,\infty) : H^1)$ such that $\hat{\theta} \in L^\infty([0,\infty) : L^1)$*

Proof:
Combine the two last Lemmas.

3. H^m and fractional derivatives decay

In this section we improve the decay of the derivatives of order β of the solution θ of (1.2), assuming the external force $f = 0$. The decay established in the last section is not optimal but does provide the stepping stone to obtain the optimal decay; that is, a decay rate which coincides with that of the underlying linear part. The main tool used is the Fourier splitting method, (see [11], [12]). The solutions considered here are supposed to be smooth. The assumption that the external force is zero is not essential. The same results can be obtained when $f \neq 0$, provided $\|\Lambda^{\beta-\alpha} f\|_2$ decays sufficiently fast (see Remark 2.6 above and Corollary 3.4 at the end of this section). The proof is the same as the one presented below, with the addition of a term that decays sufficiently fast by hypothesis.

We assume throughout this section that $\alpha \in (\frac{1}{2}, 1]$, $m \geq \alpha$ and θ is the solution of (1.2) (with $f = 0$ until further notice) such that $\theta_0 = \theta(0)$ satisfies $\theta_0 \in L^1(\mathbb{R}^2) \cap H^m(\mathbb{R}^2)$. The hypotheses of Theorem 2.4 are thus satisfied for any $\beta \in [\alpha, m]$. The numbers p, q are as in the previous section; $\frac{1}{p} + \frac{1}{q} = \frac{1}{2}$, $0 < \frac{1}{q} < \alpha - \frac{1}{2}$.

Before improving the rate of decay of the derivatives of θ, we state some of the immediate consequences of Theorem 2.4.

Corollary 3.1. Under the assumptions mentioned above, the following estimates hold for $t \geq 0$:

$$\|\theta(t)\|_{H^m} \leq C(1+t)^{-\frac{1}{2\alpha}}, \tag{3.1}$$

$$\|u(t)\|_{H^m} \leq C(1+t)^{-\frac{1}{2\alpha}},, \tag{3.2}$$

C a constant depending only on norms of the initial datum; moreover, if $m \geq 1$, r any exponent in $[2, \infty)$, then

$$\|\theta(t)\|_r \leq C_r(1+t)^{-\frac{1}{2\alpha}}, 0 \leq \gamma \leq \beta - 1, \tag{3.3}$$

$$\|\Lambda^\gamma u(t)\|_r \leq C_r(1+t)^{-\frac{1}{2\alpha}}, 0 \leq \gamma \leq \beta - 1, \tag{3.4}$$

C_r a constant depending only on norms of the initial datum and r.

Proof: Since $H^m = \{g \in L^2 : \Lambda^m g \in L^2\}$, inequality (3.1) is immediate from Theorem 2.4; inequality (3.2) follows then from Remark 2.3. Inequalities (3.3), (3.4) follow from these and Sobolev's Theorem.

The next Theorem will give the optimal decay rate of decay for the derivatives in the sense that it coincides with the decay rate of the underlying linear part.

Theorem 3.2. *Assume θ is a solution of (2.2) with data $\theta_0 \in L^1 \cap H^m$. Then*

$$\|\Lambda^\beta \theta(t)\|_{L^2} \leq C(t+1)^{-\frac{\beta+1}{2\alpha}}, \tag{3.5}$$

where C is a constant which depends only on the norms of the initial datum.

Proof: The proof is based on an appropriately modified Fourier splitting method, combined with the preliminary estimates of the last section. We will assume $\alpha < 1$; referring to [5] for the case $\alpha = 1$.

Assume $\alpha \leq \beta \leq m$. We return to the derivation of inequality (2.14) in the proof of Theorem 2.4, recalling that $C(\kappa, \theta_0, f) = \frac{4}{\kappa} C(\theta_0, f)^2$ and $C(\theta_0, f)$ was a bound for $\|\theta(t)\|_q$ given by the maximum principle. If we forego this bound, we obtain directly for some constant C_1, all $t \geq 0$,

$$\frac{1}{2}\frac{d}{dt}\|\Lambda^\beta\theta(t)\|_2^2 + \frac{3\kappa}{4}\|\Lambda^{\alpha+\beta}\theta(t)\|_2^2 \leq C_1\|\theta(t)\|_q^2 \|\Lambda^{\beta-\alpha+1-\frac{2}{p}}\theta(t)\|_2^2. \tag{3.6}$$

Because $\beta < \beta - a + 1 - \frac{2}{p} < \alpha + \beta$, with $\delta > 0$ such that $\beta - a + 1 - \frac{2}{p} = (1-\delta)\beta + \delta(\alpha = \beta)$, we have

$$\|\Lambda^{\beta-\alpha+1-\frac{2}{p}}\theta(t)\|_2^2 \leq \|\Lambda^\beta\theta(t)\|_2^{2(1-\delta)}\|\Lambda^{\alpha+\beta}\theta(t)\|_2^{2\delta} \leq \frac{\kappa}{4C_0}\|\Lambda^\beta\theta(t)\|_2^2 + C_2\|\Lambda^{\alpha+\beta}\theta(t)\|_2^2,$$

where we take C_0 so that $C_1\|\theta(t)\|_q^2 \leq C_0$ for all $t \geq 0$; C_2 being determined by this choice. Inequality (3.6) can be modified to

$$\frac{1}{2}\frac{d}{dt}\|\Lambda^\beta\theta(t)\|_2^2 + \frac{\kappa}{2}\|\Lambda^{\alpha+\beta}\theta(t)\|_2^2 \leq C_2\|\theta(t)\|_q^2\|\Lambda^\beta\theta(t)\|_2^2. \tag{3.7}$$

For $t \geq 0$ set

$$S(t) = \{\xi : |\xi|^{2\alpha} \leq \frac{\mu}{\kappa(t+1)}\},$$

where μ is chosen so that $\mu > \frac{\beta+1}{\alpha} + 1$. Then

$$\begin{aligned}
\|\Lambda^{\alpha+\beta}\theta(t)\|_2^2 &= \int_{\mathbb{R}^2} |\xi|^{2(\alpha+\beta)}|\theta(t)|^2\, d\xi \\
&\geq \frac{2\mu}{3\kappa(t+1)} \int_{S(t)^c} |\xi|^{2\beta}|\theta(t)|^2\, d\xi \\
&= \frac{2\mu}{3\kappa(t+1)} \left(\|\Lambda^{\beta}\theta(t)\|_2^2 - \int_{S(t)} |\xi|^{2\beta}|\theta(t)|^2\, d\xi \right) \\
&\geq \frac{2\mu}{3\kappa(t+1)} \left(\|\Lambda^{\beta}\theta(t)\|_2^2 - \left(\frac{2\mu}{3\kappa(t+1)}\right)^{\frac{\beta}{\alpha}} \|\theta(t)\|_2^2 \right);
\end{aligned}$$

estimating $\|\theta(t)\|_2$ by const$\cdot(1+t)^{-1/(2\alpha)}$, and putting it into (3.7) we get, after we estimate the factor $\|\theta(t)|_q^2$ in (3.7) by $C(t+1)^{-\frac{1}{\alpha}}$ (Corollary 3.1),

$$\frac{1}{2}\frac{d}{dt}\|\Lambda^{\beta}\theta(t)\|_2^2 + \frac{\mu}{2(t+1)}\|\Lambda^{\beta}\theta(t)\|_2^2 \leq C(t+1)^{-\frac{1}{\alpha}}\|\Lambda^{\beta}\theta\|_2^2 + C(t+1)^{-\frac{\beta+1}{\alpha}-1}. \tag{3.8}$$

Assume proved for some λ, $0 < \lambda < (\beta+1)/\alpha$, some $C \geq 0$, and all $t \geq 0$, that

$$\|\Lambda^{\beta}\theta\|_2^2 \leq C(t+1)^{-\lambda}. \tag{3.9}$$

Then using this in (3.8) we obtain, after multiplying by the integrating factor $2(t+1)^{\mu}$,

$$\frac{d}{dt}\left((t+1)^{\mu}\|\Lambda^{\beta}\theta(t)\|_2^2\right) \leq C(t+1)^{\mu-\frac{1}{\alpha}-\lambda} + C(t+1)^{\mu-\frac{\beta+1}{\alpha}-1}.$$

Integrating from 0 to t, and then dividing by $(t+1)^{-\mu}$

$$\|\Lambda^{\beta}\theta(t)\|_2^2 \leq \|\Lambda^{\beta}\theta(0)\|_2^2 + C)(t+1)^{-\mu} + C(t+1)^{1-\frac{1}{\alpha}-\lambda} + C(t+1)^{-\frac{\beta+1}{\alpha}}.$$

It follows that in (3.9) we can replace λ by $\min(\lambda + \frac{1}{\alpha} - 1, \frac{\beta+1}{\alpha})$. Since $\frac{1}{\alpha} - 1 > 0$, we are done. □

Corollary 3.3. Under the conditions of the last theorem it follows that the solutions to DGD equations have the decay in L^p

$$\|D^j u\|_p \leq C_p(t+1)^{-\frac{1}{\alpha}[\frac{j+2}{2}-\frac{1}{p}]}$$

Proof: Use the estimates in Theorem (3.2) and [4] combined with a Gagliardo-Nirenberg inequality.

$$\|D^j u\|_p \leq C_p\|u\|_2^{1-a}\|D^{j+1}\|_2^{a} \tag{3.10}$$

where $a = 1 - \frac{2}{j+1}\frac{1}{p}$. Thus

$$\|D^j u\|_p \leq C_p(t+1)^{-[(1-a)\frac{1}{2\alpha}+a\frac{j+2}{2\alpha}]} \tag{3.11}$$

Replacing a with its definition gives the expected decay. □

In the case that $f \neq 0$ we can obtain the same results of Theorem(3.2) provided f decays at the appropriate rate. More precisely

Corollary 3.4. Under the conditions of Theorem (3.2), suppose f satisfies (2.1) and

$$\|\Lambda^{\beta-\alpha} f(\cdot,t)\|_2^2 \leq C(1+t)^{-\frac{\beta+1}{2}-1} \tag{3.12}$$

if θ is a solution to (2.2) with data θ_0 then

$$\|\Lambda^{\beta}\theta(t)\|_{L^2} \leq Ct^{-\frac{\beta+1}{2\alpha}}, \tag{3.13}$$

where C is constant which depends only on the L^2 norm of the data and f.

Proof: The proof follows the same steps of the last theorem. □

4. L^1 AND IMPROVED L^p DECAY

In this section we consider the decay in L^p spaces for $p \in [1,\infty]$. New conditions on the data will be necessary to insure decay of the solutions in the L^p-norms when $p \in [1,2)$, mainly that a Riesz potential of the data lies in the corresponding L^p space.

We first consider the L^1 decay of the solutions for the special case when $\alpha = 1/2$. In the more general case when $\alpha \in (\frac{1}{2},1)$ the L^1 decay for derivatives of higher order will be obtained. The case of $\alpha = 1$ is the easiest since the linear part is the heat equation.

4.1. **Linear asymptotics.** Let $\alpha \in (0,1]$, $\kappa > 0$. We consider the linear equation

$$\frac{\partial\theta}{\partial t} + \kappa(-\triangle)^{\alpha}\theta = 0, \tag{4.1}$$

in $\mathbb{R}^2 \times \mathbb{R}$; the solution $\theta = \theta(x,t)$ is a function of a space variable $x \in \mathbb{R}^2$ and a time variable $t \geq 0$. Without loss of generality, we assume $\kappa = 1$.

The function G_α will be defined for $\alpha \in (0,1]$ by

$$\hat{G}_\alpha(\xi,t) = e^{-|\xi|^{2\alpha}t}.$$

The solution θ of (4.1) with initial datum θ_0 is then given by

$$\theta(t) = e^{t\Lambda^{2\alpha}}\theta_0 = G_a(t) * \theta_0.$$

We recall once again that if $0 < \beta < 2$, the Riesz potential I_β is defined in the Fourier variables by

$$\widehat{(I_\beta w)}(\xi) = \frac{\widehat{w}(\xi)}{|\xi|^{\beta}}.$$

Then we can write

$$\partial^\gamma\theta(t) = (\partial^\gamma\Lambda^\beta G_a)(t) * (I_\beta\theta_0). \tag{4.2}$$

By a standard change of variables, since $n = 2$, it follows that

$$(\partial^\gamma\Lambda^\beta G_\alpha)(x,t) = t^{-(\frac{\beta}{2\alpha}+\frac{|\gamma|}{2\alpha}+\frac{1}{\alpha})}(\partial^\gamma\Lambda^\beta G_\alpha)(t^{-\frac{1}{2\alpha}}x,1) \tag{4.3}$$

hence, by the Hausdorff-Young inequality,

$$\|\partial^\gamma\theta(t)\|_p \leq t^{-(\frac{\beta}{2\alpha}+\frac{|\gamma|}{2\alpha}+\frac{1}{\alpha}[1-\frac{1}{p}])}\|\partial^\gamma\Lambda^\beta G_\alpha(1)\|_p\|I_\beta\theta_0\|_1 \tag{4.4}$$

for all $t \geq 0$, $1 \leq p \leq \infty$. Thus, in order to establish the L^p decay of $\partial^\gamma\theta(t)$ it will suffice to prove that $\partial^\gamma\Lambda^\beta G_\alpha(1)$ is in L^p. We do this in the next lemma.

Lemma 4.1. *Assume $\alpha \geq \frac{1}{2}$ and let $p \in [1,\infty]$. Then $\partial^\gamma\Lambda^\beta G_\alpha(1) \in L^p$ for all $\beta \geq 0$ and all multi-indices $\gamma = (\gamma_1,\gamma_2)$.*

Proof: Since

$$\partial^\gamma \widehat{\Lambda^\beta G_\alpha}(1)(\xi) = \xi^\gamma |\xi|^\beta e^{-|\xi|^{2\alpha}}$$

is integrable, it follows that $\partial^\gamma \Lambda^\beta G_\alpha(1) \in L^\infty$ (for all $\alpha > 0$). All that remains to be proved is that $\partial^\gamma \Lambda^\beta G_\alpha(1) \in L^1$.

We consider two cases; $\alpha > \frac{1}{2}$ and $\alpha = \frac{1}{2}$. Assume first $\alpha > \frac{1}{2}$. It is not hard to see that

$$\begin{aligned} \left|\Delta\left(\partial^\gamma \widehat{\Lambda^\beta G_\alpha}\right)(\xi, 1)\right| &= \left|\Delta\left(\xi^\gamma |\xi|^\beta e^{-|\xi|^{2\alpha}}\right)\right| \\ &\le C(1 + |\xi|^N)|\xi|^{|\gamma|+\beta+2\alpha-2} e^{-|\xi|^{2\alpha}} \end{aligned}$$

for some constants $C, N \ge 0$, all $\xi \in \mathbb{R}^2$. It follows that $\left|\Delta\left(\partial^\gamma \widehat{\Lambda^\beta G_\alpha}\right)(\xi, 1)\right|^2$ near 0 behaves like $|\xi|^{2|\gamma|+2\beta+2\alpha-4}$ which is integrable since, because $\alpha > \frac{1}{2}$, $2|\gamma| + 2\beta + 4\alpha - 4 \ge 4\alpha - 4 > -2$. It follows that $\Delta\left(\partial^\gamma \widehat{\Lambda^\beta G_\alpha}\right)(1)$ is in L^2, hence so is $|x|^2 \partial^\gamma \Lambda^\beta G_\alpha(1)$. It being clear that $G_\alpha(1) \in L^2$, it follows that $(1+|x|^2)\partial^\gamma \Lambda^\beta G_\alpha(1)$ is in L^2; since $(1 + |x|^2)^{-1}$ is in L^2, the proof that $\partial^\gamma \Lambda^\beta G_\alpha(1)$ is in L^1 is complete.

Assume now $\alpha = \frac{1}{2}$. Then

$$\begin{aligned} \Lambda^\beta G_{\frac{1}{2}}(x, 1) &= \frac{1}{2\pi} \int_{\mathbb{R}^2} |\xi|^\beta e^{ix\cdot\xi} e^{-|\xi|}\, d\xi \\ &= \frac{1}{2\pi} \int_0^{2\pi} \int_0^\infty r^{\beta+1} e^{-r(1-ix\sin\theta)}\, dr\, d\theta \\ &= \frac{\Gamma(\beta+2)}{2\pi} \int_0^{2\pi} \frac{d\theta}{(1 - ix\sin\theta)^{\beta+2}}. \end{aligned}$$

¿From the last expression it is immediate that $\partial^\gamma \Lambda^\beta G_\alpha(1) \in L^1(\mathbb{R}^2)$ if $\beta + |\gamma| > 0$. It remains to see that the same is true if $\beta = 0$, $\gamma = 0$. However, in this case the integral is easily computed by residues; one has

$$G_{\frac{1}{2}}(x, 1) = \frac{1}{2\pi} \int_0^{2\pi} \frac{d\theta}{(1 - ix\sin\theta)^2} = \frac{1}{4(1 + |x|^2)^{3/2}}.$$

The last expression is clearly integrable over $\mathbb{R}^2$.

□

Remark 4.2. The last Lemma is valid for $\alpha \in (0, 1/2)$ provided $\beta + |\gamma| \ge 1$. In fact, essentially the same proof as for the case $\alpha > \frac{1}{2}$ applies. The only relation α, β, γ had to satisfy for the argument to be valid was $2|\gamma| + 2\beta + 4\alpha - 4 > -2$, which clearly holds if $\alpha > 0$ and $\beta + |\gamma| \ge 1$.

The results in the remainder of this section are based on the ideas described in [5] to study the L^1 decay for solutions to viscous conservation laws.

Theorem 4.3. *Let $\alpha \in (0, 1]$, let $0 < \beta$ and assume that $I_\beta \theta_0 \in L^1(\mathbb{R}^2)$. Let $\gamma = (\gamma_1, \gamma_2)$ be a multi-index; assume $|\gamma| + \beta \ge 1$ if $\alpha < \frac{1}{2}$. Set*

$$A = \lim_{|\xi|\to 0} \frac{\widehat{\theta_0}(\xi)}{|\xi|^\beta} = \int_{\mathbb{R}^2} (I_\beta \theta_0)(x)\, dx. \tag{4.5}$$

Let $\theta(t) = e^{-t\Lambda^{2\alpha}}\theta_0 = G_a(t) * \theta_0$ be the solution of (4.1) with initial datum θ_0. Then, for $1 \le p \le \infty$,

$$\|\partial^\gamma \theta(t)|_p \le C t^{-\frac{\beta}{2\alpha} - \frac{|\gamma|}{2\alpha} - (1-\frac{1}{p})\frac{1}{\alpha}} \|I_\beta \theta_0\|_1 \tag{4.6}$$

for all $t > 0$ and $C = C(\beta, \gamma)$ independent of t and θ_0. Moreover,

$$t^{\frac{\beta}{2\alpha} + \frac{|\gamma|}{2\alpha} + (1-\frac{1}{p})\frac{1}{\alpha}} \|\partial^\gamma \theta(t) - A \partial^\gamma \Lambda^\beta G_\alpha(t)\|_p \to 0 \tag{4.7}$$

as $t \to \infty$.

Proof: By Lemma 4.1 (see also the remark following it) we have $\partial^\gamma \Lambda^\beta G_\alpha \in L^p$. Writing

$$\partial^\gamma \theta(t) = (\partial^\gamma \Lambda^\beta G_\alpha(t)) * I_\beta \theta_0,$$

in view of Lemma 4.1, (4.6) is immediate from (4.4) $C = \|\partial^\gamma \Lambda^\beta G_\alpha(1)\|_p$.

For the proof of (4.7), we can write

$$\begin{aligned}
&\left|\partial^\gamma \theta(x,t) - A\partial^\gamma \Lambda^\beta G_\alpha(x,t)\right| \\
&\le \int_{\mathbb{R}^2} \left|(\partial^\gamma \Lambda^\beta G_\alpha)(x-y,t) - (\partial^\gamma \Lambda^\beta G_\alpha)(x,t)\right| |I_\beta \theta_0(y)| \, dy \\
&\le \left(\int_{\mathbb{R}^2} |I_\beta \theta_0(y)| \, dy\right)^{\frac{p-1}{p}} \left(\int_{\mathbb{R}^2} \left|(\partial^\gamma \Lambda^\beta G_\alpha)(x-y,t) - (\partial^\gamma \Lambda^\beta G_\alpha)(x,t)\right|^p |I_\beta \theta_0(y)| \, dy\right)^{1/p}.
\end{aligned}$$

Raising to the power p, integrating with respect to x, and changing the variables by $z = xt^{-\frac{1}{2\alpha}}$, combined with the self-similar form of G_α (see (4.3)) leads to the following expression

$$\begin{aligned}
&\|\partial^\gamma \theta(t) - A\partial^\gamma \Lambda^\beta G_\alpha(t)\|_p^p \\
&\quad \le \|I_\beta \theta_0\|_1^{p-1} \int_{\mathbb{R}^2 \times \mathbb{R}^2} \left|(\partial^\gamma \Lambda^\beta G_\alpha)(x-y,t) - (\partial^\gamma \Lambda^\beta G_\alpha)(x,t)\right|^p |I_\beta \theta_0(y)| \, dxdy \\
&\quad = t^{-\frac{p}{2\alpha}(\beta + |\gamma| + 2(1-\frac{1}{p}))} \|I_\beta \theta_0\|_1^{p-1} \\
&\qquad \times \int_{\mathbb{R}^2 \times \mathbb{R}^2} \left|(\partial^\gamma \Lambda^\beta G_\alpha)(z - t^{-\frac{1}{2}}y, 1) - (\partial^\gamma \Lambda^\beta G_\alpha)(z,1)\right|^p |I_\beta \theta_0(y)| \, dxdy.
\end{aligned}$$

To complete the proof of (4.7) we only need to show that

$$\lim_{t\to\infty} \int_{\mathbb{R}^2 \times \mathbb{R}^2} \left|(\partial^\gamma \Lambda^\beta G_\alpha)(z - t^{-\frac{1}{2}}y, 1) - (\partial^\gamma \Lambda^\beta G_\alpha)(z,1)\right|^p |I_\beta \theta_0(y)| \, dxdy = 0. \tag{4.8}$$

The Fourier transforms of all the derivatives of the function $\partial^\gamma \Lambda^\beta G(1)$ are in L^1; it follows that this function is infinitely many times differentiable, with all derivatives bounded. Thus the integrand in (4.8) converges uniformly to 0 over compact subsets of $\mathbb{R}^2 \times \mathbb{R}^2$. Moreover, by Lemma 4.1, the function (and its derivatives) are in L^p. By this L^P-integrability, and the integrability of $I_\beta \theta_0$, one can find for each $\epsilon > 0$ a compact subset K_ϵ of $\mathbb{R}^2 \times \mathbb{R}^2$ such that

$$\int_{(\mathbb{R}^2 \times \mathbb{R}^2)\setminus K_\epsilon} \left|(\partial^\gamma \Lambda^\beta G_\alpha)(z - t^{-\frac{1}{2}}y, 1) - (\partial^\gamma \Lambda^\beta G_\alpha)(z,1)\right|^p |I_\beta \theta_0(y)| \, dxdy < \epsilon.$$

This, and the aforementioned uniform convergence, prove (4.8). This completes the proof of the theorem. $\square$

Remark 4.4. In the case $\alpha = 1$ we recall that, in [7], [6] Miyakawa obtained the L^1-decay of $e^{t\Delta}u_0$ provided the $|x|^\beta$-momentum of the data is bounded. The assumption on the Riesz potential is weaker than the one assumed by Miyakawa, see [5].

□

We also obtain, as an immediate corollary to Lemma 4.1:

Corollary 4.5. Let $\alpha \in (0, 1]$, let $0 < \beta$ and assume that $I_\beta\theta_0 \in L^p(\mathbb{R}^2)$; $1 \le p \le \infty$. Let $\gamma = (\gamma_1, \gamma_2)$ be a multi-index; assume $|\gamma| + \beta \ge 1$ if $\alpha < \frac{1}{2}$. Then there exists a constant $C \ge 0$ such that

$$\|\partial^\gamma e^{-t\Lambda^{2\alpha}}\theta_0\|_p \le Ct^{-\frac{\beta}{2\alpha}-\frac{|\gamma|}{2\alpha}}\|I_\beta\theta_0\|_p \tag{4.9}$$

for all $t > 0$.

Proof: By (4.2) and (4.3), and the Hausdorff-Young inequality,

$$\begin{aligned}\|\partial^\gamma e^{-t\Lambda^{2\alpha}}\theta_0\|_p &\le t^{-\frac{\beta}{2\alpha}-\frac{|\gamma|}{2\alpha}-\frac{1}{\alpha}}\|\partial^\gamma\Lambda^\beta G_\alpha(t^{-\frac{1}{2\alpha}}\cdot, 1)\|_1\|I_\beta\theta_0(t)\|_p \\ &= t^{-\frac{\beta}{2\alpha}-\frac{|\gamma|}{2\alpha}}\|\partial^\gamma\Lambda^\beta G_\alpha(1)\|_1\|I_\beta\theta_0(t)\|_p\end{aligned}$$

and the result follows from lemma 4.1.

□

4.2. **Nonlinear Asymptotics.**

The next step is to use the results from the last section to get the decay of the solutions to the geostrophic equations in L^1 and with that improve the decay of the solutions in L^p. The decay will be obtained by estimating the solutions via their integral representation. We note that the decay below might not be optimal. So as to be able to include the critical case $\alpha = \frac{1}{2}$, we recall the following result due to Constantin, Córdoba and Wu [1].

Theorem 4.6. *There exists a constant c_∞ such that for any $\theta_0 \in H^2(\mathbb{R}^2)$ with $\|\theta_0\|_{H^2} \le c_\infty$, the equation*

$$\theta_t + u \cdot \nabla\theta + \Lambda\theta = 0$$

has a unique global solution θ with initial datum θ_0, satisfying

$$\|\theta(t)\|_{H^2} \le \|\theta_0\|_{H^2}$$

for all $t \ge 0$.

Combining this theorem with Theorem 2.1, and using the Gagliardo- Nirenberg inequalities, one obtains for this solution θ, $u = \mathcal{R}^\perp\theta$

$$\|\theta(t)\|_\infty \le C\|\theta\|_2^{\frac{1}{2}}\|\Lambda^2\theta\|_2^{\frac{1}{2}} \le C\|\theta_0\|_{H_2}^{\frac{1}{2}}(1+t)^{-\frac{1}{2}}, \tag{4.10}$$

$$\|u(t)\|_\infty \le C\|u\|_2^{\frac{1}{2}}\|\Lambda^2 u\|_2^{\frac{1}{2}} \le C\|\theta_0\|_{H_2}^{\frac{1}{2}}(1+t)^{-\frac{1}{2}}, \tag{4.11}$$

and by Hölder,

$$\|\nabla\theta(t)\|_2 \le \|\theta(t)\|_2^{\frac{1}{2}}\|\Lambda^2\theta(t)\|_2^{\frac{1}{2}} \le \|\theta_0\|_{H_2}^{\frac{1}{2}}(1+t)^{-\frac{1}{2}}. \tag{4.12}$$

We assume θ is this solution in case $\alpha = \frac{1}{2}$.

Theorem 4.7. *Let $\beta > 0$, assume that $I_\beta\theta_0 \in L^1(\mathbb{R}^2)$, and let θ be the solution of the homogeneous DQG with initial datum θ_0.*

i: *Assume $\frac{1}{2} \leq \alpha < 1$. Then*

$$\|\theta(t)\|_1 \leq Ct^{-\nu}$$

for all $t > 0$, some constant C, where

$$\nu = \begin{cases} \min(\beta, \frac{1}{2}) & \text{if} \quad \alpha = \frac{1}{2}, \\ \min(\frac{\beta}{2\alpha}, \frac{1}{2\alpha}) & \text{if} \quad \frac{1}{2} < \alpha < 1. \end{cases}$$

ii: *Assume $\alpha = 1$. Then*

$$\|\theta(t)\|_1 \leq \begin{cases} Ct^{-\frac{\beta}{2}} & \text{if} \quad \beta < 1, \\ Ct^{-\frac{1}{2}}\log(t+1) & \text{if} \quad \beta \geq 1, \end{cases}$$

for some constant C.

Proof: Write the solution by its integral representation,

$$\theta(t) = G_\alpha(t) * \theta_0 + \int_0^t G_\alpha(s) * (u \cdot \nabla\theta)(t-s)\, ds = G_\alpha(t) * \theta_0 + I(t). \tag{4.13}$$

From the last section it follows that

$$\|G_\alpha(t) * \theta_0\|_1 \leq Ct^{-\frac{\beta}{2\alpha}}\|I_\beta\theta_0\|_1 \tag{4.14}$$

and the theorem reduces to proving that

$$I(t) = \int_0^t \|G_\alpha(s) * (u \cdot \nabla\theta)(t-s)\|_1\, ds \leq C(1+t)^{-\nu}$$

is appropriately bounded. By Hausdorff-Young, Hölder, and the fact that $u\nabla\theta = \operatorname{div}(u\theta)$,

$$\begin{aligned} I(t) &= \int_0^{t/2} \|G_\alpha(s) * (u \cdot \nabla\theta)(t-s)\|_1\, ds + \int_{t/2}^t \|\nabla G_\alpha(s) * (u\theta)(t-s)\|_1\, ds \\ &\leq \int_0^{t/2} \|G_\alpha(s)\|_1 \|u(t-s)\|_2 \|\nabla\theta(t-s)\|_2\, ds + \int_{t/2}^t \|\nabla G_\alpha(s)\|_1 \|u(t-s)\|_2 \|\theta(t-s)\|_2\, ds \\ &= J(t) + K(t). \end{aligned}$$

By the results of the last section we have

$$\|G_\alpha(s)\|_1 = \|G_\alpha(1)\|_1 = C \tag{4.15}$$

$$\|\nabla G_\alpha(s)\|_1 = s^{-\frac{1}{2\alpha}}\|\nabla G_\alpha(1)\|_1 = Cs^{-\frac{1}{2\alpha}}, \tag{4.16}$$

C a constant depending only on α. We also have

$$\|u(t-s)\|_2\|\nabla\theta(t-s)\|_2 \leq \begin{cases} (1+t)^{-3/2} & \text{if} \quad \alpha = \frac{1}{2}, \\ (1+t)^{-\frac{3}{2\alpha}} & \text{if} \quad \frac{1}{2} < \alpha \leq 1. \end{cases}$$

The estimate for $\alpha = 1/2$ comes from Theorem 2.1 and (4.12), the one for $\alpha > \frac{1}{2}$ from Theorem 3.2. Using this and (4.15) we get, with $\mu = \frac{3}{2}$ if $\alpha = \frac{1}{2}$, $\mu = \frac{3}{2\alpha}$, otherwise

$$J(t) \leq C\int_0^{t/2} (1+t-s)^{-\mu}\, ds \leq C(1+t)^{1-\mu},$$

since in all cases $\mu > 1$. Note that $\mu - 1 = \frac{1}{2}$ when $\mu = \frac{3}{2}$ and for all other μ's it follows that $\mu - 1 \geq \frac{1}{2\alpha}$, so $\mu - 1 \geq \nu$ in all cases. Thus

$$J(t) \leq C(1+t)^{-\nu} \tag{4.17}$$

and we are done with the estimate for J. To estimate $K(t)$ we use that $\|u(t-s)\|_2\|\theta(t-s)\|_2 \leq C(1+t-s)^{-\frac{1}{\alpha}}$ and (4.16) to get

$$K(t) \leq C\int_{t/2}^{t} s^{-\frac{1}{2\alpha}}(1+t-s)^{-\frac{1}{\alpha}}\,ds \leq \begin{cases} Ct^{-\frac{1}{2\alpha}} & \text{if} \quad \frac{1}{2} \leq \alpha < 1, \\ Ct^{-\frac{1}{2}}\log t & \text{if} \quad \alpha = 1. \end{cases} \tag{4.18}$$

The conclusion of the theorem follows now from (4.13) and (4.14), using (4.17) and (4.18) to bound $I(t) = J(t) + K(t)$

□

Derivatives of the solution θ can be similarly bounded, at least if $\alpha > \frac{1}{2}$. We have

Theorem 4.8. *Let $\beta > 0$. Assume $I_\beta\theta_0 \in L^1(\mathbb{R}^2)$, $\theta_0 \in H^m$ for some $m \geq 1$ and let γ be a multi-index, $|\gamma| \leq m-1$. Then*

$$\|\partial^\gamma\theta(t)\|_1 \leq \begin{cases} Ct^{-\min(\frac{\beta+|\gamma|}{2\alpha},\frac{|\gamma|+1}{2\alpha})}, & \frac{1}{2} < \alpha < 1, \\ Ct^{-\min(\frac{\beta+|\gamma|}{2\alpha},\frac{|\gamma|+1}{2\alpha})}\log(t+1), & \alpha = 1. \end{cases}$$

Proof: Proceeding as in the proof of Theorem 4.7, we get

$$\|\partial^\gamma\theta(t)\|_1 \leq \|\partial^\gamma G_\alpha(t) * \theta_0\|_1 + J_\gamma(t) + K_\gamma(t),$$

where the terms on the right hand side now have the following interpretations and bounds:

$$\|\partial^\gamma G_\alpha(t) * \theta_0\|_1 \leq Ct^{-\frac{\beta+|\gamma|}{2\alpha}},$$

by (4.6). For the second term, using the estimates in Theorem 3.2 we obtain first

$$\|\partial^\gamma(u\cdot\nabla\theta)(t-s)\|_1 \leq \sum_{|\gamma_1|+|\gamma_2|=|\gamma|+1} c_{\gamma_1,\gamma_2}\|\partial^{\gamma_1}u(t-s)\|_2\|\partial^{\gamma_2}\theta(t-s)\|_2 \leq C(1+t-s)^{-\frac{|\gamma|+3}{2\alpha}}, \tag{4.19}$$

(the coefficients c_{γ_1,γ_2} coming from Leibnitz' formula), hence (since $3 - 2\alpha \geq 1$)

$$J_\gamma(t) = \int_0^{t/2} \|G_\alpha(s)\|_1\|\partial^\gamma(u\cdot\nabla\theta)(t-s)\|_1\,ds \leq C(t+1)^{-\frac{|\gamma|+3-2\alpha}{2\alpha}} \leq C(t+1)^{-\frac{|\gamma|+1}{2\alpha}}$$

For the third term we use that $\|\nabla\partial^\gamma G_\alpha(t)\| = Ct^{-\frac{|\gamma|+1}{2\alpha}}$ and, as in Theorem 4.7, that

$$\|u(t-s)\theta(t-s)\|_1 \leq (1+t-s)^{-\frac{1}{\alpha}}$$

to get

$$K_\gamma(t) = \int_{t/2}^{t} \|\nabla\partial^\gamma G_\alpha(s)\|_1\|(u\theta)(t-s)\|_1\,ds \leq \begin{cases} Ct^{-\frac{|\gamma|+1}{2\alpha}} & \text{if} \quad \frac{1}{2} < \alpha < 1, \\ Ct^{-\frac{|\gamma|+1}{2}}\log(t+1) & \text{if} \quad \alpha = 1. \end{cases}$$

The theorem follows. □

Finally, we see that the solution θ is asymptotically equivalent to the self-similar solution of the linear equation, at least if $\beta < 1$. For a given $\beta > 0$, (4.3) shows that the self-similar solution $\partial^\gamma\Lambda^\beta G_\alpha$ of the linear equation decays in L^1-norm at the rate of $t^{-\frac{\beta+|\gamma|}{2\alpha}}$ as $t \to \infty$. Theorem 4.8 shows that the derivative ∂^γ of the solution of the non-linear equation (with datum θ_0 satisfying $I_\beta\theta_0 \in L^1$) decays (at least) at the same rate if $\beta < 1$. By asymptotic equivalence, we mean that the difference of θ and the self-similar solution of the linear equation decays at a better rate.

Theorem 4.9. *Assume* $0 < \beta < 1$ *and the hypotheses of Theorem 4.8. Then, with*

$$A = A_\beta = \int_{\mathbb{R}^2} (I_\beta\theta_0)(x)\, dx,$$

one has

$$\lim_{t\to\infty} t^{\frac{\beta+|\gamma|}{2\alpha}} \left\|\partial^\gamma\theta(t) - A\partial^\gamma\Lambda^\beta G_\alpha(t)\right\|_1 = 0.$$

Proof: We have

$$\partial^\gamma\theta(t) - A\partial^\gamma\Lambda^\beta G_\alpha(t) = \partial^\gamma(G_a(t)*\theta_0) - A\partial^\gamma\Lambda^\beta G_\alpha(t) + \int_0^t \partial^\gamma G_\alpha(s)*(u\cdot\nabla\theta)(t-s)\, ds$$

and by Theorem 4.3 it suffices to prove that

$$\lim_{t\to\infty} t^{\frac{\beta+|\gamma|}{2\alpha}} H(t) = 0 \tag{4.20}$$

where

$$H(t) = \int_0^t \|\partial^\gamma G_\alpha(s) * (u\cdot\nabla\theta)(t-s)\|_1\, ds.$$

This is, however, immediate from the proof of Theorem 4.8. In fact, we have $H \leq J_\gamma + K_\gamma$, where J_γ, K_γ are as in the proof of Theorem 4.8. It follows that $H(t)$ decays at the rate of either $t^{-\frac{|\gamma|+1}{2\alpha}}$ $(\alpha < 1)$ or $t^{-\frac{|\gamma|+1}{2\alpha}} \log t$ $(\alpha = 1)$; in either case (4.20) holds because $\beta < 1$.

□

Corollary 4.10. Under the conditions of the last Theorem if $|\gamma| = 0$ the conclusion of the theorem is also valid for the case $\alpha = \frac{1}{2}$

Proof: The proof follows the same lines as the last theorem. □

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